Math 540 Exam
January, 2011

Calculators, books and notes are not allowed!

1. Let $m^*$ be an outer measure defined by

$$m^*(E) = \inf \{ m(U) : U \text{ open and } E \subseteq U \} .$$

Here $m$ stands for the Lebesgue measure on $\mathbb{R}$. Prove that for any $E \subseteq \mathbb{R}$, there exists a Lebesgue measurable set $G$ such that $m(G) = m^*(E)$.

2. a) State Fatou’s Lemma and the monotone convergence theorem.  
   b) Use Fatou’s lemma to prove the monotone convergence theorem.

3. Let $1 \leq p < \infty$. Suppose that $f_n \to f$ in measure, and $|f_n| \leq g \in L^p$ for all $n \in \mathbb{N}$. Prove that $\lim_{n \to \infty} \|f_n - f\|_p = 0$.

4. Let $1 < p < \infty$ and $f, g \in L^p(\mathbb{R}^n)$. Prove that $\|f + g\|_p = \|f\|_p + \|g\|_p$ if and only if there exists a non-negative constant $C$ such that either $f = Cg$ a.e. or $g = Cf$ a.e.

5. Suppose $f : \mathbb{R} \to \mathbb{C}$. $F$ is said to be Lipschitz with constant $M$ if there is a constant $M$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Prove that $f$ is Lipschitz with constant $M$ iff $f$ is absolutely continuous and $|f'| \leq M$ a.e.

6. Let $f_n$'s and $f$ be measurable complex-valued functions on a measure space $(X, A, \mu)$. We say that $f_n \to f$ almost uniformly on $X$ if for every $\varepsilon > 0$, there exists $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E$. Prove that if $f_n \to f$ almost uniformly on $X$, then $f_n \to f$ a.e. and in measure.