Problem 1 (20 points) Let $R$ be a ring (with identity, as always).

(a) Explain what is meant by saying that a short exact sequence splits. Given an example of a ring $R$ and a non-split short exact sequence of $R$ modules.

(b) Let $k$ be a field. Show that every short exact sequence of vector spaces is split.

(c) With $k$ still a field, let $p, q, r$ be non-negative integers and suppose we have a short exact sequence

$$0 \to k^p \to k^q \to k^r \to 0.$$ 

What can you say about the relation between the integers $p, q, r$ in this case? Explain.

Problem 2 (20 points) Let $R$ be a ring, with identity.

(a) Let us define a projective $R$-module $P$ as one that is a direct summand of a free module: $P \oplus M = F$ for some module $M$ and a free module $F$. Show that any short exact sequence

$$0 \to A \to B \to P \to 0$$

splits if $P$ is projective.

(b) Let $0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0$ be a short exact sequence of $R$-modules. Suppose that we are given homomorphisms $\alpha_1 : P_1 \to A$, $\alpha_2 : P_2 \to C$, where $P_i$ are projective. Show that there is a projective module $P$ and a homomorphism $\alpha : P \to B$ such that the following diagram commutes for some homomorphisms $d_1, d$

$$
\begin{array}{ccc}
0 & \to & P_1 \xrightarrow{d_1} P \xrightarrow{d} P_2 \to 0 \\
\downarrow{\alpha_1} & & \downarrow{\alpha} & \downarrow{\alpha_2} \\
0 & \to & A \xrightarrow{f} B \xrightarrow{\pi} C \to 0
\end{array}
$$

and such that the top row is also exact. Explain in detail what $P$ is (and why is it projective!), and what the maps $d_1, d, \alpha$ are.
(c) Let \( M = \mathbb{Z}/3 \mathbb{Z} \) and consider \( M \) as a module over the rings \( R \) given below.
- If \( R = \mathbb{Z} \) is \( M \) free? Projective?
- If \( R = \mathbb{Z}/3 \mathbb{Z} \) is \( M \) free? Projective?
- If \( R = \mathbb{Z}/6 \mathbb{Z} \) is \( M \) free? Projective?

In each case explain how \( M \) is a module over the given ring \( R \).

**Problem 3 (20 points)** Let \( R, S, T \) be not necessarily commutative rings with identity.

(a) If \( R M_S \) and \( S N_T \) are bimodules as shown, describe in detail the module structure of \( M \otimes_S N \) and justify your answer.

(b) Given a bimodule \( R M_S \), prove that there is an isomorphism of left \( R \)-modules \( M \otimes_S S \simeq M \).

(c) Given bimodules \( L_R, R M_S, N_S \), show that there is an isomorphism of abelian groups

\[
\text{Hom}_S(L \otimes_R M, N) \simeq \text{Hom}_R(L, \text{Hom}_S(M, N)).
\]

**Problem 4 (20 points)** Let \( R \) be a ring with identity and let \( M \) be an \( R \)-module such that \( M = M_1 + M_2 + \cdots + M_k \) where the \( M_i \) are simple submodules of \( M \).

(a) Prove that \( M \) is the direct sum of certain of the \( M_i \).

(b) If \( N \) is a submodule of \( M \), prove that \( N \) is a direct summand of \( M \), i.e., \( M = N \oplus L \) for some submodule \( L \).

(c) Prove that every submodule and quotient module of \( M \) is a direct sum of simple submodules isomorphic with certain \( M_i \)'s.

**Problem 5 (20 points)** Let \( G \) be a finite group and \( \mathbb{C} \) the complex field.

(a) Explain in detail how the structure of the group algebra \( \mathbb{C} G \) determines the irreducible \( \mathbb{C} \)-representations of \( G \).

(b) Prove that \( |G| = n_1^2 + n_2^2 + \cdots + n_k^2 \) where the \( n_i \) are the degrees of the irreducible \( \mathbb{C} \)-representations of \( G \).

(c) Let \( G \) be the symmetric group of degree 3. Find the degrees of the irreducible \( \mathbb{C} \)-representations of \( G \) and describe the corresponding representations.