Math 542, Comprehensive Examination
January 18, 2012

Solve all eight problems. Each problem is worth 10 points.

Notation: \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Re} \, z > 0 \} \), \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \).

1) Is there an analytic function \( f : \mathbb{H} \to \mathbb{C} \) such that
\[
\text{Re} \, f(z) = x \arctan \left( \frac{y}{x} \right) + \frac{y \log(x^2 + y^2)}{2} \quad \text{for all } z = x + iy \in \mathbb{H}.
\]
Justify your claim.

2) Evaluate the integral
\[
\int_{\gamma} \frac{e^{-z}}{z^2 - 2} \, dz,
\]
where \( \gamma \) is the imaginary axis with positive upward orientation.

3) Find a conformal map of \( \mathbb{H} \setminus \{ z = x + iy : x \geq 1, \ y = 0 \} \) onto \( \mathbb{H} \).

4) Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of entire functions. Assume that this sequence converges to a polynomial \( f \) of degree \( d \geq 1 \) uniformly on compact subsets of \( \mathbb{C} \).
   (i) Prove that there exists \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \), \( n \geq N \), the function \( f_n \) has at least \( d \) zeroes, counting multiplicity.
   (ii) Is it true that there must exist \( N \in \mathbb{N} \) such that each \( f_n \), \( n \geq N \), has exactly \( d \) zeroes? Justify your claim.

5) Let \( f \) be an analytic function in \( \mathbb{D} \), and assume that
\[
\left| f\left( \frac{1}{n} \right) \right| \leq \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}, \ n \geq 2.
\]
Prove that \( f \) vanishes identically in \( \mathbb{D} \).

6) Let \( f \) be an analytic function in \( \mathbb{D} \) with \( |f(z)| \leq M \) for some \( M > 0 \) and all \( z \in \mathbb{D} \).
Prove that \( |f'(1/2)| \leq 4M/3 \). Is this bound sharp? Justify your claim.

7) Let \( G(z) \) be defined by the infinite product
\[
G(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n}.
\]
   (i) Show that \( G(z) \) defines an entire function.
   (ii) Show that \( \pi z G(z) G(-z) = \sin(\pi z) \) for all \( z \in \mathbb{C} \).

8) Let \( h(e^{i\theta}) \) be a continuous function on the unit circle \( \mathbb{T} \). Show that
\[
\tilde{h}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} h(e^{i\varphi}) \, d\varphi
\]
defines a harmonic function in \( \mathbb{D} \) and that \( \lim_{z \to z_0} \tilde{h}(z) = h(z_0) \) for all \( z_0 \in \mathbb{T} \).