Math 542 Comprehensive Examination
January 2013

Each problem is worth 10 points. Justify all the claims that you make.

For $z \in \mathbb{C}$, let $\Re z$ and $\Im z$ denote the real and imaginary parts of $z$, respectively, so that $z = \Re z + i\Im z$. Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} : \Im z > 0\}$.

1. Find all entire functions $f$ that satisfy the inequality
   $$|f(z)| \leq |z|^{3/2}, \quad \forall z \in \mathbb{C}.$$  

2. Find a conformal map of the domain
   $$D = \mathbb{D} \setminus \{z = \Re z + i0 : \Re z \in [0, 1)\},$$
   obtained by removing the half-open interval $[0, 1)$ from the unit disc, onto the unit disc $\mathbb{D}$.

3. Find all entire functions whose set of zeroes coincides with the set of all non-negative integers, and so that each zero has order two.

4. Use the Residue Theorem to calculate the integral
   $$\int_0^\infty \frac{x^{-1/6}}{x + 1} \, dx.$$  

5. How many solutions does the equation
   $$z^4 - z^3 - 3z^2 + 8z + 2 = 0$$
   have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$?

6. Suppose that $D$ is a domain in $\mathbb{C}$. Is there a sequence $(u_n)_{n \in \mathbb{N}}$ of harmonic functions in $D$ that converges uniformly on compacta in $D$ to the function $u(x, y) = x^3 - 2xy^2$?

7. Prove that if $f : \mathbb{H} \to \mathbb{H}$ is an analytic function and $t$ is a positive real number, then $|f'(it)| \geq 1$ implies $\Im f(it) \geq t$.

8. Prove the following statement if true, or give a counterexample if it is false. If $K$ is a compact subset of a domain $D \subseteq \mathbb{C}$ and $f$ is an analytic function in $D$, then there exists a sequence of polynomials $(p_n(z))_{n \in \mathbb{N}}$ that converges to $f$ uniformly on $K$. 