Comprehensive exam

(1) Consider the smooth vector field on $\mathbb{R}^2$, $V = -2x \partial_x + 2y \partial_y$. Find the 1-parameter group of diffeomorphisms corresponding to $V$. That is, compute the flow of $V$.

(2) In $\mathbb{R}^3$, with coordinates $(x, y, z)$, let

$$V = xy \partial_x - \partial_y,$$

and

$$\omega = z \, dx \wedge dy + dy \wedge dz$$

Compute the following:
(a) $i_V \omega$,  
(b) $d \omega$,  
(c) $\mathcal{L}_V \omega$,  
(d) $\Phi^* \omega$ where $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the map $(s, t) \mapsto (s + t, s, t)$.

(3) Consider the pair of vector fields $V = \partial_y$ and $W = y \partial_x - \partial_z$ on $\mathbb{R}^3$, where we use the standard coordinates $(x, y, z)$. Is it possible to find a 2-dimensional submanifold of $\mathbb{R}^3$ with the property that both $V$ and $W$ are tangent to it at all points? If so, construct one; if not, why not?

(4) Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$

$$\alpha = -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy.$$  

Prove that $d \alpha = 0$ and hence that the integral of $d \alpha$ over the unit disk

$$D := \{(x, y) \mid x^2 + y^2 \leq 1\}$$

is 0. Show that the integral of $\alpha$ over the unit circle

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$$

is not zero. Does this contradict Stokes theorem? Explain.