**MATH 530 - Comprehensive Examination - May 2006**

**Instructions:** Do four of the following six problems. Select two problems from 1, 2, 3 and two problems from 4, 5, 6. Indicate clearly which problems you have selected. Each question is worth 25 points, Maximum score is 100 points.

Here is a theorem that you may wish to quote in your solutions:

**Minkowski's bound:** Let $K$ be a finite extension of the rational numbers with degree $n$ and let $O_K$ be the set of algebraic integers in $K$ and $\Delta$ the discriminant of $O_K$ over the integers. Assume $K$ has $r$ embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal $I$ in $O_K$ that satisfies

$$|N_{K/Q}(I)| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$ 

**** Select two problems from 1, 2, 3 ****

1) Let $K = \mathbb{Q}(\sqrt{61})$. Show that the ring of integers $O_K$ of $K$ is a principal ideal domain.

2) Let $K$ be a number field with ring of integers $O_K$ and group of units $E_K$.
   a) State Dirichlet’s Unit Theorem for $E_K$.
   b) Let $K = \mathbb{Q}(\sqrt[3]{2})$ and let $L/\mathbb{Q}$ be the normal closure of $K/\mathbb{Q}$. Describe $E_K$ and $E_L$ as abstract abelian groups.
   c) Find subgroups of finite index in $E_K$ and $E_L$ with explicit generators.

3) Let $K = \mathbb{Q}(\sqrt{21}, \sqrt{33})$.
   a) For each of the primes $p = 2, 3$ give the decomposition parameters $e, f, r$ for the decomposition of $p$ in $K/Q$.
   b) For each of the primes $p = 2, 3$ give the decomposition field and the inertia field for $K/Q$. 
4) Let $K$ be a number field with ring of integers $O_K$. Given any ideal class $c$ of $K$ and any ideal $I$ of $O_K$, show that there exists an ideal $J$ of $O_K$ in the class $c$ that is relatively prime to $I$.

5) Let $K$ be a number field with ring of integers $O_K = \mathbb{Z}[\alpha]$. Let $f$ be the minimum polynomial for $\alpha$ over $\mathbb{Q}$. For a prime $p$ in $\mathbb{Z}$ let

$$\bar{f} = f_1 \cdot f_2 \cdots f_r \pmod{p}$$

be the factorization of $f$ modulo $p$ into irreducible factors. Denote by $I_1$ the ideal $(p, f_1(\alpha)) \subset \mathbb{Z}[\alpha]$, where $f_1 \in \mathbb{Z}[x]$ is a polynomial that reduces modulo $p$ to $f_1$. Assume that $I_1 \neq \mathbb{Z}[\alpha]$.

Show that $I_1$ is a prime ideal of $O_K$ dividing $p$.

6) Show that the quotient group $\mathbb{Q}_5^*/(\mathbb{Q}_5^*)^2$ is finite, where $\mathbb{Q}_5$ is the complete $5$—adic field.