Math 540 Comprehensive Examination
May 16, 2006

Solve all five problems. All problems have equal value.

$m$ denotes the Lebesgue measure on $\mathbb{R}$.

**I.** Prove the following equality by using an infinite series expansion. Justify the term-by-term integration.

$$\int_{[0,1]} \frac{\ln x}{x-1} \, dm(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$ Does this equality hold if the integral is regarded as a Riemann integral?

**II.** Suppose $f_n : [0, 1] \to [0, \infty)$ are measurable, $\| f_n \|_2 \leq 1$ for all $n$, and $f_n \to f$ a.e. on $[0, 1]$.
(i) Show $f \in L^2([0, 1])$.
(ii) Show $\| f_n - f \|_1 \to 0$ as $n \to \infty$.
Hint. One way of solving (ii) is by using Egoroff’s theorem.

**III.** Suppose $f \in L^1(\mathbb{R}, m)$. Prove that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\int_E |f| \, dm < \varepsilon \text{ whenever } E \text{ is measurable and } m(E) < \delta.$$

**IV.** (i) Let $f \in L^p([0, 1], m)$, $1 < p < \infty$. Show

$$\lim_{y \to 0^+} y^{\frac{1-p}{p}} \int_{[0,y]} f(x) \, dm(x) = 0.$$ (ii) Is it true that

$$\bigcap_{1 \leq p < \infty} L^p(\mathbb{R}, m) \subseteq L^\infty(\mathbb{R}, m)?$$ Justify your answer.

**V.** Consider the Banach space

$$\ell^p = \left\{ f : \mathbb{N} \to \mathbb{C} : \| f \|_p = \left( \sum_{n=1}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 < p < \infty.$$ Prove directly that for every element $\phi$ in the dual Banach space $(\ell^3)^*$ there is a unique element $g \in \ell^3$ such that

$$\phi(f) = \sum_{n=1}^{\infty} f(n) g(n).$$ What is the relation between $\| g \|_2$ and $\| \phi \|$?