Problem 1 Let $R$ be a ring.
   a. (10 points) Define when an $R$-module $M$ is Artinian and when it is Noetherian.
   b. (10 points) Let $R = \mathbb{Z}$, the ring of integers. Show that every Noetherian $\mathbb{Z}$-module is Artinian, or give a counter example.
   c. (10 points) Let $M$ be an $R$-module and $N \subset M$ be a submodule. Show that if both $N$ and $M/N$ are Artinian then $M$ is Artinian.

Problem 2
   a. (10 points) Let $G$ be an Abelian group of order $mn$, where $m$ and $n$ are arbitrary positive integers. Show that there is a subgroup and a quotient group of $G$ of order $m$.
   b. (10 points) Let $R = \mathbb{Z}/45\mathbb{Z}$. Find all finitely-generated $R$-modules (list without repetitions).

Problem 3 Let $M$ and $N$ be $\mathbb{Z}$-modules. In this problem all $\otimes$'s and Hom's are over $\mathbb{Z}$.
   a. (10 points) Does $M$ being projective imply $M \otimes N$ being projective? Justify.
   b. (10 points) Does $M$ being injective imply $M \otimes N$ being injective? Justify.
   c. (10 points) Simplify
      \[
      \text{Hom}(((\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Q}) \otimes ((\mathbb{Z}/9\mathbb{Z}) \oplus 2\mathbb{Z}), \mathbb{R} \oplus (\mathbb{Z}/9\mathbb{Z})).
      \]

Problem 4 Let $\mathbb{C}[t]$ be the ring of polynomials in $t$ with complex coefficients. A $\mathbb{C}[t]$-module can be described by giving a vector space $V$ (a $\mathbb{C}$-module) together with a linear operator in $V$ (the $t$-action).
   a. (10 points) Find all simple $\mathbb{C}[t]$-modules.
   b. (10 points) Find a Jordan-Hölder filtration of the $\mathbb{C}[t]$-module
      \[
      \left( \mathbb{C}^3, t \mapsto \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right).
      \]