Math 531 Comprehensive Exam
May 2007

1. Without using the Prime Number Theorem (you may use Chebyshev and/or Mertens type estimates), determine whether the series \( \sum_{p \geq 3} (p \log \log p)^{-1} \) converges or diverges. If it converges, obtain an estimate for the tails \( R(x) = \sum_{p > x} (p \log \log p)^{-1} \); if it diverges, obtain an asymptotic formula for the partial sums \( S(x) = \sum_{n \leq x} (p \log \log p)^{-1} \).

2. Let \( S(x) = \sum_{n \leq x} \log(x/p) \). Obtain an asymptotic estimate for \( S(x) \) with an error term that is by a factor \( O(1/\log x) \) smaller than the main term. (You may use any standard version of the Prime Number Theorem.)

3. Let \( d_{\text{even}}(n) \), resp. \( d_{\text{odd}}(n) \), denote the number of even, resp. odd, (positive) divisors of \( n \), and let \( f(n) = d_{\text{even}}(n) - d_{\text{odd}}(n) \).
   (i) Evaluate the Dirichlet series \( F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \), and express it in terms of the Riemann zeta function.
   (ii) Show that \( \sum_{n \leq x} f(n) = o \left( \sum_{n \leq x} d(n) \right) \) as \( x \to \infty \).

4. Show that there exists a positive constant \( c \) such that for all positive integers \( n \),
\[
\sum_{d \mid n} d \geq c \frac{n^2}{\phi(n)},
\]
and determine, with proof, the best-possible value for this constant. In other words, compute
\[
L = \inf_{n \in \mathbb{N}} \frac{\phi(n)}{n^2} \sum_{d \mid n} d.
\]
(Note that \( L \) should be given as a simple explicit expression involving familiar constants, and not, for example, sums or products over primes.)

5. Let
\[
F(s) = \sum_{n=1}^{\infty} \left\{ \frac{n}{3} \right\} \log n \frac{1}{n^s},
\]
where \( \{t\} = t - \lfloor t \rfloor \) is the fractional part of \( t \).
   (i) Express \( F(s) \) in terms of the Riemann zeta function and/or Dirichlet \( L \)-series.
   (ii) Show that \( F(s) \) has a meromorphic continuation to the half-plane \( \sigma > 0 \) and determine all its poles (if any) in this half-plane, along with their order. (I.e., determine for each pole whether it is simple, double, etc.; there is no need to compute residues.)