Solve all four problems in Part I and one problem in Part II. Indicate your choice. Credit will be given only for one problem in Part II. Each problem is worth 20 points.

\( m \) denotes the Lebesgue measure on \( \mathbb{R} \).

Part I

I. Let \( f \in L^1([0,1]) \). For \( t \geq 0 \), let \( F(t) = \int_0^1 x e^{-t/x} f(x) \, dm(x) \).

(i) (6 points) Prove that \( F(t) \) is finite for all \( t \geq 0 \).

(ii) (7 points) Prove that \( F : [0, \infty) \to \mathbb{R} \) is continuous.

(iii) (7 points) Is \( F \) differentiable? If so, calculate \( F'(t) \) for \( t > 0 \) and \( \lim_{h \to 0^+} \frac{F(h) - F(0)}{h} \).

II. Decide whether each of the following statements is true or false. Justify your answer with a short proof if the statement is true or a counterexample if it is false. (5 points each)

(a) Let \( (f_n) \) be a sequence in \( L^p([0,1]) \) which converges in \( L^p \) to \( f \in L^p([0,1]) \). Then \( (f_n) \) converges to \( f \) in measure. (A sequence \( (g_n) \) is said to converge in measure to \( g \) if \( \lim_{n \to \infty} m(\{x : |g_n(x) - g(x)| \geq \epsilon\}) = 0 \) for every \( \epsilon > 0 \).)

(b) If \( f : [a, b] \to \mathbb{R} \) is absolutely continuous and one-to-one, then \( f^{-1} \) is absolutely continuous.

(c) If \( f : [0, 1] \to \mathbb{R} \) is a measurable function so that \( \int_E f \, dm = 0 \) for all measurable sets \( E \subset [0,1] \), then \( f = 0 \) almost everywhere.

(d) If \( f : [0,1] \to \mathbb{R} \) is continuous, then \( f \) is of bounded variation.

III. For \( n \in \mathbb{N} \), let \( h_n = \sum_{j=1}^n (-1)^j \chi_{[(j-1)/n,j/j/n]} \). Show that \( \lim_{n \to \infty} \int_{[0,1]} f h_n \, dm = 0 \) for every \( f \in L^1([0,1]) \). Here \( \chi_E \) denotes the characteristic function of the set \( E \).

IV. (i) (10 points) Let \( f : [0,1] \to \mathbb{R} \) be a continuous function which is absolutely continuous on \([\epsilon, 1]\) for every \( 0 < \epsilon < 1 \), and of bounded variation on \([0,1]\). Prove that \( f \) is absolutely continuous on \([0,1]\).

(ii) (10 points) Let \( f : [0,1] \to \mathbb{R} \) satisfy \( |f(x) - f(y)| \leq |x^{1/3} - y^{1/3}| \) for all \( x, y \in [0,1] \). Must \( f \) be absolutely continuous? Justify your answer.
Part II

V. Let $f \in L^\infty([0, 1])$. Prove the following statements:

(i) (10 points) The function $p \mapsto \|f\|_p$ is nondecreasing for $1 \leq p < \infty$.
(ii) (10 points) $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

VI. Let $\mathcal{H}$ be the Hilbert space $L^2([0, 1])$.

(i) (5 points) Prove the Parallelogram Identity:

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2) \quad \forall f, g \in \mathcal{H}.$$ 

(ii) (15 points) Let $K \subset \mathcal{H}$ be a nonempty, closed, convex set, and let $f \in \mathcal{H}$. Prove that there exists a unique element $h \in K$ so that $\|f - h\| = \inf_{g \in K} \|f - g\|$. 