COMBINATORICS COMPREHENSIVE - Spring 2007

Submit only FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on each Part. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

1. During $2n$ flips of a fair coin, a running total of heads and tails is kept. Compute the probability that the lead changes (one type leads and later the other type leads), given that each outcome appears $n$ times.

2. Let $a_n$ be the number of $n$-tuples in $[4]^n$ that have at least one 1 and have no 2 appearing before the first 1 (note that $\langle a \rangle$ begins $0, 1, 6, \ldots$). Obtain and solve a recurrence for $\langle a \rangle$. Give a direct counting argument (without using summations) to prove the resulting simple formula.

3. Let $\langle a \rangle$ be the sequence whose generating function is $\frac{3-3x}{1-x-2x^2}$. Without obtaining a formula for $a_k$, obtain from the generating function a simple formula for $\sum_{k=0}^{n} a_k$ as a function of $n$.

4. Derive a summation formula whose value is the number of permutations of $[n]$ with no cycles of length 2. Explain why the formula is correct.

5. A tournament is an orientation of a complete graph, and the outdegree of a vertex in a tournament is the number of edges leaving it. Apply Hall's Theorem to prove that there is a tournament with outdegrees $d_1, \ldots, d_n$ if and only if for each $k \in [n]$ the $k$ smallest of these numbers sum to at least $\binom{k}{2}$, with equality when $k = n$. (Hint: Construct a graph $H$ with $2 \binom{n}{2}$ vertices such that a perfect matching in $H$ yields the desired tournament.)

6. The Kneser graph $K(n, k)$ has vertex set $\binom{[n]}{k}$, with two vertices adjacent if they are disjoint $k$-sets. Prove that $\chi(K(n, k)) \leq n - 2k + 2$ by covering the vertices with $n - 2k + 2$ independent sets. Prove that this is optimal when $n = 2k + 1$. (Example: The Petersen graph is $K(5, 2)$. Comment: In fact equality always holds.)

7. Use Euler's Formula to count the regions determined by a configuration of $n$ lines in the plane, where no three lines have a common point.

Part II

8. Let $b_1, \ldots, b_n$ be distinct integers. Show that some nonempty subset of $\{b_1, \ldots, b_n\}$ has sum divisible by $n$. Determine whether the same conclusion holds for every set of $n - 1$ distinct integers.

9. Use partially ordered sets to prove that every list of $mn + 1$ distinct integers has an increasing sublist with more than $m$ elements or a decreasing sublist with more than $n$ elements. (Note: A sublist of a list need not occupy consecutive positions.)

10. Consider the random graph $G$ generated with edge probability $p$, where $p$ is a function of $n$. Prove that if $pn \to 0$, then almost always $G$ has no cycles. Use a different random variable to prove that if $pn \to \infty$, then almost always $G$ has a cycle.

11. Explain how to construct a pair of orthogonal latin squares of order 15. Include all needed building blocks, but do not write out the final pair of squares.