Math 531 Comprehensive Exam
May 2008

Problem 1
Let \( \Psi(x,y) \) be the number of positive integers \( n \leq x \), all of whose prime factors are \( \leq y \).
(i) Show, for \( y \geq \sqrt{x} \), that

\[
\Psi(x,y) = \lfloor x \rfloor - \sum_{y < p \leq x} \left\lfloor \frac{x}{p} \right\rfloor ,
\]

where \( \lfloor x \rfloor \) is the greatest integer \( \leq x \).
(ii) Without using the Prime Number Theorem (you may use the Chebyshev and Mertens estimates), find an approximate formula for \( \Psi(x,y) \), for \( \sqrt{x} \leq y \leq x \), in terms of elementary functions of \( x \) and \( y \) and with an error term no worse than \( O(x/\log x) \).

Problem 2
(i) Describe precisely a number-theoretic function \( f(n) \) such that

\[
\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad (\Re s > 1).
\]

Compute \( f(35), f(105), f(315) \).
(ii) Let \( s = \sigma + it \), where \( \sigma > 1 \) and \( t \) is real. Prove that

\[
\left| \frac{1}{\zeta(s)} \right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)} .
\]

(iii) State a formula, without proof, for \( \sum_{n \leq x} f(n) \) as an integral in the complex plane involving \( \zeta(s)/\zeta(2s) \).

Problem 3
Define \( \pi(x) \) to be the number of primes \( \leq x \), \( p_n \) the \( n \)-th prime number, \( \psi(x) = \sum_{n \leq x} \Lambda(n) \) and \( M(x) = \sum_{n \leq x} \mu(n) \), where \( \Lambda \) is the von Mangoldt function and \( \mu \) is the Möbius function.
(i) Prove that \( \Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) \)
(ii) Give, without proof, statements about \( \pi(x), p_n, \psi(x) \) and \( M(x) \) which are equivalent to the Prime Number Theorem.
(iii) State, without proof, a result about the zeros of $\zeta(s)$ which is needed in the analytic proof of the Prime Number Theorem.

(iv) State the Riemann Hypothesis about $\zeta(s)$.

(v) Give a statement about one of $\pi(x)$, $p_n$, $\psi(x)$, $M(x)$ that is equivalent to the Riemann Hypothesis.

**Problem 4**

Without using the Prime Number Theorem, prove the equivalence of the statements given in Problem 3 (ii) for $\pi(x)$ and $\psi(x)$.

**Problem 5**

Let $\chi$ be a Dirichlet character mod $q$, and put $L(s, \chi) = \sum_{n \leq x} \chi(n)n^{-s}$.

(i) Evaluate

$$f(\chi) = \lim_{s \to 1} \frac{L(s, \chi)}{\zeta(s)}.$$

Make sure your answer covers all possibilities for $\chi$.

(ii) Show how

$$\sum_{\chi \text{ mod } q} \log L(s, \chi)$$

can be written as a multiple sum that does not involve Dirichlet characters.