Math 531 Comprehensive Exam
May 2009

Problem 1
(a) Let \( f(n) = \sum_{p^n | n} \frac{1}{p} \). Prove an asymptotic for \( \sum_{n \leq x} f(n) \).

(b) Let \( g \) be the multiplicative function satisfying \( g(p^a) = p^{a-1}(p + 1) \) for prime \( p \) and \( a \geq 1 \). Prove a formula for \( \sum_{n \leq x} g(n) \) with error \( O(x \log x) \).

Express the constant in terms of values of the Riemann zeta function.

Problem 2
For this problem, any form of the prime number theorem may be used.
(a) When \( x \) is very large, determine which function is larger,

\[
A(x) = x \sum_{p \leq x} \frac{1}{p}, \quad \text{or} \quad B(x) = \frac{3}{5} \sum_{x < p \leq 2x} p.
\]

(b) Determine asymptotically how many positive integers \( \leq x \) are odd, squarefree and have an even number of prime factors.

Problem 3
(i) Suppose \( t_n \) are complex numbers and \( T(x) = \sum_{n \leq x} t_n \) satisfies \( T(x) = O(x^a) \), where \( a \geq 0 \). Prove that the Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} \frac{t_n}{n^s}
\]

represents an analytic function in the half-plane \( \text{Re} \, s > a \).

(ii) Let \( t_n = \Lambda(n) - 1 \). Give a plausible estimate for \( T(x) \) that implies the Riemann Hypothesis, and prove the implication. By plausible, we mean an estimate which is not known to be false, such as \( T(x) = O(1) \). In other words, use the weakest estimate for \( T(x) \) that still implies RH.

Problem 4
Let \( \chi \) be a nonprincipal Dirichlet character modulo \( q \) and let

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{(Re} \, s > 1)\text{.}
\]
(i) Explain why $L(s, \chi)$ has an analytic continuation to the half-plane $\text{Re } s > 0$.

(ii) Let

$$H(s) = \frac{\zeta(s)}{\phi(q)} \sum_{\chi \mod q} L(s, \chi) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$ 

Find the smallest number $n$ (as a function of $q$) for which $h(n) \neq 1$. 