Comprehensive Exam — PDEs (Math 553) — May 2009

Total points 100. Do 4 problems.

Instructions Show ALL your working and make your explanations as full as possible. Calculators are not allowed on this exam, and neither are books or notes. Unless otherwise stated, \( T > 0 \) is fixed and \( \Omega \) denotes a smoothly-bounded domain in \( \mathbb{R}^n, n \geq 2 \).

You may use Green’s Formulas:

\[
\int_{\Omega} \left[ u \Delta v + \nabla u \cdot \nabla v \right] \, dx = \int_{\partial \Omega} u \frac{\partial v}{\partial n} \, dS
\]

\[
\int_{\Omega} \left[ u \Delta v - v \Delta u \right] \, dx = \int_{\partial \Omega} \left[ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] \, dS
\]

(1) (25 points) (First order equations) Consider the semilinear equation \( xu_x + y^2 u_y = 3u \).

(a) Use the method of characteristics to solve the PDE using the Cauchy data \( u(x, 1) = k(x) \), where \( k \) is an arbitrary function.

(b) Explain how the method of characteristics can fail for Cauchy data of the form \( x = s, y = g(s), z = h(s) \).

(2) (25 points) (Laplace’s equation)

Let \( u \) be a smooth function on \( \mathbb{R}^n, n \geq 2 \).

(a) Prove \( u \) is harmonic if and only if

\[
u(x) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u \, dS
\]

for all \( x \in \mathbb{R}^n, r > 0 \). Here \( |\partial B(x, r)| \) denotes the surface area of the sphere \( \partial B(x, r) \).

(b) Briefly discuss the one-dimensional case.
(3) (25 points) *(Eigenvalues of the Laplacian)* Let $u \neq 0$ be a smooth function on $\Omega$. Suppose $u$ is an eigenfunction of the negative Laplacian with eigenvalue $\lambda$, meaning $-\Delta u = \lambda u$ in $\Omega$ and $u = 0$ on $\partial \Omega$.

(a) Apply Green's Formula to prove $\lambda > 0$. (Justify your deductions.)

(b) Use the Maximum Principle to prove $\lambda > 0$. (Be careful to verify the hypotheses.)

(c) Explain the meaning of $\sqrt{\lambda}$ in terms of separation of variables and the wave equation.

(4) (25 points) *(Wave equation)*

(a) Solve the initial value problem for the wave equation in one dimension

$$
\begin{cases}
  u_{tt} - u_{xx} = 0, & x \in \mathbb{R}, \ t \in (0, \infty), \\
  u(x, 0) = f(x), u_t(x, 0) = g(x), & x \in \mathbb{R},
\end{cases}
$$

where $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

Do not use a solution formula. You should derive the solution from first principles.

(b) Apply your solution formula to $g(x) \equiv 0$ and

$$
f(x) = \begin{cases}
  0 & \text{for } x < 0, \\
  1 & \text{for } x \geq 0.
\end{cases}
$$

(Obviously this $f$ is not $C^2$-smooth.) Evaluate $u(x, t)$ in the different regions of the $xt$-plane.

(c) Briefly explain how to show $u$ in part (b) is a weak solution of the wave equation. *(Hint: transmission conditions.)* You do not need to carry out the calculations.
(5) (25 points) (Heat equation in bounded domain) Assume $u(x, t)$ is smooth on $\Omega \times [0, T]$ and solves:

\[
\begin{align*}
    u_t &= \Delta u, \quad x \in \Omega, \quad t > 0, \\
    u(x, 0) &= g(x), \quad x \in \Omega, \\
    u(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align*}
\]

where $g$ is smooth and $g \leq 0$.

(a) Explain why $u(x, t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Suppose in addition $g$ has compact support in $\Omega$, and $g(x) < 0$ for some $x \in \Omega$. It can be shown $u(x, t) < 0$ for all $x \in \Omega, 0 < t < T$.

Use this fact to justify the claim that “the heat equation has infinite propagation speed”.