Problem 1 (25 points) Let $A$ be a commutative ring with 1 and let $M$ be an $A$-module. $M$ is called divisible if for all $a \neq 0 \in A$ the multiplication map $a \cdot M \rightarrow M$ is surjective.

a. Let $A$ be an integral domain. Show that every injective module over $A$ is divisible.

b. Show that any divisible module over a PID is injective. Deduce that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective over $\mathbb{Z}$.

Problem 2 (25 points) Let $f: A \rightarrow B$ be a ring homomorphism of commutative rings. This makes $B$ into an $A$-module, using $f$. Suppose $B$ is $A$-flat.

a. Let $I \subseteq J$ be two ideals of $A$. Prove that $J/I \otimes_A B \cong JB/IB$, where $JB = f(J)B$, and $IB = f(I)B$.

b. Moreover, suppose that $B$ has the following property: if $N$ is an $A$-module then $N \otimes_A B = 0$ implies that $N = 0$. Show then that for every ideal $I \subseteq A$ we have $f^{-1}(IB) = I$.

Problem 3 (25 points) Consider a category $I$ with three objects and (besides the identities) morphisms given as follows:

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a. Let $C$ be a category and $\mathcal{F}: I \rightarrow C$ a covariant functor. Explain what a limit of $\mathcal{F}$ in $C$ is.

b. Show that if $C = R$-Mod, for some ring $R$, then the limit of a covariant functor $\mathcal{F}: I \rightarrow C$ as above always exists. Give an explicit module and morphisms that represents the limit. (Hint: it might be useful to think about the kernel of an appropriate morphism.)

See next page for the fourth problem!
Problem 4 (25 points) (Permutation representations). Let $X$ be a finite set with an action of a finite group $G$. Let $\mathcal{F}(X)$ be the vector space (over the complex numbers $\mathbb{C}$) with basis $e_x, x \in X$, and define and action of $G$ on $\mathcal{F}(X)$ by $g \cdot e_x = e_{gx}$. $\mathcal{F}(X)$ is called the permutation representation of $X$.

1. Show that the character of the permutation representation counts the elements fixed by $g$:
   \[ \chi_{\mathcal{F}(X)}(g) = \# \{ x \in X \mid gx = x \} \]

2. Consider the group $S_3$ of permutations of 3 letters. Let $S_3$ act on $V = \mathbb{C}^3$ by permuting the elements of a basis. Find the character of $V$. Is $V$ irreducible?

3. Give the character table of $S_3$, i.e., the values of the irreducible characters on all conjugacy classes. Explain.

4. Determine the decomposition of the permutation representation $V = \mathbb{C}^3$ over $S_3$ in irreducibles: write $V$ as a direct sum $n_1 V_1 \oplus n_2 V_2 \oplus n_3 V_3$, and find $n_1, n_2, n_3$. 