Math 540 Exam  
May, 2010

Calculators, books and notes are not allowed!

1. Show that for $p > 1$,

$$\lim_{n \to \infty} \int_1^n \frac{(1 - \frac{t}{n})^n}{tp} dm(t) = \int_1^\infty \frac{e^{-t}}{tp} dm(t).$$

Here $m$ is the Lebesgue measure on $\mathbb{R}$.

2. Let

$$f(x) = \begin{cases} x \sin(1/x) & \text{for } 0 < x \leq \infty \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Is $f$ is uniformly continuous on $[0, \infty)$? Prove your answer!
(b) Is $f$ of bounded variation on $[0, \infty)$? Prove your answer!

3. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{\delta \to 0} \int |f(x + \delta) - f(x)|^p dx = 0.$$  

(Hint: Use the fact that $C_0^\infty$ is dense in $L^p$. Here the space $C_0^\infty$ is the set consisting of all continuous functions with compact support.)

4. (a) State Egoroff's theorem.
(b) State the Dominated Convergence Theorem.
(c) Prove the Dominated Convergence Theorem.

5. Let $m$ be Lebesgue measure on $\mathbb{R}$. A sequence $\{f_n\}$ of measurable functions on $\mathbb{R}$ is said to converge in measure to the measurable function $f$ if, given $\varepsilon > 0$, there exists an $N$ such that

$$m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n \geq N$. Prove that
a) If $f_n \in L^p(\mathbb{R})$ and $\|f_n - f\|_p \to 0$ for some $1 \leq p \leq \infty$, then $f_n \to f$ in measure.
b) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to $f$ a.e.

6. Let $\mathbb{Q}$ be the set of all rational numbers. A coset of $\mathbb{Q}$ in additive group $\mathbb{R}$ is a set $x + \mathbb{Q} = \{y \in \mathbb{R} : y = x + r \text{ for some } r \in \mathbb{Q}\}$. Let $E$ be a set that contains exactly one point from each coset of $\mathbb{Q}$ in $\mathbb{R}$. Prove that
(a) $(r_1 + E) \cap (r_2 + E) = \emptyset$ if $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$
(b) $\mathbb{R} = \cup_{r \in \mathbb{Q}} (r + E)$.
(c) Prove that if $F \subset \mathbb{R}$ is a set such that every subset of $F$ is Lebesgue measurable, then Lebesgue measure of $F$ is 0.