Problem 1  

a. (10 points) Let $A$ be a (possibly non commutative) ring with 1 and with the property that any element $a$ of $A$ is either invertible or nilpotent ($a^N = 0$ for some $N \geq 0$). Show that $A$ is a local ring, i.e., $A$ has a unique maximal ideal. (Hint: show that if $x$ is nilpotent, then $1 - x$ is invertible.)

b. (10 points) Let $M$ be an $R$-module such that $\text{End}_R(M) (= \{ f \mid f: M \to M \text{ is } R\text{-linear} \})$ is local. Prove that $M$ is indecomposable (i.e., non-zero and not the direct sum of proper nonzero submodules).

Problem 2  

Let $A$ be a commutative ring with 1 and let $M$ be an $A$-module. $M$ is called divisible if for all $a \neq 0 \in A$ the multiplication map $M \xrightarrow{a} M$ is surjective.

a. (10 points) Let $A$ be an integral domain. Show that every injective module over $A$ is divisible.

b. (10 points) Show that any divisible module over a PID is injective. Deduce that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective over $\mathbb{Z}$.

Problem 3  

a. (8 points) Let $V$ be a finite dimensional vector space over a field $k$. Let $T \in \text{End}_k(V)$ and let $f(x) = (x - \alpha)^d$, $\alpha \in k$, $d \geq 1$ be its minimal polynomial. Assume that $V \cong k[x]/(f(x))$. Show that $V$ has a basis over $k$ such that the matrix for $T$ with respect to this basis is given by

$$
\begin{pmatrix}
\alpha & 0 & \cdots & \cdots & 0 \\
1 & \alpha & 0 & \cdots & \vdots \\
0 & 1 & \alpha & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & \alpha 
\end{pmatrix}
$$

(1)
b. (4 points) Let, as above, $T$ be a linear operator on a finite dimensional vector space, and suppose it has characteristic polynomial $f(x) = (x - \alpha_1)^{d_1}(x - \alpha_2)^{d_2} \cdots (x - \alpha_k)^{d_k}$. Write down at least one possible form for the minimal polynomial for $T$.

c. (8 points) With $T$ as in part b. show that there exists a basis for $V$ such that matrix of $T$ consists of blocks, each one of which is of the form (1) in part a.

Problem 4 Let $f: A \rightarrow B$ be a ring homomorphism of commutative rings. This makes $B$ into an $A$-module, using $f$. Suppose $B$ is $A$-flat.

a. (10 points) Let $I \subset J$ be two ideals of $A$. Prove that $J/I \otimes_A B \cong JB/IB$, where $JB = f(J)B$, and $IB = f(I)B$.

b. (10 points) Moreover, suppose that $B$ has the property that for any $A$-module $N$, $N \otimes_A B = 0$ implies that $N = 0$. Then show that for every ideal $I \subset A$ we have $f^{-1}(IB) = I$.

Problem 5 (20 points) Let $C$ be a cyclic group of order a prime number $p$. Prove that $\mathbb{Q}[C]$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}(\epsilon)$, where $\epsilon$ is a $p$-th root of unity, and $\mathbb{Q}[C]$ is the rational group algebra of $C$. 