Do your best. We are as much interested in your ability to think and reason as the correct answers.

1. 50 points Let \( \{X_n; n \in \mathbb{N}\} \) be independent and identically distributed random variables with common law given by \( \mathbb{P}\{X_1 = 1\} = p \in (0, 1) \setminus \{\frac{1}{2}\} \) and \( \mathbb{P}\{X_1 = -1\} = 1 - p \). Define \( S_n = \sum_{i=1}^{n} X_i \). For any integer \( x \), define \( T_x = \inf\{n : S_n = x\} \). Let \( \varphi(x) = \left(\frac{1-x}{p}\right)^x \).

   Fix also integers \( a < 0 < b \) and define \( T \overset{\text{def}}{=} \min\{T_a, T_b\} \).

   (a) 10 points Show that \( \varphi(S_n) \) is a martingale with respect to the filtration

   \[ \mathcal{F}_n = \sigma\{X_m; m \leq n\}. \]

   (b) 20 points Show that \( \mathbb{P}\{T < \infty\} = 1 \). Hint: compute \( \lim_{n \to \infty} \frac{1}{n} \ln \varphi(S_n) \).

   (c) 20 points Show that

   \[ \mathbb{P}\{T_a < T_b\} = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}. \]

   Hint: Consider the quantity \( \varphi(S_T) \).

2. 50 points Let \( X \) be a bounded or nonnegative random variable, and let \( \mathcal{G} \) be a sub sigma-algebra of \( \mathcal{F} \). Let \( \mathbb{P}' \) be a second probability measure on \( (\Omega, \mathcal{F}) \) which is absolutely continuous with respect to \( \mathbb{P} \), and let \( \mathbb{E}' \) be the expectation operator associated with \( \mathbb{P}' \).

   (a) 10 points Prove that \( \mathbb{P}'\text{-a.s., } \mathbb{E}'\left[\frac{d\mathbb{P}}{d\mathbb{P}'}|\mathcal{G}\right] > 0 \).

   (b) 40 points Prove that

   \[ \mathbb{E}'[X|\mathcal{G}] = \frac{\mathbb{E}'\left[X\frac{d\mathbb{P}}{d\mathbb{P}'}|\mathcal{G}\right]}{\mathbb{E}'\left[\frac{d\mathbb{P}}{d\mathbb{P}'}|\mathcal{G}\right]}. \]

3. 50 points Let \( \{\mu_i; i \in \mathcal{I}\} \) be a collection of probability measures on \( \mathbb{R} \), where \( \mathcal{I} \) is some index set. For each \( i \in \mathcal{I} \), define the characteristic function

   \[ \varphi_i(\theta) \overset{\text{def}}{=} \int_{\mathbb{R}} e^{\sqrt{-1}\theta x} \mu_i(dx), \quad \theta \in \mathbb{R} \]

   Recall that \( \{\mu_i; i \in \mathcal{I}\} \) is said to be tight if for every \( \varepsilon > 0 \), there is a compact subset \( K \) of \( \mathbb{R} \) such that

   \[ \sup_{i \in \mathcal{I}} \mu_i(K^c) < \varepsilon. \]

   Show that \( \{\mu_i; i \in \mathcal{I}\} \) is tight if \( \{\varphi_i; i \in \mathcal{I}\} \) is equicontinuous in a neighborhood of the origin. The point here is that control of the characteristic function near the origin controls the tail behavior of the \( \mu_i \)'s. We will break this up into several parts.

   (a) 10 points First prove that

   \[ \frac{1}{2\delta} \int_{-\delta}^{\delta} \left\{ 1 - \frac{\varphi_i(\theta) + \varphi_i(-\theta)}{2} \right\} d\theta = \int_{\mathbb{R}\setminus\{0\}} \left\{ 1 - \frac{\sin(\delta x)}{\delta x} \right\} \mu_i(dx). \]
(b) 20 points Show that

\[ \mu_1([-L, L]) \leq \frac{L}{4} \int_{\theta=-2/L}^{2/L} \left\{ 1 - \frac{\varphi_1(\theta) + \varphi_1(-\theta)}{2} \right\} d\theta. \]

Hint: you might first understand the structure of the function \( f(x) \equiv 1 - \frac{\sin(x)}{x} \). You might separately consider the cases \(|x| \geq 2\) and \(|x| \leq 2\).

(c) 20 points Show that indeed \( \{\mu_i; i \in \mathcal{I}\} \) is tight if \( \{\varphi_i; i \in \mathcal{I}\} \) is equicontinuous in a neighborhood of the origin.

4. 50 points Suppose that \( \{X_1, X_2, \ldots\} \) is an independent collection of random variables such that \( \mathbb{E}[X_n] = 0 \) and \( \mathbb{E}[X_n^4] \leq 1 \). Show that \( \lim_{n \to \infty} \frac{S_n}{n} = 0 \) almost surely.