Math 542 Comprehensive Examination
May 2012

Solve **EIGHT** of the following **NINE** problems. Indicate the problems you want graded.

For $z \in \mathbb{C}$, let $\Re z$ and $\Im z$ denote the real and imaginary parts of $z$, respectively.

Let $\mathbb{D}$ denote the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ and $\mathbb{T}$ the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$ in the complex plane $\mathbb{C}$.

1. Is there an analytic function $f$ in the punctured disc $\mathbb{D} \setminus \{0\}$ such that $f'$ has a simple pole at 0? Justify your claim.

2. Use the Residue Theorem to calculate the integral

$$
\int_0^\infty \frac{x^2 \sqrt{x}}{(x^2 + 1)^2} \, dx.
$$

3. Let $f$ be an entire function such that there exist positive constants $C, R > 0$ with

$$
|\Re f(z)| \leq C|\Im f(z)|
$$

for all $z$, $|z| > R$. Prove that $f$ is a constant.

4. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Assuming that there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, such that $f(z_1) = z_1$ and $f(z_2) = z_2$, prove that $f(z) = z$ for all $z \in \mathbb{D}$.

5. What is the number of solutions of

$$
z^4 - 4z^3 + 6z - 3 = 0
$$

in $\{ z \in \mathbb{C} : |z| < 2, \ \Im z > 0 \}$? Justify your claim.

6. Suppose that $(f_n)$ is a sequence of analytic functions on a domain $D$. Prove that

$L^2$-convergence of $(f_n)$ on compact subsets of $D$ implies normal convergence (i.e., uniform convergence on compacts) on $D$ together with all derivatives.

7. Prove or disprove the following statement:

There exists a sequence of polynomials $(p_n(z))_n$ such that $p_n(z)$ converges uniformly on the unit circle $\mathbb{T}$ to the function $f(z) = \bar{z}^2$. 

8. Let $u$ be a harmonic function in $\mathbb{R}^2$ such that

$$u(x, y) + y^2 \geq x^2, \quad \forall (x, y) \in \mathbb{R}^2.$$ 

Prove that there exists a non-negative constant $C$ such that

$$u(x, y) + y^2 = x^2 + C, \quad \forall (x, y) \in \mathbb{R}^2.$$ 

9. Construct an entire function $f(z)$ that has simple zeros at $\sqrt{n}$, $n = 0, 1, 2, 3, \ldots$ and at $\pm im^2$, $m = \pm 1, \pm 2, \ldots$, and no other zeros. If $g(z)$ is another function with the same properties, what is the relation between $f(z)$ and $g(z)$?