Comprehensive Exam in Algebra (500)
May, 2013.

Each question is worth 20 points.

1. (a) Let $n \geq 3$. Show that the alternating group $A_n$ is generated by 3-cycles.
   (b) Let $n \geq 5$. Let $H \leq S_n$ be a subgroup, and let $H_1 \leq H$ be a normal subgroup such that $H/H_1$ is abelian. If $H$ contains all 3-cycles, then show that $H_1$ also contains all 3-cycles.
   (c) Deduce that $S_n$ is not solvable for $n \geq 5$. Also show that the commutator subgroup of $S_n$ is $A_n$.

2. (a) Let $f : G \to G'$ be an epimorphism of groups. Let $H$ be a Sylow $p$-subgroup of $G$. Then $f(H)$ is either the trivial group or a Sylow $p$-subgroup of $G'$.
   (b) Let $G_2$ be a finite group and let $p$ a prime dividing $|G|$, the order of $G$. Suppose $H \leq G$ is a normal subgroup such that $p$ does not divide $[G : H]$. Show that all Sylow $p$-subgroups of $G$ are contained in $H$.

3. (a) Deduce from the structure theorems for modules over a PID the following: Given any finite dimensional vector space $E \neq 0$ over the field $k$ and $A \in \text{End}_k(E)$, there exists a direct sum decomposition

   $$E = E_1 \oplus \cdots \oplus E_r,$$

   where each $E_i$ is a principal $k[A]$-submodule with invariant $q_i \neq 0$ such that $q_1 | q_2 | \cdots | q_r$. The sequence $(q_1, \ldots, q_r)$ is uniquely determined by $E$ and $A$, and $q_r$ is the minimal polynomial of $A$. (Note: The invariant of a principal $k[A]$-module $M$ is the monic polynomial $q(t)$ of minimal degree such that $q(A)M = 0$.)
   (b) Let $k'$ be an extension field of $k$ and $A$ be an $n \times n$ matrix with entries in $k$. Show that the invariants of $A$ over $k$ are the same as its invariants over $k'$.

Continue to page 2.
4. Prove that $f(x) = x^p - x - 1$ is irreducible in $\mathbb{Z}[x]$. (Hint: use Problem #5.)

5. Let $k$ be a field of characteristic $p > 0$, and let $a \in k$. Show that the polynomial $f(x) = x^p - x - a$ either (i) splits into linear factors over $k$, or (ii) is irreducible over $k$.

6. Let $k$ be a field of some characteristic $p$ (which could be 0) and let $n$ be a positive integer; in the case that $p > 0$, assume also that $n$ is prime to $p$. Let $\zeta$ be a primitive $n$th root of unity in $\overline{k}$, the algebraic closure of $k$.

   (a) Show that $k(\zeta)$ is a normal extension of $k$.

   (b) Let $G = \text{Aut}_k(k(\zeta))$. Prove that $G$ can be identified as a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ (the multiplicative group of units in $\mathbb{Z}/n\mathbb{Z}$). Deduce that $G$ is abelian.

   (c) Let $k = \mathbb{Q}$ the field of rational numbers. Assume in this case $G = (\mathbb{Z}/n\mathbb{Z})^\times$ and deduce that $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. (Hint: may use $\phi(mn) = \phi(m)\phi(n)$ when $m, n$ relatively prime.)