Math 540 Comprehensive Examination
May 17, 2013

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by \( m \).

1. Let \( f \in L^p(\mathbb{R}) \). Prove that for any \( \varepsilon > 0 \), there exists a measurable set \( E \) such that \( m(E) < \infty \) and \( \|f\|_p \leq \|f \chi_E\|_p + \varepsilon \).

2. Let \( \{f_n\} \) be a sequence of complex-valued measurable functions on a measure space \((X, \mathcal{A}, \mu)\). Determine whether the following statements are true. For the false statement, provide a counterexample. For the true one, prove it.
   a) \( \{f_n\} \) converges to \( f \) in \( L^1 \), then \( f_n \to f \) in measure.
   b) \( f_n \to f \) a.e., then \( f_n \to f \) in measure.
   c) \( f_n \to f \) a.e. and \( \mu(X) < \infty \), then \( f_n \to f \) in measure.

3. Let \( f \) be a measurable function on \((X, \mathcal{A}, \mu)\). Determine whether the following statements are true. For the false statement, provide a counterexample. For the true one, prove it.
   a) if \( f \in L^\infty \), then \( \|f\|_\infty = \lim_{p \to \infty} \|f\|_p \).
   b) if \( f \in L^p \) for all \( \infty \geq p \geq 1 \), then \( \|f\|_\infty = \lim_{p \to \infty} \|f\|_p \).

4. Let \( E \) be a Lebesgue measurable subset of \( \mathbb{R} \). Prove that
   \[ \lim_{x \to 0} m(E \cap (E + x)) = m(E). \]
   Here \( E + x = \{y + x : y \in E\} \).

5. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is nondecreasing,
   \[ \int_{\mathbb{R}} f' = 1, \quad \lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to \infty} f(x) = 1. \]
   Prove that \( f \) is AC on any interval \([a, b]\).

6. Let \( f_n \) be a sequence of Lebesgue measurable functions on the interval \([0, 1]\). Assume that \( f_n \) converges to a function \( f \) \( m \) almost everywhere, and that
   \[ \int_{[0,1]} |f_n|^2 \, dm \leq 1 \]
   for each \( n \). Prove that \( f_n \) converges to \( f \) in \( L^1 \).
   Hint: Use Egoroff’s thm.