Math 540 Comprehensive Examination
May 20, 2014

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by \( m \).

1. Let \( m^* \) be defined by, for any \( E \subset \mathbb{R} \),
   \[
   m^*(E) = \sup\{m(K) : K \subset E \text{ and } K \text{ is closed}\}.
   \]
   Here \( m \) is Lebesgue measure. Show that there exists a Lebesgue measurable set \( F \) such that \( m(F) = m^*(E) \).

2. Let \((X, \mathcal{A}, \mu)\) be a measure space. For each statement give a counterexample or a proof/explanation.
   a) \( f_j \to f \) a.e. and \( \sup_j \|f_j\|_p \leq 1 \) for some \( p > 1 \), then \( f_j \) converges to \( f \) in \( L^1 \).
   b) \( f_j \to f \) a.e., \( \sup_j \|f_j\|_p \leq 1 \) for some \( p > 1 \) and \( \mu(X) < \infty \), then \( f_j \) converges to \( f \) in \( L^1 \).

3. Let \( \mu \) be a measure on \( X \), \( 0 < p < \infty \), \( f \in L^p \). Suppose that \( (f_j) \) is a sequence of \( L^p \) functions such that \( f_j \to f \) a.e. and \( \lim_{j \to \infty} \|f_j\|_p = \|f\|_p \). Prove that \( f_j \) converges to \( f \) in \( L^p \).

4. Prove the following particular case of the change of variable theorem for the Lebesgue integral:
   If \( f \in L^1(\mathbb{R}, m) \), then for any \( a > 0 \) and for any \( b \in \mathbb{R} \),
   \[
   \int_{\mathbb{R}} f(ax + b)dm(x) = \frac{1}{a} \int_{\mathbb{R}} f(x)dm(x).
   \]

5. Let \( g \in L^1(\mathbb{R}, m) \) be nonnegative. Fix \( p \in [1, \infty) \). For \( f \in L^p(\mathbb{R}, m) \), let \( T(f) = f * g \) (convolution of \( f \) and \( g \)). Prove that
   \[
   \|T\|_{L^p \to L^p} = \|g\|_{L^1}.
   \]
   Here
   \[
   \|T\|_{L^p \to L^p} := \sup_{f : \|f\|_{L^p} = 1} \|T(f)\|_{L^p}.
   \]

6. Let \( E \subset \mathbb{R} \) be Lebesgue measurable. Define
   \[
   f(x) = \text{dist}(x, E) = \inf\{|x - e| : e \in E\}.
   \]
   Prove that for \( m \) a.e. \( x \in E \), \( \lim_{r \to 0^+} \frac{f(x+r)}{r} = 0 \).