Math 540 Comprehensive Examination
May 22, 2015

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by \( m \). Calculators, books and notes are not allowed.

1. Suppose that \( f_n : X \to [0, \infty] \) is measurable for any \( n \in \mathbb{N} \), \( f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0 \) and \( \lim_{n \to \infty} f_n(x) = f(x) \) for every \( x \in X \).
   For each statement, give a counterexample or a short proof/explanation.
   a) \( \lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu \).
   b) if \( f_1 \in L^1(\mu) \), then a) holds.

2. Let \( 1 < p < \infty \), \( f \in L^p((0, \infty)) \) and define
   \[
   Tf(x) = \frac{1}{x} \int_0^x f(t) \, dm,
   \]
   for \( x \in (0, \infty) \). Here \( m \) is Lebesgue measure.
   Prove Hardy's inequality
   \[
   \|Tf\|_p \leq \frac{p}{p-1} \|f\|_p.
   \]
   and the equality holds if and only if \( f = 0 \) a.e.

3. Suppose that \( \mu \) is a measure on \( X \) with \( \mu(X) < \infty \), \( f_n \in L^1(\mu) \), and \( f_n(x) \to f(x) \) a.e. There exists \( p > 1 \) and a constant \( C \) such that
   \[
   \sup_{n \in \mathbb{N}} \int_X |f_n|^p d\mu \leq C.
   \]
   Prove that \( (f_n) \) converges to \( f \) in \( L^1(\mu) \).

4. Let \((X, \mathcal{M}, \mu)\) be a finite measure space. Fix \( p > 0 \), and suppose that a sequence \( E_n \) of measurable subsets satisfies
   \[
   \sum_n \left[ \mu(E_n) \right]^p < \infty.
   \]
   i) Prove that \( \mu(\lim \sup E_n) = 0 \) provided that \( p \leq 1 \),
   ii) Give a counterexample to the statement in part i, when \( p > 1 \).

5. Prove or give a counterexample: If \( f \in L^1(\mathbb{R}, m) \), then
   \[
   \text{esssup}_{x \in I} |f(x)| < \infty
   \]
   for some open interval \( I \).

6. Let \( f \) be a function on \([a, b]\) of total variation \( T_a^b f < \infty \).
   i) Prove that \( \int_{[a,b]} |f'| \leq T_a^b f \).
   (ii) Prove that if \( f \) is absolutely continuous then equality holds in (i).