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WEIGHT FILTRATIONS IN ALGEBRAIC K-THEORY

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Abstract. We survey briefly some of the K-theoretic background related to the theory of mixed motives and motivic cohomology.

1. Introduction.

The recent search for a motivic cohomology theory for varieties, described elsewhere in this volume, has been largely guided by certain aspects of the higher algebraic K-theory developed by Quillen in 1972. It is the purpose of this article to explain the sense in which the previous statement is true, and to explain how it is thought that the motivic cohomology groups with rational coefficients arise from K-theory through the intervention of the Adams operations. We give a basic description of algebraic K-theory and explain how Quillen's idea [42] that the Atiyah-Hirzebruch spectral sequence of topology may have an algebraic analogue guides the search for motivic cohomology.

There are other useful survey articles about algebraic K-theory: [50], [46], [23], [56], and [40].

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2. Constructing topological spaces.

In this section we explain the considerations from combinatorial topology which give rise to the higher algebraic K-groups. The first principle is simple enough to state, but hard to implement: when given an interesting group (such as the Grothendieck group of a ring) arising from some algebraic situation, try to realize it as a low-dimensional homotopy group (especially π_0 or π_1) of a space X constructed in some combinatorial way from the same algebraic inputs, and study the homotopy type of the resulting space.

The groups which can be easily described as π_0 of some space are those which are presented as the set of equivalence classes for some equivalence relation on a set S. We may then take for the space T the graph that has S as its set of

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vertices, and as edges some subset of $S \times S$ which generates the given equivalence relation.

The groups which can be easily described as π_1 of some space are those which are presented by generators and relations. One may then take for T the connected cell-complex constructed with one vertex, one edge for each generator of the group, and one 2-cell for each relation (appropriately glued in to edges to impose the desired relations).

Neither of the two spaces T mentioned above has particularly interesting homotopy groups, due to the absence of higher dimensional cells. One hopes that in the algebraic situation at hand there is some particularly evident and natural way of adding cells of higher dimension to T. The most natural and fruitful framework for adding higher-dimensional cells to such spaces hinges on the notion of geometric realization of a simplicial set, as invented by John Milnor in [36]. The cells used are simplices (triangles, tetrahedra, etc.), and a *simplicial set* is a sort of combinatorial object which amounts to a convenient way of labeling the faces of the simplices in preparation for gluing. For each integer $n \geq 0$ we give ourselves a set X_n and regard it as the set of labels for the simplices of dimension n to be used in the gluing construction. Then for each labeled simplex of dimension n we assign labels of the appropriate dimension to each of the faces.

The faces of a simplex of dimension n may be accounted for as follows. Let $\underline{\underline{n}}$ denote the ordered set $\{0 < 1 < 2 < \cdots < n\}$, and write the points of the standard n dimensional simplex Δ^n as formal linear combinations $\sum_{i=0}^{n} a_i \langle i \rangle$, where $\langle i \rangle$ is simply a symbol, and where the coefficients a_i are nonnegative real numbers with $\sum_{i=0}^{n} a_i = 1$. The faces of Δ^n are the affine-linear spans of subsets of $\{\langle 0 \rangle, \ldots, \langle n \rangle\}$. We can index the m-dimensional faces of Δ^n by the injective maps $s : \underline{\underline{m}} \to \underline{\underline{n}}$ which are *increasing* in the sense that $i \leq j \Rightarrow s(i) \leq s(j)$. We consider the unique affine-linear map $s_* : \Delta^m \to \Delta^n$ satisfying $s_*(\langle i \rangle) = \langle s(i) \rangle$, which embeds Δ^m as a face of Δ^n. If $x \in X_n$ is a label for an n-dimensional simplex, then the label we assign to the face given by the image of s_* should be an element of X_m which we will dub $s^*(x)$. We have to do this for each s and for each x. It turns out to be convenient to do this also for increasing maps s which are not necessarily injective.

The total system of compatibilities which this system of labels must satisfy is codified as follows. Let Ord denote the category of finite nonempty ordered sets $\underline{\underline{n}}$, where the arrows are the increasing maps. Then the collection of sets X_n together with the collection of maps $s^* : X_n \to X_m$ should constitute a contravariant functor from Ord to the category of sets. Such a functor X is called a *simplicial set*, and the corresponding space $|X|$ obtained by gluing simplices together is called the *geometric realization* of X.

A label $x \in X_n$ is called an *n-simplex* of X.

3. Nerves of categories.

The primary example of a simplicial set arises from a category $\mathcal{C}$ in the following way. Interpret the ordered set $\underline{\underline{n}}$ as a category $0 \to 1 \to 2 \to \cdots \to n$, and let $\mathcal{C}_n = \mathcal{C}(\underline{\underline{n}})$ denote the set $\mathrm{Hom}(\underline{\underline{n}}, \mathcal{C})$ of functors from $\underline{\underline{n}}$ to $\mathcal{C}$. The resulting simplicial set, which we may also write as $\mathcal{C}$ without too much fear of confusion, is called the *nerve* of $\mathcal{C}$. The space $|\mathcal{C}|$ will have a vertex for each object of $\mathcal{C}$, an edge for each arrow of $\mathcal{C}$, and so on.

Geometric realization is a functor from the category of simplicial sets to the category of spaces, and taking the nerve is a functor from the category of small categories to the category of simplicial sets. There is a dictionary of corresponding notions in these three categories which are related by these two functors. Thus, the nerve of a functor $F : \mathcal{C} \to \mathcal{C}'$ is a natural transformation, i.e., a map of simplicial sets, and the geometric realization of that is a continuous map. The geometric realization of the category $0 \to 1$ can be identified with the unit interval $I = [0, 1]$, and investigating the extent to which compatibility with products holds shows that the nerve of a natural transformation $\mathcal{C} \times \underline{\underline{1}} \to \mathcal{C}'$ is a simplicial homotopy, and the geometric realization of that is a homotopy $|\mathcal{C}| \times I \to |\mathcal{C}'|$. In particular, if $\mathcal{C}$ has an initial object, then contraction along the cone of initial arrows amounts to a null-homotopy of $|\mathcal{C}|$, so that $|\mathcal{C}|$ is a contractible space.

4. Classifying spaces of groups.

The simplest examples of categories for which it is useful to consider the geometric realization arise from groups. Let G be a group, and let $G[1]$ denote the category with one object $*$ and with G as its monoid of arrows. One finds that

$$\pi_i |G[1]| = \begin{cases} G & i = 1 \\ 0 & i \neq 1 \end{cases} \tag{4.1}$$

To prove this one considers the category $\tilde{G}$ whose set of objects is G, and in which there is for each $g, h \in G$ a unique arrow $g \to h$ labeled hg^{-1}. The labels are there only for the purpose of describing the map $\tilde{G} \to G[1]$, which sends an arrow of $\tilde{G}$ labeled hg^{-1} to the arrow hg^{-1} of $G[1]$. One sees that G acts freely on $|\tilde{G}|$ on the right, and that the map $|\tilde{G}| \to |G[1]|$ is the covering map corresponding to the quotient by this action. Moreover, the category $\tilde{G}$ has an initial object, so $|\tilde{G}|$ is contractible. Putting this information together yields the result.

Another name for the space $|G[1]|$ is BG, and it is called the *classifying space* of the group G.

An explicit calculation with simplicial chains shows that the singular homology group $H_n(BG, \mathbb{Z})$ is isomorphic to the group cohomology group $H^n(G, \mathbb{Z})$, as calculated using the bar resolution. In fact, the difference between the normalized bar resolution and the ordinary bar resolution amounts to the homeomorphism $|\tilde{G}|/G \cong |G[1]|$.

5. Simplicial abelian groups.

There is a third important class of examples of simplicial sets. Consider a simplicial abelian group A, which by definition is a contravariant functor from Ord to the category of abelian groups. If we forget the abelian group structure on each A_n, we are left with a simplicial set whose geometric realization $|A|$ we may consider. The striking fact here is that $\pi_i|A| = H_i NA$, where by NA we mean the normalized chain complex associated to A. The functor $A \mapsto NA$ is an equivalence of categories from the category of simplicial abelian groups to the category of homological chain complexes of abelian groups ([11] and [28]). (The appropriate definition of the inverse functor to N is easy to deduce from Yoneda's lemma.) This Dold-Kan equivalence allows us to embed the theory of homological algebra into homotopy theory.

We remark that if X is a simplicial set, and we let $\mathbb{Z}[X]$ denote the simplicial abelian group whose group of n-simplices is the free abelian group $\mathbb{Z}[X_n]$ on the set X_n, then the homotopy groups of $|\mathbb{Z}[X]|$ are the homology groups of $|X|$. Thus $\mathbb{Z}[X]$ is a drastic form of abelianization for simplicial sets.

6. Eilenberg-MacLane spaces.

Here is the fourth important class of examples of simplicial sets, obtained as a special case of the third. Let G be an abelian group, and let $n \geq 0$ be an integer. Then consider the homological chain complex which has G in dimension n and the group 0 in all the other dimensions. Let $G[n]$ denote the corresponding simplicial abelian group, obtained according to the Dold-Kan equivalence. We get a space $|G[n]|$ which has the abelian group $G = \pi_n|G[n]|$ as its only nonvanishing homotopy group; it is an *Eilenberg-MacLane space*.

Consider pointed spaces (CW-complexes) V and W, and let $[V, W]$ denote the set of homotopy classes of (base-point preserving) maps from V to W. There is a natural isomorphism $[V, |G[n]|] \cong H^n(V, G)$, which we will use later. To see the plausibility of the isomorphism, consider the case where V is a sphere, and identify the two sides.

7. The lower K-groups.

Grothendieck considered the group $K(R)$ generated by the isomorphism classes of finitely generated projective R-modules, modulo relations $[P] = [P'] + [P'']$ coming from short exact sequences $0 \to P' \to P \to P'' \to 0$. The group $G\ell(R)$ is the group of invertible matrices over R with a countable number of rows and columns, equal to the identity outside of some square. Bass defined $K_1(R)$ to be the abelianization $G\ell(R)^{\mathrm{ab}}$ of the infinite general linear group $G\ell(R)$, and renamed the Grothendieck group to $K_0(R)$ because of six-term exact sequences he was able to prove involving the Grothendieck group and his new group [2]. Milnor [37] found the correct definition for $K_2(R)$, and supported its correctness by extending the exact sequences of Bass.

8. The construction of the higher K-groups.

We can now describe Quillen's first construction of algebraic K-theory for a ring R.

By adding a single two-cell and a single three-cell to the space $BG\ell(R)$ Quillen was able to construct a space $BG\ell(R)^+$ with the property that the map $BG\ell(R) \to BG\ell(R)^+$ induces an isomorphism on homology groups with integer coefficients, and induces the map $G\ell(R) \twoheadrightarrow G\ell(R)^{\mathrm{ab}}$ on fundamental groups. (The construction provides a functor from rings to spaces, because the cells' attaching maps used in the case $R = \mathbb{Z}$ work for any ring R.) Quillen also proved that the space $BG\ell(R)^+$ is an abelian group in the homotopy category of pointed spaces. This construction therefore serves as a modest form of abelianization for spaces such as this one, whose commutator subgroup is perfect.

Quillen's first definition of the higher algebraic K-groups is given in terms of this plus-construction by setting $K_i(R) = \pi_i BG\ell(R)^+$ for $i > 0$.

The Hurewicz mapping

$$K_i(R) \to H_i(BG\ell(R)^+, \mathbb{Z}) \cong H_i(BG\ell(R), \mathbb{Z}) \cong H_i(G\ell(R), \mathbb{Z}) \tag{8.1}$$

gives an initial stab at the relationship between the higher K-groups and the homology groups of the general linear group.

9. K-groups of exact categories.

The plus-construction is a curious creature. It permits some explicit computations to be performed, by virtue of its close connection with the general linear group. For example, using it Quillen proved that $K_1(R)$ agrees with the group of the same name defined by Bass, and that $K_2(R)$ agrees with the group of the same name defined by Milnor in terms of the Steinberg group [37]. But it suffers from two drawbacks. First, one would prefer to have a space $K(R)$ which has all of the K-groups appearing as its homotopy groups, including $K_0(R)$. The naive construction of such a space, $K_0(R) \times BG\ell(R)^+$, has such homotopy groups, but when regarded as an abelian group in the homotopy category, it fails to mix π_0 with π_1 appropriately, unless it happens that $K_0(R) = \mathbb{Z}$. Second, the Grothendieck group is defined for almost any sort of category with a notion of exact sequence; the higher K-groups should be defined for such categories, too.

The definitions of K-theory which don't suffer from these two drawbacks deal with an *exact* category $\mathcal{M}$. An *exact* category $\mathcal{M}$ is an additive category equipped with a set of sequences $0 \to M' \to M \to M'' \to 0$ called *exact* sequences, which arises as a full subcategory $\mathcal{M} \subseteq \mathcal{A}$ closed under extensions in some (unspecified) abelian category $\mathcal{A}$, equipped with the collection of all short sequences of $\mathcal{M}$ which are exact in $\mathcal{A}$. As examples of exact categories, we mention the category $\mathcal{P}(R)$ of finitely generated projective left R-modules, the category $\mathcal{M}(R)$ of finitely generated left R-modules, the category $\mathcal{P}(Z)$ of locally free $\mathcal{O}_Z$-Modules of finite type on a scheme Z, and the category $\mathcal{M}(Z)$ of quasi-coherent $\mathcal{O}_Z$-Modules of finite type on a scheme Z. The corresponding higher K-groups are all of interest.

The first such definition of K-theory I intend to discuss is Quillen's Q-construction, [43]. The category $Q\mathcal{M}$ has the same objects as does $\mathcal{M}$, but an arrow $M' \to M$ of $Q\mathcal{M}$ is an isomorphism of M' with an admissible subobject of an admissible quotient object of M. Admissibility of a subobject refers to the requirement that the corresponding quotient object also lies in $\mathcal{M}$, or more precisely, that the inclusion map for the subobject is part of a short exact sequence in $\mathcal{M}$. One may check eventually that the connected space $|Q\mathcal{M}|$ satisfies $\pi_1|Q\mathcal{M}| \cong K_0\mathcal{M}$, and one defines $K_i(\mathcal{M}) = \pi_{i+1}|Q\mathcal{M}|$ for all $i \geq 0$. It is then a theorem of Quillen [22] that $K_i(R) \cong K_i(\mathcal{P}(R))$. We define the K-theory space $K(\mathcal{M})$ to be the loop space $\Omega|Q\mathcal{M}|$ of $|Q\mathcal{M}|$, so that $K_i(\mathcal{M}) = \pi_i(K(\mathcal{M}))$.

If X is a scheme, we may define $K_i(X) = K_i(\mathcal{P}(X))$, where $\mathcal{P}(X)$ denotes the category of locally free $\mathcal{O}_X$-Modules of finite type on X. Much of what I say below for commutative rings R applies equally well to schemes X.

If R is a ring, we define $K'_i(R) := K_i(\mathcal{M}(R))$, where $\mathcal{M}(R)$ denotes the category of finitely generated R-modules. If X is a scheme, we define $K'_i(X) := K_i(\mathcal{M}(X))$, where $\mathcal{M}(X)$ denotes the category of quasi-coherent $\mathcal{O}_X$-Modules locally of finite type.

A second definition of K-theory is provided by Waldhausen's S-construction [57]. The simplicial set $S\mathcal{M}$ is defined in such a way that its set of n-simplices consists of chains $0 = M_0 \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail M_n$ of admissible monomorphisms of $\mathcal{M}$, together with objects of $\mathcal{M}$ representing all the quotient objects. One sees that there is exactly one vertex in the space $|S\mathcal{M}|$, one edge for each object M of $\mathcal{M}$, and one triangle for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{M}$. If we let $[M]$ denote the class in $\pi_1\mathcal{M}$ arising from the edge labeled by M, then the triangles are glued to the edges so that the relation $[M] = [M'] + [M'']$ is imposed. It follows that $\pi_1 S\mathcal{M} \cong K_0\mathcal{M}$ – this fits in well with the earlier remark about expressing groups given by generators and relations as π_1 of a space. It is a theorem of Waldhausen that $|S\mathcal{M}|$ is homotopy equivalent to $|Q\mathcal{M}|$, so that $\pi_{i+1}|S\mathcal{M}| \cong K_i\mathcal{M}$ for all $i \geq 0$.

The S-construction or the Q-construction can be used to show that the space $K(\mathcal{M})$ is naturally an infinite loop space; the deloopings obtained thereby are increasingly connected, so the homotopy groups in negative dimension that arise thereby are all zero. This makes the space $K(\mathcal{M})$ into an Ω-*spectrum* (or an infinite loop space), which is essentially a space Z, together with compatible choices of deloopings $\Omega^{-n}Z$ for every $n > 0$. The sort of spectrum just obtained, in which the deloopings are increasingly connected, with no new homotopy groups arising in low degrees, is called *connective*.

Another equivalent definition for K-theory is presented in [20] by Gillet and Grayson. It hinges on the elementary fact that every element of $K_0(\mathcal{M})$ can be expressed as a difference $[P] - [Q]$, where P and Q are objects of $\mathcal{M}$, and $[P]$ denotes the class in $K_0(\mathcal{M})$. Thus $K_0(\mathcal{M})$ is a quotient of the set of pairs (P, Q) of objects of $\mathcal{M}$ by the equivalence relation where $(P, Q) \sim (P', Q')$ if and only if $[P] - [Q] = [P'] - [Q']$. According to the earlier remark about groups whose elements are equivalence classes of a relation, we may try to express $K_0(\mathcal{M})$ as π_0 of some space. That space would have vertices which are pairs (P, Q) of objects, its edges would generate the equivalence relation, and its higher dimensional simplices should be defined in a natural and simple way. It is an exercise to show that our equivalence relation is generated by the requirement that $(P, Q) \sim (P', Q')$ whenever there are admissible monomorphisms $P' \rightarrowtail P$ and $Q' \rightarrowtail Q$ whose cokernels are isomorphic. This suggests that an edge of our simplicial set should be a triple consisting of two such monomorphisms and an isomorphism of their cokernels. The simplicial set $G\mathcal{M}$ is defined by saying that an n-simplex is a pair of chains of admissible monomorphisms $P_0 \rightarrowtail P_1 \rightarrowtail \dots \rightarrowtail P_n$ and $Q_0 \rightarrowtail Q_1 \rightarrowtail \dots \rightarrowtail Q_n$, together with a commutative diagram of isomorphisms of quotients as illustrated here.

$$\begin{array}{ccccccc} P_1/P_0 & \longrightarrow & P_2/P_0 & \longrightarrow & \dots & \longrightarrow & P_n/P_0 \\ \cong\downarrow & & \cong\downarrow & & & & \cong\downarrow \\ Q_1/Q_0 & \longrightarrow & Q_2/Q_0 & \longrightarrow & \dots & \longrightarrow & Q_n/Q_0 \end{array}$$

It is then a theorem that $K_i(\mathcal{M}) = \pi_i|G\mathcal{M}|$ for all $i \geq 0$, and that $|G\mathcal{M}|$ is a loop space of $|S\mathcal{M}|$. It turns out that having $K_0(\mathcal{M})$ appear as π_0 rather than as π_1 offers a technical advantage when constructing operations on K-groups that are homomorphisms on K_i only for $i > 0$, for then the operations may arise from maps of spaces [24].

10. Some theorems of algebraic K-theory.

Using the Q-construction, Quillen [43] was able to prove the following four foundational theorems (among others).

The first theorem is the additivity theorem. It states that if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of exact functors $\mathcal{M} \to \mathcal{M}'$, then the map $F_* : K_n(\mathcal{M}) \to K_n(\mathcal{M}')$ satisfies the formula $F_* = F'_* + F''_*$.

The second theorem is the Jordan-Hölder theorem for higher K-theory. It states that if $\mathcal{M}$ is an artinian abelian category (i.e., every object has a composition series), then $K_n(\mathcal{M}) = \bigoplus_V K_n(\mathrm{End}(V))$, where the direct sum runs over the isomorphism classes of simple objects V of $\mathcal{M}$. An important application of this theorem is the following. Let R be a noetherian ring, and let $\mathcal{M}^p(R)$ denote the category of finitely generated R-modules whose support has codimension $\geq p$. Then the quotient abelian category $\mathcal{M}^p/\mathcal{M}^{p+1}$ is artinian, and the theorem implies that $K_n(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x K_n(k(x))$, where x runs over points $x \in \mathrm{Spec}(R)$ of height p, and $k(x)$ denotes the residue field at x. If we take $n = 0$ then we see that $K_0(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x \mathbb{Z}$ is the group of algebraic cycles of codimension p. This is the way that algebraic cycles are related to algebraic K-theory.

The third theorem is the resolution theorem, which implies that if X is a regular noetherian scheme, then $K_n(\mathcal{P}(X)) = K_n(\mathcal{M}(X))$. It hinges on the fact that any $M \in \mathcal{M}(X)$ has a resolution of finite length by objects of $\mathcal{P}(X)$. For X affine, that resolution is simply a projective resolution.

The fourth theorem is the localization theorem for abelian categories. It says that if $\mathcal{B}$ is a Serre subcategory of an abelian category $\mathcal{A}$, (which means that it is closed under taking subobjects, taking quotient objects, and taking extensions), then there is a fibration sequence $K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})$, where $\mathcal{A}/\mathcal{B}$ denotes the quotient abelian category. The main import of being a fibration sequence is that there results the following long exact sequence of K-groups.

$$\cdots \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to K_{n-1}(B) \to \dots$$

This theorem can be applied notably in the case above where $\mathcal{A} = \mathcal{M}^p(R)$ and $\mathcal{B} = \mathcal{M}^{p+1}(R)$, or in the case where X is a noetherian scheme, Y is a closed subscheme, $A = \mathcal{M}(X)$, $\mathcal{B} = \mathcal{M}(Y)$, and $\mathcal{A}/\mathcal{B} \cong \mathcal{M}(X - Y)$.

11. Some computations.

We now present some explicit computations of some algebraic K-groups.

There is a map $R^\times = G\ell_1(R) \subseteq G\ell(R) \to K_1(R)$; let's use $\{u\}$ to denote the image of a unit u under this map, so that we can write the group law in $K_1(R)$ additively, $\{u\} + \{v\} = \{uv\}$. When R is commutative this map is split by the determinant map, so that $K_1(R)$ has $R^\times$ as a direct summand. It is known that when R is a field, a local ring, or a Eulidean domain, then $K_1(R) \cong R^\times$.

Concerning the K-groups of a field F, we know that

$$\begin{aligned} K_0 F &= \mathbb{Z} \\ K_1 F &= F^\times \\ K_2 F &= (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle a \otimes (1-a) \mid a \in F - \{0,1\}\rangle \end{aligned} \tag{11.1}$$

The standard notation for the image of $a \otimes b$ in $K_2 F$ is $\{a, b\}$, and the relation $\{a, 1-a\} = 0$ is called the Steinberg relation. From the Steinberg relation one can deduce that $\{a, b\} = -\{b, a\}$ [37, p.95].

For finite fields, the K-groups are completely known [45].

$$\begin{aligned} K_0 \mathbb{F}_q &= \mathbb{Z} \\ K_{2i+1} \mathbb{F}_q &\cong \mathbb{Z}/(q^{i+1} - 1) \\ K_{2i+2} \mathbb{F}_q &= 0 \end{aligned}$$

In fact, what motivated Quillen's original definition of algebraic K-theory was the discovery of a space (the homotopy fixed-point set of the Adams operation ψ^q acting on topological K-theory) that has these homotopy groups, whose homology groups are isomorphic to the homology groups of the general linear group.

Suslin's important work on the K-groups of algebraically closed fields [54] shows that the torsion subgroup of $K_n\mathbb{C}$ is $\mathbb{Q}/\mathbb{Z}$ for n odd, and is 0 for n even and nonzero. The quotient modulo the torsion is a uniquely divisible group. (For example, $K_1(\mathbb{C}) = \mathbb{C}^\times$, so the torsion subgroup is the group of roots of unity, thus isomorphic to $\mathbb{Q}/\mathbb{Z}$.) For a general exposition of these matters and others in the K-theory of fields, see [23].

As for the field $\mathbb{Q}$ of rational numbers, Tate proved [37, 11.6] that $K_2\mathbb{Q} \cong \mathbb{Z}/2 \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}/p)^\times$.

The higher K-groups of $\mathbb{Q}$ are not known exactly, but the ranks are known. In fact, the ranks are known for any ring of algebraic integers, so consider now the case where F is a number field and $\mathcal{O}_F$ is the ring of integers in F. Write $F \otimes \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, so that r_1 is the number of real places of F, and r_2 is the number of complex places.

For $\mathcal{O}_F$, and indeed, for any Dedekind domain, one knows that $K_0\mathcal{O}_F = \mathbb{Z} \oplus \operatorname{Pic}(\mathcal{O}_F)$, where $\operatorname{Pic}(\mathcal{O}_F)$ denotes the ideal class group of $\mathcal{O}_F$.

It is a theorem of Bass, Milnor, and Serre [3] that $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$. This is not a general fact about Dedekind domains.

Quillen [44] has shown that $K_n(\mathcal{O}_F)$ is a finitely generated abelian group for all $n \geq 0$. (In fact it is conjectured by Bass that $K_n(R)$ is a finitely generated group whenever R is a finitely generated regular commutative ring.) We list the ranks.

$$\operatorname{rank} K_n\mathcal{O}_F = \begin{cases} 1 & \text{if } n = 0 \\ r_1 + r_2 - 1 & \text{if } n = 1 \\ 0 & \text{if } n = 2k \text{ and } k > 0 \\ r_1 + r_2 & \text{if } n = 4k+1 \text{ and } k > 0 \\ r_2 & \text{if } n = 4k+3 \text{ and } k \geq 0 \end{cases} \tag{11.2}$$

In this table, the case $n = 0$ amounts to the finiteness of the ideal class group; the case $n = 1$ is Dirichlet's unit theorem; Borel's theorem, [10], handles the case $n \geq 2$ through a detailed study of harmonic forms on symmetric spaces associated to arithmetic groups, and depends on earlier work of Borel and Serre on compactifying these symmetric spaces. It is customary now to refer to a non-torsion element of $K_{2i-1}(\mathcal{O}_F)$ as a *Borel class*.

The interesting fact about these ranks is that in the odd cases, the rank of $K_{2i-1}(\mathcal{O}_F)$ is equal to the order of vanishing of the Dedekind zeta function $\zeta_F(s)$ at $s = 1-i$. It was Lichtenbaum who spawned the current endeavor by predicting this coincidence, well before Borel computed the ranks of the K-groups.

Only the first four K-groups of $\mathbb{Z}$ are known. We know that $K_0(\mathbb{Z}) = \mathbb{Z}$, and $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2$. Milnor showed [37, 10.1] that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$. Lee and Szczarba [32] proved that $K_3\mathbb{Z} \cong \mathbb{Z}/48$.

12. Products in K-theory.

Henceforth we shall deal with rings R that are commutative. In that case the tensor product operation $P \otimes_R Q$ on finitely generated projective R-modules leads to an operation

$$K_m R \otimes K_n R \to K_{m+n} R \tag{12.1}$$

which endows $\bigoplus_{n=0}^{\infty} K_n R$ with the structure of a skew-commutative graded ring, by virtue of the essential commutativity and associativity of the tensor product operation. The unit element of the ring is $1 = [R] \in K_0(R)$. In the case $m = n = 1$ the product agrees with the Steinberg symbol (at least up to a possible sign), in the sense that $\{u\} \cdot \{v\} = \{u, v\}$.

One defines the Milnor ring of a field F to be the quotient of the tensor algebra of $F^\times$ by the ideal generated by the Steinberg relations. This ring is a graded ring, and we let $K_n^M F$ denote its degree n part. It follows from what we have said that there is a natural map $K_n^M F \to K_n F$ which is an isomorphism for $0 \le n \le 2$, and which is known to be a non-isomorphism in general for $n > 2$.

13. Nonlinear operations on K-theory.

If $F : \mathcal{M} \to \mathcal{M}'$ is an exact functor, then there is a natural map $K(\mathcal{M}) \to K(\mathcal{M}')$ induced by F. The exterior power operation $\Lambda^k P$ on finitely generated projective R-modules P gives rise to operations $\lambda^k : K_0(R) \to K_0(R)$, as shown by Grothendieck [26], even though Λ^k is not an exact functor. These *lambda* operations are defined by letting $\lambda^k([P] - [Q])$ be the coefficient of t^k in $\lambda_t([P])/\lambda_t([Q])$, where $\lambda_t([P]) := \sum_{k=0}^{\infty} [\Lambda^k P] t^k$ in the power series ring $K_0(R)[[t]]$. One uses the identity $\lambda_t([P \oplus Q]) = \lambda_t([P])\lambda_t([Q])$ to show that λ_t (and hence λ^k) is well defined. When R is a field, then $K_0(R) = \mathbb{Z}$, and $\lambda^k(n) = \binom{n}{k}$; the usual definition of the binomial coefficient for $n < 0$ is the one we are led to using the artifice above.

We point out, for later use, that when $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i is a projective module of rank 1, then $\lambda_t([P]) = \prod_{i=1}^n \lambda_t([L_i]) = \prod_{i=1}^n (1 + t[L_i]) = \sum_{k=0}^n t^k \sigma_k([L_1], \ldots, [L_n])$, where σ_k is the elementary symmetric polynomial of degree k. We deduce that $\lambda^k([\mathcal{P}]) = \sigma_k([L_1], \ldots, [L_n])$. We will see that the operations λ^k are useful for the same reason the elementary symmetric polynomials are: we can write other symmetric functions in terms of them.

In [27] is presented Quillen's method for defining the lambda operations on the higher K-groups. The resulting functions $\lambda^k : K_n(R) \to K_n(R)$ are group homomorphisms except when $n = 0$ and $k \ne 1$. Perhaps it is startling at first glance that functions which are decidedly not additive on K_0 are closely related to functions on the higher K-groups which are, but there is no other possibility. Any sort of operation on higher homotopy groups of a space will presumably have to arise from a map of spaces, so can't avoid being a homomorphism.

It is possible to repair the nonadditivity of the lambda operations on K_0, thereby increasing their utility, by means of a natural sort of abelianization procedure which produces new *Adams operations* $\psi^k : K_n(R) \to K_n(R)$ which are group homomorphisms, even for $n = 0$. On K_0, the salient feature of the Adams operations, aside from being homomorphisms, is that if L is a projective module of rank 1, then $\psi^k([L]) = [L^{\otimes k}]$. Consequently, if $P \cong L_1 \oplus \cdots \oplus L_n$ is a direct sum of projective modules of rank 1, then $\psi^k([P]) = [L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$. The symmetric polynomial $x_1^k + \cdots + x_n^k$ can be expressed as a polynomial with integer coefficients in the elementary symmetric polynomials, so there is a formula for $[L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$ in terms of the exterior powers of P. It doesn't really matter what this formula is, but it may be compactly recorded in terms of generating functions as follows.

$$\sum_{k=0}^{\infty} \psi^k(x)(-t)^k = \operatorname{rank} x - t\frac{d}{dt} \log \lambda_t(x)$$

This formula serves as the definition of $\psi^k(x)$ for any $x \in K_0(R)$.

A unit u of the ring R arises in $K_1(R)$ by virtue of being an automorphism of the free module $L = R$ of rank 1. As such it gives rise to the automorphism $u^k = u^{\otimes k}$ of $L = L^{\otimes k}$, so we see that $\psi^k(u) = u^k$, or, writing it additively, $\psi^k u = ku$.

Heuristically speaking, ψ^k raises the functions (i.e., the elements of R) entering into a construction of an element of K-theory, to the k-th power. To the extent that constructions of elements of K-theory from several functions of R involve those functions in a multi-multiplicative way, the effect of ψ^k on an element of K-theory can be used to count the number of functions entering into its construction.

Here we summarize some properties of the Adams operations.

(1) If $x \in K_n(R)$ and $y \in K_n(R)$ then $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$.
(2) If x is the class $[L]$ of a line bundle (rank 1 projective R-module) L in $K_0(R)$, then $\psi^k(x) = [L^{\otimes k}]$.
(3) If x is the class in $K_0(R)$ of a free module, then $\psi^k(x) = x$.
(4) If $x \in K_p(R)$ and $y \in K_q(R)$ then $\psi^k(xy) = \psi^k(x)\psi^k(y) \in K_{p+q}(R)$.
(5) If $x \in R^\times$ then $\psi^k(\{x\}) = \{x^k\} = k\{x\}$.
(6) $\psi^k \circ \psi^\ell = \psi^{k\ell}$
(7) If R is a regular noetherian ring, and M is a finitely generated R-module whose support has codimension $\geq p$, then $\psi^k([M]) = k^p[M]$ modulo torsion and classes of modules of codimension greater than p.

The last item above is related to the fact [21] that in K-theory with supports, if we have a regular sequence $t_1, \ldots, t_p$ in R, and $M = R/(t_1, \ldots, t_p)$, then $\psi^k([M]) = [R/(t_1^k, \ldots, t_p^k)]$.

In the first part of [25] the reader may find a concrete description of the k-th Adams operation on K_0 as the secondary Euler characteristic of the Koszul complex; the Koszul complex is introduced there by taking the mapping cone of the identity map on $P \in \mathcal{P}(R)$ and considering the k-th symmetric power of that complex of length 1.

14. Weight filtrations in the K-groups.

The Adams operations were used by Grothendieck to provide an answer (up to torsion) to the following question. Suppose that R is a regular noetherian ring, and that M is a finitely generated R-module with the codimension of its support at least p. We get a class $[M] \in K_0(\mathcal{M}(R))$. By the resolution theorem, we have an isomorphism $K_0(R) \cong K_0(\mathcal{M}(R))$, so we get a class $[M] \in K_0(R)$. Let $F^p_{\mathrm{top}} K_0(R)$ denote the subgroup of $K_0(R)$ generated by such classes; the collection of these subgroups is called the filtration by codimension of support. The question is, is there an algebraic construction of such a filtration which makes sense for any commutative ring R, not necessarily regular? The construction would have to be expressed in terms of projective modules alone, in the absence of the resolution theorem. In the next few paragraphs, we explore the construction of such a filtration.

Let's assume that R is a commutative domain. Then we have the *rank* homomorphism $K_0(R) \to \mathbb{Z}$ defined by $[P] \mapsto \operatorname{rank} P$. This map is an isomorphism when R has dimension 0, i.e., is a field. The kernel F^1_{alg} of the rank homomorphism consists of elements of the form $[P] - \operatorname{rank} P$, and we declare such elements to have *weight* ≥ 1.

The Picard group $\operatorname{Pic}(R)$ is the group of isomorphism classes $[L]$ of rank 1 projective R-modules L, with tensor product as the group operation. Given $P \in \mathcal{P}(R)$, we let $\det P$ denote the highest exterior power of P, and we let $[\det P]$ denote its isomorphism class in $\operatorname{Pic}(R)$. From the formula $\det(P \oplus P') \cong (\det P) \otimes (\det P')$ we find that there is a group homomorphism $K_0(R) \to \operatorname{Pic}(R)$. There is also a function $\operatorname{Pic}(R) \to K_0(R)$, defined by sending $[L] \in \operatorname{Pic}(R)$ to $[L] \in K_0(R)$, which is a group homomorphism from $\operatorname{Pic}(R)$ to the group of units $K_0(R)^\times$ of the ring $K_0(R)$. (We shall see shortly that $[L] - 1$ is a nilpotent element of the ring $K_0(R)$.)

Let's consider the simple case where R is a regular domain of dimension 1, i.e., is a Dedekind domain. In this case it is known that the homomorphism $K_0(R) \to \mathbb{Z} \oplus \operatorname{Pic}(R)$ given by $[P] \mapsto (\operatorname{rank} P, [\det P])$ is an isomorphism. The proof depends on the algebraic fact that any $P \in \mathcal{P}(R)$ of rank n is the direct sum of a free module R^{n-1} and a projective module L of rank 1, from which it follows that $L \cong \det P$, allowing us to recover $[P]$ from $\operatorname{rank} P$ and $\det P$ by the formula $[P] = [\det P] - 1 + \operatorname{rank} P$. (The nilpotence of $[L] - 1$ results from the isomorphism $L \oplus L \cong L^{\otimes 2} \oplus R$, which allows us to prove that $([L] - 1)^2 = 0$.)

For an arbitrary commutative domain R the kernel F^2_{alg} of the surjective homomorphism $K_0(R) \to \mathbb{Z} \oplus \operatorname{Pic}(R)$ is the subgroup generated by elements of the form $([\det P] - 1) - ([P] - \operatorname{rank} P)$, for the vanishing of such elements is all that's required for the proof from the previous paragraph. We declare such elements to be of *weight* ≥ 2.

What ought we to use for the elements of weight ≥ 3? One problem confronting us is the lack of a function analogous to $\operatorname{rank} P$ and $\det P$ which would vanish on such elements. (Using the second Chern class associated with some cohomology theory seems unnatural and unprofitable.)

We try to get an idea by examining $([\det P] - 1) - ([P] - \operatorname{rank} P)$ when P is decomposable. For example, when $P \cong L_1 \oplus L_2$, where L_1 and L_2 have rank 1, we see that

$$\begin{aligned}([\det P] - 1) - ([P] - \operatorname{rank} P) &= [L_1 \otimes L_2] + 1 - [L_1 \oplus L_2] \\ &= ([L_1] - 1)([L_2] - 1),\end{aligned}$$

showing that our element of weight ≥ 2 is a product of two elements of weight ≥ 1. This suggests that the weight filtration we are constructing ought to be compatible with multiplication. That in turn fits in well with our original topological filtration, for if D_1 is a divisor whose defining ideal is isomorphic to L_1, then we have $[D_1] = 1 - [L_1]$, and if D_2 is a divisor whose idea is isomorphic to L_2, and D_1 and D_2 intersect regularly, then $D_1 \cap D_2$ has codimension 2 and $[D_1 \cap D_2] = (1 - [L_1])(1 - [L_2])$.

A first attempt at a weight filtration might involve declaring a product such as

$$([L_1] - 1) \cdot \ldots \cdot ([L_k] - 1),$$

where each L_i is a rank 1 projective module, to have weight $\geq k$. This isn't quite enough because it fails to assign weight ≥ 2 to the element $([\det P] - 1) - ([P] - \operatorname{rank} P)$ when P is an arbitrary projective module, for P may not decompose as a direct sum of rank 1 modules; P may even fail to have a filtration whose successive quotients are rank 1 projectives.

Supposing $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i is a rank 1 projective, we let $x_i = [L_1] - 1$ and compute

$$\begin{aligned}([\det P] - 1) - ([P] - \operatorname{rank} P) &= ([L_1 \otimes \cdots \otimes L_n] - 1) - ([L_1 \oplus \cdots \oplus L_n] - n) \\ &= (x_1 + 1) \cdot \ldots \cdot (x_n + 1) - (1 + (x_1 + \cdots + x_n)) \\ &= \sum_{k=2}^{n} \sigma_k(x_1, \ldots, x_n).\end{aligned}$$

One can show that each term in the latter sum has trivial rank and determinant, and that suggests singling them out. Setting $x = x_1 + \cdots + x_n = [P] - n$, we define $\gamma^k(x) := \sigma_k(x_1, \ldots, x_n)$. We justify the notation by showing that $\gamma^k(x)$ is a well-defined function of x as follows. Using the definition of the elementary symmetric polynomials, we compute

$$\begin{aligned}\sum_{k=0}^{n} t^k\gamma^k(x) &= \sum_{k=0}^{n} t^k\sigma_k(x_1,\dots,x_n)\\ &= \prod_{i=1}^{n}(1+tx_i)\\ &= \prod_{i=1}^{n}(1-t+t[L_i])\\ &= (1-t)^n\prod_{i=1}^{n}(1+\frac{t}{1-t}[L_i]).\end{aligned}$$

The latter product is $\lambda_t([P])$ with t replaced by $u := \frac{t}{1-t}$, and we use $\lambda_u([P])$ to denote it. We find that

$$\begin{aligned}\sum_{k=0}^{n} t^k\gamma^k(x) &= (1-t)^n\lambda_u([P]) = (1+\frac{t}{1-t})^{-n}\lambda_u([P])\\ &= \lambda_u(1)^{-n}\lambda_u([P]) = \lambda_u([P]-n) = \lambda_u(x).\end{aligned}$$

This formula shows that $\gamma^k(x)$ depends only on x, and also provides a definition of $\gamma^k(x)$ which is useable for any element $x \in K_0(R)$, without assuming that x has the form $\sum([L_i]-1)$.

We observe that $\gamma^0(x) = 1$ and $\gamma^1(x) = x$. When $\operatorname{rank} x = 0$, from the equation $\gamma_t(x)\gamma_t(-x) = 1$ and the fact that both $\gamma_t(x)$ and $\gamma_t(-x)$ are polynomials with constant term 1, we see that the coefficients $\gamma^k(x)$ must be nilpotent for $k \geq 1$.

One can show that $\det(\gamma^k(x)) = 0$ when $k \geq 2$ and $\operatorname{rank} x = 0$, using the splitting principle. Thus $F^2_{\mathrm{alg}}K_0(R)$ is generated by such elements $\gamma^k(x)$.

We now have sufficient motivation to define the gamma filtration on $K_0(R)$. For $k \geq 1$ we let $F^k_\gamma K_0(R)$ be the subgroup generated by all products of the form $\gamma^{k_1}(y_1)\cdot\ \dots\ \cdot\gamma^{k_m}(y_m)$ where each $y_i \in K_0(R)$ has $\operatorname{rank} y_i = 0$, and $\sum_{i=1}^m k_m \geq k$. In particular, $F^1_\gamma K_0(R) = \{x \in K_0(R) | \operatorname{rank} x = 0\}$. It is evident from the preceding discussion that for a domain R we have isomorphisms $F^0_\gamma K_0(R)/F^1_\gamma K_0(R) \cong \mathbb{Z}$ and $F^1_\gamma K_0(R)/F^2_\gamma K_0(R) \cong \operatorname{Pic}(R)$.

Now we take a look at the effect of the Adams operations on the gamma filtration. With the notation as above, we have

$$\begin{aligned}\psi^k(x_i) &= \psi^k([L_i]-1) = [L_i^{\otimes k}]-1 = [L_i]^k-1\\ &= (1+x_i)^k-1 = kx_i+\binom{k}{2}x_i^2+\dots+x_i^k,\end{aligned}$$

so that $\psi^k(x_i) \equiv kx_i$ modulo $F_\gamma^2 K_0(R)$. Then

$$\begin{aligned}\psi^k(\gamma^r(x)) &= \psi^k \sigma_r(x_1, \dots, x_n) = \sigma_r([L_1]^k - 1, \dots, [L_n]^k - 1) \\ &= \sigma_r(kx_1 + \cdots, \dots, kx_n + \cdots) = \sigma_r(kx_1, \dots, kx_n) + g(x_1, \dots, x_n) \\ &= k^r \sigma_r(x_1, \dots, x_n) + g(x_1, \dots, x_n) = k^r \gamma_r(x) + g(x_1, \dots, x_n),\end{aligned}$$

where $g(x_1, \dots, x_n)$ is some symmetric polynomial in n variables of degree $\geq r+1$. Writing $g(x_1, \dots, x_n)$ as a polynomial $G(\gamma^{r+1}(x), \gamma^{r+2}(x), \dots)$ yields a formula

$$\psi^k(\gamma^r(x)) = k^r \gamma^r(x) + G(\gamma^{r+1}(x), \gamma^{r+2}(x), \dots)$$

which is true for every $x \in F_\gamma^1 K_0(R)$, by the splitting principle. It follows that $\psi^k(\gamma^r(x)) \equiv k^r \gamma^r(x)$ modulo F_γ^{r+1}, and from this it follows (by multiplication) that $\psi^k(z) \equiv k^r z$ modulo F_γ^{r+1} for any $z \in F_\gamma^r K_0(R)$.

If for any $x \in F_\gamma^1 K_0(R)$ we consider the expression $(\psi^k - k^r) \circ (\psi^k - k^{r-1}) \circ \cdots \circ (\psi^k - k)(x)$ for large r, we see that it is a sum of monomials in $\gamma^1(x), \dots, \gamma^n(x)$ of high degree, so by the nilpotence of the elements $\gamma^i(x)$, the expression will be zero when r is sufficiently large. Combining this with the fact that for any $x \in K_0(R)$, the element $(\psi^k - 1)(x)$ lies in $F_\gamma^1 K_0(R)$, we see (from linear algebra) that an element of $K_0(R)_\mathbb{Q} := K_0(R) \otimes \mathbb{Q}$ is a sum of eigenvectors of ψ^k, and that the eigenvalues occuring are nonnegative powers of k. We let $K_0(R)_\mathbb{Q}^{(i)}$ denote the eigenspace for ψ^k with the eigenvalue k^i, so that

$$K_0(R)_\mathbb{Q} = \bigoplus_{i=0}^{\infty} K_0(R)_\mathbb{Q}^{(i)}.$$

This eigenspace can be shown to be independent of the choice of k by considering an expression like $(\psi^k - k^r) \circ (\psi^k - k^{r-1}) \circ \cdots \circ (\psi^{k'} - (k')^i) \circ \cdots \circ (\psi^k - 1)(x)$, which must be zero for sufficiently large r, for the same reason as before.

Finally, because we know that ψ^k acts as k^i on $F_\gamma^i / F_\gamma^{i+1}$, we see that the natural map $K_0(R)_\mathbb{Q}^{(i)} \to F_\gamma^i K_0(R)_\mathbb{Q} / F_\gamma^{i+1} K_0(R)_\mathbb{Q}$ is an isomorphism.

Quillen's method for extending the facts about the gamma filtration and Adams operations to the higher K-groups is described in [27]. Let $K_n(R)_\mathbb{Q}$ denote $K_n(R) \otimes_\mathbb{Z} \mathbb{Q}$, and fix a value $k > 1$. For each $i \geq 0$ define $K_n(R)_\mathbb{Q}^{(i)}$ to be the eigenspace of $\psi^k : K_n(R)_\mathbb{Q} \to K_n(R)_\mathbb{Q}$ for the eigenvalue k^i; we call this group the *weight i part* of $K_n(R)$. The *gamma filtration* on the higher K-groups is defined as follows. Let F_γ^1 denote the kernel of the rank homomorphism $K_*(R) \to K_0(R) \to H^0(\mathrm{Spec}(R), \mathbb{Z})$. Let F_γ^i denote the subgroup of $K_*(R)$ generated by elements $\gamma^{r_1}(x_1) \cdot \ldots \cdot \gamma^{r_m}(x_m)$ with $x_1, \dots, x_m \in F_\gamma^1$, at most one x_i not in $K_0(R)$, and $\sum r_j \geq i$. (The values of the gamma operations are multiplied using the product operation (12.1).) The main hurdle Quillen had to overcome in showing that $K_n(R)_{\mathbb{Q}} = \bigoplus_{i=0}^{\infty} K_n(R)_{\mathbb{Q}}^{(i)}$ using the techniques analogous to those above was to show that for an element $x \in K_n(R)$ with $n > 0$, the expression $\gamma^k(x)$ vanishes for sufficiently large k. Any such x arises from a pointed map $S^n \to BG\ell_N(R)^+$ for some N, and he showed that $\gamma^k(x) = 0$ for $k > N$.

There are two other plausible definitions of the gamma filtration on the K-groups $K_n(R)$ which agree up to torsion with the one above ([51, 1.5-1.6] and [29]). In one, we allow more than one x_i to lie outside K_0. In the other, we insist that $m = 1$. It is not important that there is no compelling reason to choose one of these filtrations over the other (integrally), because the goal is to define the gamma filtration on the space, rather than on its homotopy groups.

Let X be a nonsingular quasiprojective variety over a field, and let $CH^i(X)$ denote the Chow group codimension i algebraic cycles modulo rational equivalence. Grothendieck's theorem [26], answering the question mentioned earlier, asserts that the filtration of $K_0(X)$ by codimension of support agrees with the gamma filtration up to torsion and

$$K_0(X)_{\mathbb{Q}}^{(i)} \cong CH^i(X)_{\mathbb{Q}}. \tag{14.1}$$

15. Computations and conjectures about weights.

Let's return to the explicit computations of K-groups mentioned earlier, and add the weights to the description of the K-groups. Classes of free modules in K_0 have weight 0, classes $\{u\}$ of units u in $K_1(R)$ have weight 1, and a product $\{u_1\} \cdot \ldots \cdot \{u_n\} \in K_n(R)$ of units has weight n. Thus, for a field F, the image of $K_n^M(F) \to K_n(F)$ has weight n. In particular, $K_n(F)$ has weight n for $n \leq 2$, but this is not true for $n > 2$. Kratzer has shown [29] that in $K_*(R)$ the weight 0 occurs only in $K_0(R)$, and the weight 1 occurs only in $K_0(R)$ and $K_1(R)$; moreover, the units $R^\times$ account for the all of the weight 1 part of $K_1(R)$.

Soulé has shown [51, Corollary 1, p. 498] that

$$K_n(F)_{\mathbb{Q}}^{(i)} = 0 \qquad \text{for } i > n, \tag{15.1}$$

or in other words, $K_n(F)_{\mathbb{Q}}$ has all its weights $\leq n$. The result depends on stability results due to Suslin [52]; the proof has roughly the following flavor: elements in $K_n(F)$ come from n by n matrices, and the k-th exterior power of a vector space of dimension n is zero if $k > n$. Soulé shows, more generally, that for noetherian rings R of dimension d, the group $K_n(R)_{\mathbb{Q}}$ has all its weights $\leq n + d$.

Soulé [51, 2.9] and Beilinson [5, 2.2.2] have conjectured independently that $K_{2i}(R)$ and $K_{2i-1}(R)$ have all their weights $\geq i$. This conjecture is not known even for fields, except for finite fields, and fields closely related to those. Explicit computations involving cyclic cohomology [17] have shown recently that we must append the hypothesis of regularity of the ring R to the conjecture. It would be very useful to have a proof of this conjecture, for Marc Levine has shown that it implies the existence of an interesting construction for a category of mixed motives. In [4] the conjecture is strengthened to assert that $K_{2i}(F)$ has all of its weights $\geq i+1$.

We would like to say that $K_{2i-1}(\mathbb{F}_q)$ has weight i, but this statement is vacuously true because the group is finite. Nevertheless, Hiller [27] and Kratzer [29] showed that the Adams operation ψ^k acts on $K_{2i-1}(\mathbb{F}_q)$ via multiplication by k^i, so it plausible to think of i as the weight.

In the decomposition $K_0(\mathcal{O}_F) = \mathbb{Z} \oplus \operatorname{Pic}(\mathcal{O}_F)$, the factor $\mathbb{Z}$ accounts for the free modules and has weight 0, and the factor $\operatorname{Pic}(\mathcal{O}_F)$ accounts for the line bundles and has weight 1. The groups $K_{2i-1}(\mathcal{O}_F)$ are weight i; this follows from examining the proof of Borel's theorem, and eventually boils down to the fact that the unit sphere in $\mathbb{C}^i$ is of dimension $2i-1$.

16. The Atiyah-Hirzebruch spectral sequence.

Now let X be a finite cell complex, and let $\mathbb{C}(X^{\text{top}})$ denote the topological ring of continuous functions $X \to \mathbb{C}$. It turns out that there is a way to take the topology of a ring into account when defining the algebraic K-groups, yielding the topological K-groups $K_n(X^{\text{top}}) := K_n(\mathbb{C}(X^{\text{top}}))$, [41]. One can do this with any of the definitions above, and let $K(X^{\text{top}})$ denote the space obtained, so that $K_n(X^{\text{top}}) = \pi_n K(X^{\text{top}})$. It turns out that the plus-construction no longer is needed, and one may define $K_n(X^{\text{top}}) = \pi_{n+1} BG\ell(\mathbb{C}(X^{\text{top}}))$ for $n > 0$, and prove this is the same as $\pi_n G\ell(\mathbb{C}(X^{\text{top}}))$. Thus $G\ell(\mathbb{C}(X^{\text{top}}))$ is the connected component of the identity in $K(X^{\text{top}})$. As mentioned before (during the discussion of the S-construction), the space $K(X^{\text{top}})$ is naturally an infinite loop space, with the deloopings getting more and more connected. The connective spectrum corresponding to this infinite loop space structure on $K(*^{\text{top}})$ is denoted by bu.

Let $*$ denote the one point space. Bott computed the homotopy groups of $K(*^{\text{top}})$.

$$K_{2i}(*^{\text{top}}) = \pi_{2i} K(*^{\text{top}}) = \mathbb{Z}(i)$$
$$K_{2i-1}(*^{\text{top}}) = \pi_{2i-1} K(*^{\text{top}}) = 0$$

I write $\mathbb{Z}(i)$ above to mean simply the group $\mathbb{Z}$, together with an action of weight i by the Adams operations. The identification $\mathbb{Z}(1) = K_2(*^{\text{top}})$ can be decomposed as a sequence of isomorphisms.

$$K_2(*^{\text{top}}) = \pi_2 BG\ell(\mathbb{C}^{\text{top}}) \cong \pi_2 BG\ell_1(\mathbb{C}^{\text{top}})) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}(1).$$

The group $\pi_1(\mathbb{C}^\times)$ is weight 1 for the Adams operations, and hence we write it as $\mathbb{Z}(1)$, essentially because the map $\psi^k : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $u \mapsto u^k$ multiplies homotopy classes by k.

Picking a generator for $\pi_1(\mathbb{C}^\times)$ gives us a generator β for $K_2(*^{\text{top}})$. Bott's theorem includes the additional statement that multiplication by β gives a homotopy equivalence of spaces $K(*^{\text{top}}) \to \Omega^2 K(*^{\text{top}})$. This homotopy equivalence gives us a non-connected delooping $\Omega^{-2}K(*^{\mathrm{top}})$ of $K(*^{\mathrm{top}})$, which is $K(*^{\mathrm{top}})$ itself. These deloopings can be composed to give deloopings of every order, and hence yields an Ω-spectrum BU that has $K(*^{\mathrm{top}})$ as its underlying infinite loop space, and whose homotopy group in dimension $2i$ is $\mathbb{Z}$, for every integer i. The spectrum BU can also be thought of as arising from the connective spectrum bu by inverting the action of multiplication by β.

Let X_+ denote X with a disjoint base point adjoined. There is a homotopy equivalence of the mapping space $K(*^{\mathrm{top}})^X$ with $K(X^{\mathrm{top}})$ and from this it follows that $K_n(X^{\mathrm{top}}) = [X_+, \Omega^n BU]$. When $n < 0$ there might be a bit of ambiguity about what we might mean when we write $K_n(X^{\mathrm{top}})$; we let it always denote $[X_+, \Omega^n BU]$, so that $K_n(X^{\mathrm{top}}) = K_{n+2}(X^{\mathrm{top}})$ for all $n \in \mathbb{Z}$.

We introduce the notation $F^0_\gamma K_n(X)$ for $[X_+, \Omega^n bu]$. We see that

$$F^0_\gamma K_n(X) = \begin{cases} K_n(X^{\mathrm{top}}) & n \geq 0 \\ 0 & n < -\dim X. \end{cases}$$

The topological K-groups behave nicely with respect to the Adams operations: it is a theorem of Atiyah and Hirzebruch that $K_n(X^{\mathrm{top}})^{(i)}_{\mathbb{Q}} = H^{2i-n}(X, \mathbb{Q})$. For $n = 0$ this formula should be compared with (14.1). It was obtained in [1] from a spectral sequence known as the Atiyah-Hirzebruch spectral sequence. The construction of the spectral sequence uses the skeletal filtration $\operatorname{sk}_p X$ of X as follows.

A cofibration sequence $A \subseteq B \twoheadrightarrow B/A$ of pointed spaces and an infinite loop space F give rise to a long exact sequence $\cdots \to [A, \Omega^1 F] \to [B/A, F] \to [B, F] \to [A, F] \to [B/A, \Omega^{-1}F] \to \ldots$.

We construct an exact couple $D_1^{p-1,q} \to E_1^{pq} \to D_1^{pq} \to D_1^{p-1,q+1} \to \ldots$ by setting $E_1^{pq} := [\operatorname{sk}_p X / \operatorname{sk}_{p-1} X, \Omega^{-p-q} BU]$ and $D_1^{pq} := [(\operatorname{sk}_p X)_+, \Omega^{-p-q} BU]$. The explicit computation of the homotopy groups of BU presented above, together with fact that the space $\operatorname{sk}_p X / \operatorname{sk}_{p-1} X$ is a bouquet of the p-cells from X, leads to the computation that

$$E_1^{pq} = \begin{cases} C^p(X, \mathbb{Z}(-q/2)) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases}$$

where C^p denotes the group of cellular cochains. (We abbreviate this conclusion by regarding $\mathbb{Z}(-q/2)$ as zero when q is odd.) The differential $d_1 : E_1^{pq} \to E_1^{p+1,q}$ is seen to be the usual differential for cochains, so that $E_2^{pq} = H^p(X, \mathbb{Z}(-q/2))$. The exact couple gives rise to a convergent spectral sequence because X is a finite dimensional cell complex. The abutment is $[X_+, \Omega^{-p-q} BU] = K_{-p-q}(X)$, so the resulting spectral sequence may be displayed as

$$E_2^{pq} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow K_{-p-q}(X).$$

This spectral sequence is concentrated in quadrants I and IV, is nonzero only in the rows where q is even, and is periodic with respect to the translation $(p,q) \mapsto (p,q-2)$ (if you ignore the action of the Adams operations). Using the Chern character map Atiyah and Hirzebruch show that the differentials in this spectral sequence vanish modulo torsion, and obtain a canonical isomorphism $K_n(X^{\text{top}})_{\mathbb{Q}} \cong \bigoplus_i H^{2i-n}(X,\mathbb{Q}(i))$. It follows that $K_n(X^{\text{top}})_{\mathbb{Q}}^{(i)} \cong H^{2i-n}(X,\mathbb{Q}(i))$. Again, the notation $\mathbb{Q}(i)$ simply denotes $\mathbb{Q}$ together with the action of Adams operations defined by $\psi^k(x) = k^i x$.

17. An alternate approach.

In [12] Dwyer and Friedlander faced the problem of constructing an Atiyah-Hirzebruch spectral sequence for a scheme X. In this rarified algebraic environment, the notions of triangulation and skeleton are no longer available. One of the ingredients of their solution to the problem was to use the Postnikov tower of BU. In this section we describe that idea, but we remain in the topological situation. We use bu instead of BU, and produce an exact couple involving $[X_+, bu]$ that arises from fibration sequences involving bu rather than cofibration sequences involving X_+. This leads naturally to a weight filtration on the K-theory space $K(X^{\text{top}})$.

In the construction of the Atiyah-Hirzebruch spectral sequence presented above, we could have used bu instead of BU. The resulting spectral sequence converges just as well, but must be written as

$$E_2^{pq} = H^p(X,\mathbb{Z}(-q/2)) \Rightarrow F_\gamma^0 K_{-p-q}(X^{\text{top}}), \tag{17.1}$$

and be accompanied the announcement that this is a fourth quadrant spectral sequence, i.e., this time we interpret $\mathbb{Z}(-q/2)$ as zero if $q > 0$ or q is odd. This spectral sequence degenerates modulo torsion just as the other one does, and one sees that $F_\gamma^0 K_{-n}(X^{\text{top}})_{\mathbb{Q}} = H^n(X,\mathbb{Q}(0)) \oplus H^{n+2}(X,\mathbb{Q}(1)) \oplus \cdots$.

It turns out that the gamma filtration exists as a filtration on the space $K(*^{\text{top}})$, but we must interpret the notion of "filtration" homotopy theoretically: the notion of "inclusion" doesn't survive modulo homotopy. So for our purposes, a filtration on a space T is simply a diagram of spaces $T = T^0 \leftarrow T^1 \leftarrow \dots$. We will interpret the symbol T^p/T^{p+1} to denote that space which fits into a fibration sequence $T^{p+1} \to T^p \to T^p/T^{p+1}$, if such a space exists. Such a space almost exists, in the sense that its loop space exists as the homotopy fiber of the map $T^{p+1} \to T^p$. If, moreover, the spaces T^p and T^{p+1} are infinite loop spaces and the map $T^{p+1} \to T^p$ is an infinite loop space map, then a delooping of the homotopy fiber will provide a representative for T^p/T^{p+1}. (The K-theory spaces we will be dealing with are all infinite loop spaces.) This notation is suggestive and convenient: for example, we may rephrase the localization theorem for abelian categories as saying that $K(\mathcal{A}/\mathcal{B}) = K(\mathcal{A})/K(\mathcal{B})$.

As we saw above, the space $K(*^{\text{top}})$ is a simple sort of space, for it has the nice property that each homotopy group is all of one weight for the the Adams operations, and each weight occurs in just one homotopy group. This helps us find a weight filtration for it, for there is a standard filtration of any space T, called the Postnikov tower, which peels off one homotopy group at a time.

The Postnikov tower is constructed inductively as follows. (There is also a combinatorial construction in [35].) Suppose that T^p is a space with π_p as its lowest nonvanishing homotopy group. For simplicity of exposition we suppose $p > 1$. By the Hurewicz theorem, there is an isomorphism $\pi_p(T^p) \cong H_p(T^p, \mathbb{Z})$; let G denote this group. The identity map $H_p(T^p, \mathbb{Z}) \to G$ corresponds to a cohomology class in $H^p(T^p, G)$, and thus to a map $\phi : T^p \to |G[p]|$ which induces an isomorphism on π_p. We let T^{p+1} be the homotopy fiber of the map and observe that

$$\pi_i(T^{p+1}) = \begin{cases} 0 & i \leq p \\ \pi_i(T^p) & i \geq p+1. \end{cases}$$

The "quotient" T^p/T^{p+1} is identified with $|G[p]|$, and has exactly one nonvanishing homotopy group, namely $\pi_p(T^p)$.

In the case of the space $K(*^{\mathrm{top}})$, alternate homotopy groups vanish, so we can strip off homotopy groups at double speed, getting a filtration

$$K(*^{\mathrm{top}}) = W_0 \leftarrow W_{-1} \leftarrow W_{-2} \leftarrow \dots$$

with $W_{-i}/W_{-i-1} = |\mathbb{Z}(i)[2i]|$. Since the pieces of the filtration are of distinct pure weights with respect to Adams operations, it is reasonable to call this the weight filtration of $K(*^{\mathrm{top}})$.

We remark that if $F \to E \to B$ is a fibration sequence of pointed spaces, then each adjacent triple of spaces in the long sequence $\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$ is a fibration sequence. (Here ΩE denotes the loop space of the pointed space E.) If we are dealing with infinite loop spaces, this sequence can be extended to the right with $F \to E \to B \to \Omega^{-1}F \to \Omega^{-1}E \to \Omega^{-1}B \dots$, where $\Omega^{-1}E$ denotes the chosen delooping of the space E. We will do this sort of thing below to ensure that our long exact sequences continue onward to the right, thereby obtaining legitimate exact couples. This is a concern whenever the map $\pi_0 E \to \pi_0 B$ is not surjective.

If $F \to E \to B$ is a fibration sequence of infinite loop spaces, and X is a space, then the mapping spaces fit into a fibration sequence $F^X \to E^X \to B^X$. Taking homotopy groups of this fibration gives a long exact sequence

$$\cdots \to \pi_n F^X \to \pi_n E^X \to \pi_n B^X \to \pi_{n-1} F^X \to \dots$$

which can be rewritten as

$$\cdots \to [X_+, \Omega^n F] \to [X_+, \Omega^n E] \to [X_+, \Omega^n B] \to [X_+, \Omega^{n-1} F] \to \dots.$$

As before, we regard this sequence as continuing onward through negative values of n.

The identity $K(*^{\mathrm{top}})^X \cong K(X^{\mathrm{top}})$ can be used to construct a weight filtration for the space $K(X^{\mathrm{top}})$ from the weight filtration for $K(*^{\mathrm{top}})$ considered above. We simply define

$$W_q K(X^{\mathrm{top}}) := (W_q K(*^{\mathrm{top}}))^X$$

and

$$\begin{aligned} W_q K(X^{\mathrm{top}})/W_{q-1}K(X^{\mathrm{top}}) &:= (W_q K(*^{\mathrm{top}})/W_{q-1}K(*^{\mathrm{top}}))^X \\ &= |\mathbb{Z}(-q)[-2q]|^X. \end{aligned}$$

Using this filtration of $K(X^{\mathrm{top}})$ we define $D_2^{pq} := \pi_{-p-q} W_q K(X^{\mathrm{top}})$ and

$$\begin{aligned} E_2^{pq} &:= \pi_{-p-q} W_q K(X^{\mathrm{top}})/W_{q-1}K(X^{\mathrm{top}}) \\ &= [X_+, \Omega^{-p-q}(W_q K(*^{\mathrm{top}})/W_{q-1}K(*^{\mathrm{top}}))] \\ &= [X_+, \Omega^{-p-q}|\mathbb{Z}(-q)[-2q]|] \\ &= [X_+, |\mathbb{Z}(-q)[p-q]|] \\ &= H^{p-q}(X, \mathbb{Z}(-q)) \end{aligned}$$

and produce an exact couple

$$\cdots \to D_2^{p+1,q-1} \to D_2^{pq} \to E_2^{pq} \to D_2^{p+2,q-1} \to \cdots.$$

The resulting spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow F_\gamma^0 K_{-p-q}(X^{\mathrm{top}})$$

is related to the spectral sequence (17.1). Since we were able to strip off the homotopy groups of bu at double speed, the rows in (17.1) with q odd (which are all zero) do not appear here. (In fact, there is a simple renumbering scheme that will accomplish the same thing for any spectral sequence whose odd numbered rows are zero.)

18. Motivic cohomology.

Now return to the case where X is an algebraic variety. There is no suitable cohomology theory for X with integer coefficients, but taking the Atiyah-Hirzebruch theorem as a guide, we may guess a definition of a cohomology theory with rational coefficients. Following Beilinson [5, 2.2.1] we define *motivic* cohomology groups

$$H_{\mathcal{M}}^{2i-n}(X, \mathbb{Q}(i)) := K_n(X)_{\mathbb{Q}}^{(i)},$$

or equivalently,

$$H_{\mathcal{M}}^{m}(X, \mathbb{Q}(i)) := K_{2i-m}(X)_{\mathbb{Q}}^{(i)}. \tag{18.1}$$

The rational motivic cohomology groups defined in (18.1) have the correct functorial properties and localization theorems expected for groups which we like to call "cohomology" groups, for in [51] Soulé defines motivic cohomology groups with supports and motivic homology groups, and proves part of what is required form them to form a "Poincaré duality theory with supports" in the sense of Bloch and Ogus [8]. The proof uses the localization theorems of Quillen for the algebraic K-groups, as well as the Riemann-Roch theorem without denominators for algebraic K-theory proved by Gillet [19]. Soulé's definition of a motivic homology theory for a variety Z, possibly singular, involves embedding Z in a nonsingular variety X. Using the Adams operations for X and the complement $X - Z$, together with the localization theorem, allows one to define Adams operations on $K'_m(Z)_{\mathbb{Q}}$ which are ultimately independent of the choice of X. The corresponding eigenspaces $K'_m(Z)_{(j)}$ then give the proposed homology groups. There is a cap product operation $K_m(Z) \otimes K'_n(Z) \xrightarrow{\cap} K'_{m+n}(Z)$ arising from the tensor product operation, but it is not yet known how to show this cap product is compatible with the Adams operations, in the sense that it induces pairings

$$K_m(Z)^{(i)} \otimes K'_n(Z)_{(j)} \xrightarrow{\cap} K'_{m+n}(Z)_{(j-i)}.$$

The theorem of Soulé mentioned before (15.1), when rephrased in terms of motivic cohomology groups, becomes a statement about cohomological dimension. It says that for a field F, that $H^j_{\mathcal{M}}(\mathrm{Spec}(F), \mathbb{Q}(i)) = 0$ for $j > i$. More generally, if X is an affine scheme of dimension d, then $H^j_{\mathcal{M}}(X, \mathbb{Q}(i)) = 0$ for $j > i + d$.

Let X be a (nonsingular) variety. The conjecture of Beilinson and Soulé about weights, when rephrased in terms of the motivic cohomology groups, says that if $j < 0$ then $H^j_{\mathcal{M}}(X, \mathbb{Q}(i)) = 0$. (The strengthened form includes the statement that $H^0_{\mathcal{M}}(X, \mathbb{Q}(i)) = 0$ for $i > 0$.) It is expected that there is an alternate construction of these motivic cohomology groups so that $H^j_{\mathcal{M}}(X, \mathbb{Q}(i))$ appears as a Yoneda-Ext group $\mathrm{Ext}^j(\mathbb{Q}, \mathbb{Q}(i)) = 0$ of certain objects (Tate motives) in an abelian category (yet to be defined) of "mixed motives". The vanishing conjecture would be an immediate consequence of such a construction, or an obstacle to performing the construction.

One could ask the following question. Given a nonsingular algebraic variety X, consider the algebraic K-theory space $K(X)$; does it have a *weight* filtration $K(X) = W_0K(X) \leftarrow W_{-1}K(X) \leftarrow W_{-2}K(X) \leftarrow \dots$ whose successive quotients are of pure weight for the Adams operations? (It might also be necessary to view all the spaces in sight as spectra with negative homotopy groups, and to sheafify things with respect to the topology on X.) Were there such a filtration we could define motivic cohomology groups with integer coefficients $H^m_{\mathcal{M}}(X, \mathbb{Z}(i)) := \pi_{2i-m}(W_{-i}K(X)/W_{-i-1}K(X))$ and assemble these groups into an Atiyah-Hirzebruch-type spectral sequence, as we did above. In addition, in order for the motivic cohomology groups to be "cohomology" groups in the usual sense of the word, it would be necessary for the space $W_{-i}K(X)/W_{-i-1}K(X)$ to be homotopy equivalent to the geometric realization of a simplicial abelian group. The normalized chain complex associated to that simplicial abelian group would be the motivic complex sought by Beilinson in [6, 5.10D], and by Lichtenbaum in [33] and [34]. We reverse the numbering of the complex and shift its numbering scheme by $2i$ so that $H^m_{\mathcal{M}}(X, \mathbb{Z}(i))$ is the m-th cohomology group of a cohomological complex; compatibility with the vanishing conjecture of Soulé and Beilinson would then require the complex to be exact except in degrees $1, \dots, i$ when $i > 0$ and X is the spectrum of a field or a local ring.

19. Constructing motivic chain complexes.

There has been a flurry of activity centering upon attempts to build these motivic chain complexes directly.

A promising direct construction of a chain complex aiming to fulfill this role is presented by Bloch in [9]; the construction involves algebraic cycles on the standard cosimplicial affine space over X. The resulting *higher Chow groups* bear the same relation to the usual Chow groups as the higher K-groups bear to the Grothendieck group.

Various other constructions of such chain complexes are presented in [7] by Beilinson, MacPherson, and Schechtmann: their *Grassmannian complex* involves "linear" algebraic cycles, i.e., formal linear combinations of linear subspaces of vector spaces satisfying a transversality condition. Gerdes [18] proves that the map from the homology of the Grassmannian complex to K-theory is a surjection rationally, but unfortunately is not an isomorphism rationally; thus the Grassmannian complex needs to be repaired in some as yet unknown way.

In [30] and [31] Landsburg modifies Bloch's approach by introducing the categories of modules of codimension $\geq p$ on the standard cosimplicial affine space over X.

In [56] Thomason presents an adelic construction of motivic complexes which has the virtue of being directly connected to K-theory, but is not based upon algebraic cycles. He shows how to combine the gamma filtration on rational algebraic K-theory with the canonical weight filtration on ℓ-adic étale topological K-theory to get a good weight filtration on the integral spectrum of algebraic K-theory localized (in the sense of Bousfield) at BU.

In [16] Friedlander and Gabber introduce a complex $A_r(Y, X)$, called the algebraic bivariant cycle complex, which is a possible candidate for a motivic complex. It is related to Lawson homology, but has the advantage that it incorporates rational equivalence in its construction rather than algebraic equivalence.

20. The motivic spectral sequence.

In the context of topological K-theory and singular cohomology, the rational cohomology groups, were they to be defined as the appropriate weight spaces in rational topological K-theory, would immediately be seen to be computable in terms of singular cochains by considering the influence of the skeletal filtration of the finite cell complex X on the K-theory. For an algebraic variety X, the nearest thing we have to a skeletal filtration on X is the filtration by codimension of support of the category of coherent sheaves on X. As mentioned above, the group $K_0(\mathcal{M}^p/\mathcal{M}^{p+1})$ is the group of algebraic cycles of codimension p, so it is expected that algebraic cycles on the variety X will play the same role that singular cochains play on a finite cell complex.

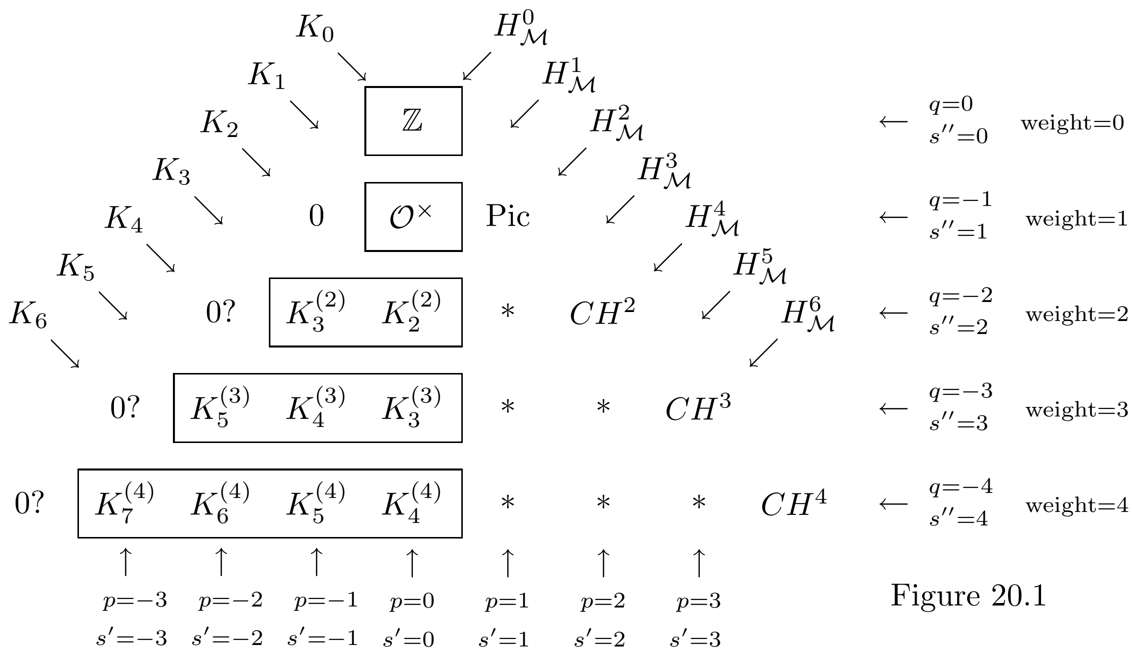

Figure 20.1

In figure 20.1 is a handy chart of the prospective motivic Atiyah-Hirzebruch spectral sequence,

$$E_2^{pq} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

Beilinson conjectured its existence in [6, 5.10B]. The spectral sequence is drawn so that the degree of the d_2 differential is the customary $(2, -1)$, but we don't draw the arrows for these differentials, as they are expected to die modulo controllable torsion anyway. Let $n = -p - q$ and $i = -q$, so that we are concerned with the weight i part of K_n. If we define $s' = p$ and $s'' = -q$, and take X to be a scheme of finite type, then the rank of the group E_2^{pq} is conjecturally [49] associated with the order of zero of the L-function for H_{et}^{p-q-1} at the point s', which is to the left of the critical line. The corresponding regulator map to Deligne cohomology should involve a polylogarithm function of order $-q$. The point s'', to the right of the critical line, is the one related to s' by the conjectural functional equation for the L-series. The real point on the critical line is $(s' + s'')/2 = (p - q)/2$, and the distance from it to either of the other two points is $(s'' - s')/2 = n/2$. When $n = 0$ we are at the center of the critical strip, and the Beilinson conjectures are like the Birch and Swinnerton-Dyer conjectures. When $n = 1$ we are at the edge of the critical strip, and the Beilinson conjectures are like the Tate conjecture. When $n \geq 2$ we are outside the critical strip, and s'' is in the region of absolute convergence. When $p = 0$, then we are concerned with the weight i part of K_i; this is the part corresponding to Milnor K-theory, so the line $p = 0$ is called the Milnor line. When $p - q = 1$, then we are concerned with the motivic cohomology groups $H^1_{\mathcal{M}}(X, \mathbb{Z}(i))$, which is where the Borel classes in the higher algebraic K-theory of algebraic number rings appear; thus the line $p - q = 1$ is called the Borel line.

The boxes in the diagram contain the groups which are expected to be nonzero when X is the spectrum of a field, according to the conjecture of Beilinson and Soulé. In addition, one can imagine the E_1 term of the spectral sequence underlying the diagram, so that the groups in the boxes would be cohomology groups of the motivic complexes lying in each row, with the differential maps going to the right.

The stars in the right hand portion of the diagram, together with the indicated Picard and Chow groups, represent the groups which are known to be zero when $\dim X < p$, according to the vanishing theorem of Soulé. We may say that a group in column p reflects the structure of X in codimension p. (Will this phrase eventually have a meaning for negative values of p?)

The question marks on the left side of the diagram together with the cells off the chart to the left are the locations where the (strengthened) vanishing conjecture is not yet known.

21. Related spectral sequences.

Let A be a finitely generated regular ring. In [42], motivated by conjectures of Lichtenbaum relating K-theory to étale cohomology, Quillen hopes that there is an Atiyah-Hirzebruch spectral sequence of the form

$$E_2^{pq} = H^p_{et}(\mathrm{Spec}(A[\ell^{-1}]), \mathbb{Z}_\ell(-q/2)) \Rightarrow K_{-p-q}(A) \otimes \mathbb{Z}_\ell \tag{21.1}$$

converging at least in degrees $-p - q > \dim(A) + 1$; it would degenerate in case A is the ring of integers in a number field, and either ℓ is odd or A is totally imaginary. This conjecture is now known as the Quillen-Lichtenbaum conjecture; it is not known yet, but inspired by work of Soulé, three approximations to this conjecture have been proved. We describe them now.

In [15] Friedlander defined étale K-theory in terms of mapping spaces from the étale homotopy type of X to a BU, and used the approach of section 17 to produce an Atiyah-Hirzebruch spectral sequence connecting étale cohomology to étale K-theory. But he was limited to considering the case of varieties over algebraically closed fields. In order to treat more general schemes X it is necessary to produce a suitable algebraic model for bu, and to account for the way it changes from point to point in the space X. This involves using the simplicial scheme $BG\ell(\mathcal{O}_X)$ as the algebraic model for bu, and defining the étale ℓ-adic K-theory space as a space of sections for the fibrewise ℓ-adic completion of the map $BG\ell(\mathcal{O}_X)_{et} \to X_{et}$. The resulting étale K-groups depend on the choice of a prime ℓ which is invertible on X, and are denoted by $K_n^{et}(X;\mathbb{Z}_\ell)$. Under the assumption that X is a scheme of finite $\mathbb{Z}/\ell$-cohomological dimension, Dwyer and Friedlander [12, 5.1] construct a fourth-quadrant spectral sequence

$$E_2^{pq} = H^p_{et}(X,\mathbb{Z}_\ell(-q/2)) \Rightarrow K^{et}_{-p-q}(X;\mathbb{Z}_\ell) \tag{21.2}$$

converging in positive degrees. (Presumably, it is possible to ensure convergence in all degrees by replacing the abutment by the zero-th stage of its gamma filtration, as we did with (17.1).) Dwyer and Friedlander rephrase the Quillen-Lichtenbaum conjecture as the assertion that the natural map $K_n(X;\mathbb{Z}_\ell) \to K_n^{et}(X;\mathbb{Z}_\ell)$ from algebraic K-theory to étale K-theory (with $\mathbb{Z}_\ell$ coefficients) is an isomorphism for all $n \geq 0$, for the case where $X = \operatorname{Spec}(\mathcal{O})$, with $\mathcal{O}$ the ring of integers in a number field, and X has finite cohomological dimension. When X has infinite cohomological dimension, but has a finite étale covering which has finite cohomological dimension, (as is the case for $\operatorname{Spec}(\mathbb{Z})$ at $\ell = 2$), they provide [13] a modification to the definition of $K_n^{et}(X;\mathbb{Z}_\ell)$ which makes use of this covering, and provides an analogous rephrasing of the Quillen-Lichtenbaum conjecture. The homotopy type of the corresponding space $K^{et}(\operatorname{Spec}(\mathbb{Z});\mathbb{Z}_\ell)$ is obtained for the case $\ell = 2$ in [13], and for ℓ a regular prime this is done in [14]. Building on this work, S. Mitchell obtained the following completely explicit conjectural calculation of the K-groups of $\mathbb{Z}$, for $n \geq 2$; its validity depends on the truth of the Quillen-Lichtenbaum conjecture, together with the conjecture of Vandiver which states that for every prime ℓ, if $\zeta_\ell = e^{2\pi i/\ell}$, the class number of the totally real number ring $\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}]$ is not divisible by ℓ.

$$K_n(\mathbb{Z}) \cong \begin{cases} 0 & n \equiv 0 \pmod 8 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 1 \pmod 8 \\ \mathbb{Z}/c_k \oplus \mathbb{Z}/2 & n \equiv 2 \pmod 8 \\ \mathbb{Z}/8d_k & n \equiv 3 \pmod 8 \\ 0 & n \equiv 4 \pmod 8 \\ \mathbb{Z} & n \equiv 5 \pmod 8 \\ \mathbb{Z}/c_k & n \equiv 6 \pmod 8 \\ \mathbb{Z}/4d_k & n \equiv 7 \pmod 8 \end{cases} \tag{21.3}$$

Here c_k and d_k are defined so that c_k/d_k is the fraction in lowest terms representing the number B_k/k, with B_k being the k-th Bernoulli number, and with k being chosen so that n is $4k-1$ or $4k-2$. For example, from $B_6 = -691/2730$ we may read off the conjectural equation $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691$. (Warning: in certain other numbering schemes for the Bernoulli numbers, it is B_{12} which is $-691/2730$.)

Let X be a nice scheme on which the prime ℓ is invertible. If $\ell = 2$, assume further that $\sqrt{-1}$ exists on X. In favorable cases an element $\beta \in K_2(X;\mathbb{Z}/\ell^\nu)$ arises via the long exact sequence

$$\cdots \to K_2(X;\mathbb{Z}/\ell^\nu) \to K_1(X) \xrightarrow{\ell^\nu} K_1(X) \to \dots$$

from a primitive ℓ^ν-th root of unity. This element serves as a replacement for the Bott element of topological K-theory, which is missing in this algebraic context. Thomason ([55] and [56]) constructs a spectral sequence

$$E_2^{pq} = H^p_{et}(X, \mathbb{Z}/\ell^\nu(-q/2)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \tag{21.4}$$

in the first and fourth quadrants which succeeds admirably in linking algebra and topology. The abutment is something obtained from K-theory itself by inverting the action of β on the level of topological spectra.

The latest exciting result in this direction is due to Mitchell [38]. (See also [39].) He strengthens Thomason's theorem by identifying the abutment with something closer to algebraic K-theory. Thomason's spectral sequence then becomes

$$E_2^{pq} = H^p_{et}(X, \mathbb{Z}/\ell^\nu(-q/2)) \Rightarrow (L_\infty K)_{-p-q}(X; \mathbb{Z}/\ell^\nu) \tag{21.5}$$

converging for $-p-q \gg 0$. The abutment $L_\infty K$ is the *harmonic* localization of algebraic K-theory, which is a much weaker localization of K-theory than the one obtained by inverting the Bott element. I refer the reader to the highly readable survey paper [40] for further details.

22. Maps from K-theory to cohomology.

According to the motivic philosophy, the motivic cohomology groups ought to be universal in some sense, so there ought to be maps from the motivic cohomology groups to any other reasonable cohomology theory of algebraic varieties. For the rational cohomology groups defined above, such maps were constructed by Soulé in [47] for étale cohomology, and by Gillet in [19] for any cohomology theory $H^*_\alpha(—, \mathbb{Z}(i))$ satisfying a certain general set of axioms. These maps hinge on the construction of Chern class maps

$$c_i : K_n(X) \to H^{2i-n}_\alpha(X, \mathbb{Z}(i)). \tag{22.1}$$

Starting with an element $x \in K_n(X)$ apply the Hurewicz map to get an element in homology, $x' \in H_n(BG\ell(R), \mathbb{Z})$. Since $G\ell(R) = \bigcup_m G\ell_m(R)$, we may choose m large enough so that x' comes from an element $x'' \in H_n(BG\ell_m(R), \mathbb{Z})$. The functor $R \mapsto G\ell_m(R)$ from rings to groups is represented by the algebraic group $G\ell_m$, and the functor $R \mapsto BG\ell_m(R)$ is represented by the simplicial scheme $BG\ell_m$. For each R, the free module R^m is a $G\ell_m(R)$-module; these modules assemble into a simplicial vector bundle $\mathcal{O}^m$ on $BG\ell_m$. Grothendieck's theory of Chern classes gives elements $c_i(\mathcal{O}^m) \in H^{2i}_\alpha(BG\ell_m, \mathbb{Z}(i))$. For any scheme Z there is a natural evaluation map $Z(R) \times \mathrm{Spec}(R) \to Z$. Let $f : BG\ell_m(R) \times \mathrm{Spec}(R) \to BG\ell_m$ denote the corresponding evaluation map of simplicial schemes. There is a cap product operation [53, 5.1.6]

$$H_n(BG\ell_m(R), \mathbb{Z}) \otimes H^{2i}_\alpha(BG\ell_m(R) \times X, \mathbb{Z}(i)) \xrightarrow{\cap} H^{2i-n}_\alpha(X, \mathbb{Z}(i)).$$

Using it, we obtain an element $c_i(x) := x'' \cap f^* c_i(\mathcal{O}^m) \in H_\alpha^{2i-n}(X, \mathbb{Z}(i))$. One checks that this procedure is independent of the choices involved, yielding the map c_i of (22.1).

As an example, if we take $\alpha = \mathcal{D}$, where $H^*_{\mathcal{D}}$ is the Deligne cohomology theory, one may show that the map

$$c_1 : K_1(X) \to H^1_{\mathcal{D}}(X, \mathbb{Z}(i)) \cong H^0(X, \mathcal{O}_X^\times)$$

is the usual determinant map.

The usual methods yield Chern *character* maps

$$\mathrm{ch}_i : K_n(X)_{\mathbb{Q}} \to H_\alpha^{2i-n}(X, \mathbb{Q}(i))$$

whose direct product is a graded ring homomorphism. By compatibility of ch_i with the Adams operations, it turns out that ch_i vanishes on the part of $K_n(X)_{\mathbb{Q}}$ of weight unequal to i, so it is natural to consider the induced map

$$\mathrm{ch}_i : K_n(X)_{\mathbb{Q}}^{(i)} \to H_\alpha^{2i-n}(X, \mathbb{Q}(i))$$

as the object of primary importance. Letting $m = 2i - n$ and rewriting the source of this map in terms of the motivic cohomology groups defined earlier allows us to present this map as

$$\mathrm{ch}_i : H^m_{\mathcal{M}}(X, \mathbb{Q}(i)) \to H^m_\alpha(X, \mathbb{Q}(i)).$$

Gillet has proved [19] an important Grothendieck-Riemann-Roch theorem for these Chern class maps and Chern character maps.

If X is a regular scheme of finite type over $\mathbb{Z}$, then the Chern character map

$$\mathrm{ch}_i : H^m_{\mathcal{M}}(X, \mathbb{Q}(i)) \to H^m_{\mathcal{D}}(X \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{R}(i))^+$$

is known as a *regulator* map. The target group (in which the superscript $+$ denotes taking the invariants under complex conjugation) is a finite dimensional real vector space. These regulator maps are central to the theorem of Borel (11.2) about ranks of K-groups of rings of algebraic integers, and to the conjecture of Beilinson.

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