A Comparison of Continuously Controlled and Controlled K-theory.

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1 Introduction.

1.1 The main results.

The main results of this preprint are the following theorems which appear in Sections 5 and 6, respectively.

Theorem 1.1 *For any map $p : E \to B$ from a topological space E into a compact metric space B, and for each $j \in \mathbb{N}$, there are natural short exact sequences*

$$0 \to \lim^1 K_{2-j}(B;p)_\epsilon \to K^{cc}_{2-j}(B \times (-\infty, 0], B \times 0; p \times id) \to K_{1-j}(B;p)_c \to 0,$$

and

$$0 \to \lim^1 \tilde{K}_{2-j}(B;p)_\epsilon \to \tilde{K}^{cc}_{2-j}(B \times (-\infty, 0], B \times 0; p \times id) \to \tilde{K}_{1-j}(B;p)_c \to 0.$$

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Theorem 1.2 *Let $(\overline{X}, B)$ be a compact, metric pair with B tame in $\overline{X}$, and let p be the evaluation map $p : Holink(\overline{X}, B) \to B$. There are natural chain complexes of abelian groups*

$$K_2(X) \xrightarrow{i_2} K_2^{cc}(\overline{X}, B) \xrightarrow{\Delta_2} K_1(B;p)_c \xrightarrow{\ell_1} K_1(X) \xrightarrow{i_1} K_1^{cc}(\overline{X}, B) \xrightarrow{\Delta_1} \cdots$$

and

$$Wh(B;p)_c \xrightarrow{\ell_1} Wh(X) \xrightarrow{i_1} Wh^{cc}(\overline{X}, B) \xrightarrow{\Delta_1} \tilde{K}_0(B;p)_c \xrightarrow{\ell_0} \tilde{K}_0(X) \xrightarrow{i_0} \cdots$$

For each $j \in \mathbb{N}$, these complexes are exact at $K_{1-j}(B;p)_c$ and $K_{1-j}(X)$, respectively at $\tilde{K}_{1-j}(B;p)_c$ and $\tilde{K}_{1-j}(X)$; the homology groups at $K_{2-j}^{cc}(\overline{X}, B)$ and at $\tilde{K}_{2-j}^{cc}(\overline{X}, B)$ are naturally isomorphic to $\lim^1 K_{2-j}(B;p)_\epsilon$.

Remarks: 1) In Section 7, we recall vanishing results (by Ferry and Pedersen, see [FP]) for the *lim*^{1} groups above. In the final version of the present paper, we plan to include an adaptation of their ideas to our (more general) situation.
2) We emphasize that the present preprint version proves the above theorems only for $j \in \mathbb{N}$. In the final version, we plan to include an extension to the case $j = 0$. □

The groups that enter into the above theorems can briefly be described as follows. For each $\epsilon > 0$, *ϵ-controlled, reduced* K-groups $Wh(B;p)_\epsilon$ $(= \tilde{K}_1(B;p)_\epsilon)$ and $\tilde{K}_0(B;p)_\epsilon$ have been defined in a geometric way by Chapman, [Ch], and in an algebraic one by Quinn, [Q3], see also [RY]. The ϵ decorated groups that appear in the above theorems are unreduced versions of the groups defined in [RY] and [Q3]. They are connected by homomorphisms $K_{1-j}(B;p)_\epsilon \to K_{1-j}(B;p)_\delta$ for $\epsilon \leq \delta$, and $K_{1-j}(B;p)_c = \lim K_{1-j}(B;p)_\epsilon$ (and similarly for reduced groups).

Let $(\overline{X}, B)$ be as in Theorem 1.2, and let R be any ring[1]. In [ACFP], the authors study the K-theory, $K_*^{cc}(\overline{X}, B; R)$, of geometric R-modules[2] in $X = \overline{X} - B$ with morphisms which are "continuously controlled" at B. Although the ideas from [ACFP] have had important applications (see e.g. [AC] and [CP]), from a geometric viewpoint it seems more natural to allow "variable coefficients" in the morphisms between geometric modules. This viewpoint is introduced in the continuously controlled setting by two of the present authors. In [AM2], they consider geometric modules in a space E which is equipped with a map $p : E \to X = \overline{X} - B$, and they take continuously controlled (= *cc*) homotopy classes of *cc* path matrix morphisms (with coefficients in the ring R) as morphisms. For technical reasons, they also need to allow X to be non-metric, and locally compact rather than compact. The resulting *continuously controlled K-theory with variable coefficients* is denoted $K_*^{cc}(\overline{X}, B; p)$. It also comes in a "germs-at-infinity" version $K_*^{cc,\infty}(\overline{X}, B; p)$ which fits into an exact sequence

$$\cdots \to K_*(E) \to K_*^{cc}(\overline{X}, B; p) \to K^{cc,\infty}(\overline{X}, B; p) \to K_{*-1}(X) \to \cdots$$

where $K_*(E)$ is the Bass-Quillen K-theory of the group(oid) ring $R[\pi_1(E)]$.

We proceed to describe the background for our interest in these results.

1.2 Background.

The type of pairs $(\overline{X}, B)$ appearing in Theorem 1.2 are precisely those that occur in the study of stratified spaces in the sense of [Q2]. Therefore, cellular chain complexes that appear in connection with such spaces tend to consist of geometric modules in the additive category $\mathcal{GM}^{cc}(\overline{X}, B)$ underlying the groups $K_*^{cc}(\overline{X}, B)$. Hence the K-theory of this category is an interesting object

[1] Actually, R must have the property that the left module R^s can be a direct summand in R^t only if $s \leq t$. We shall usually suppress R from all our notation.

[2] Actually, in [ACFP], the category of R-modules can be replaced by any additive category $\mathcal{A}$ to obtain $K_*(\overline{X}, B; \mathcal{A})$.

to study. We also note that in the special case where $\overline{X}$ is the one point compactification of X, the theorem is intimately related to Siebenmann's theory of infinite simple homotopy types in [Si]; see Remark **3)** following Theorem 6.1.

Both Quinn and Chapman were motivated by the controlled boundary problem for a manifold M, equipped with a map $p : M \to B$ with B as above. The problem is to decide under which conditions M admits a boundary in such a way that p extends over $M \cup \partial M$. Under suitable a priori conditions on the dimension and the behaviour of M near ∞, Chapman shows that the problem is completely governed by a primary obstruction and a secondary obstruction which lie in groups that are constructed (using an inverse limit and $\lim^1$) from suitable groups of the above type. Quinn, in [Q1], also gave an obstruction theory solution to the controlled boundary problem, but the obstruction group is constructed (in Sections 5 and 8 in [Q1]) in a less transparent way. In his Notre Dame CBMS-lectures in 1984, Quinn outlined an algebraic definition of groups $Wh(B;p)_\epsilon$ and $\tilde{K}_0(B;p)_\epsilon$ in terms of geometric modules in E and (ϵ-small homotopy classes of) path matrix morphisms in E with ϵ-short image paths in B. These groups can be directly related to the cellular chains on E (assuming a "controlled" cellular structure), and this introductory part of Quinn's lecture series was apparently geared towards a direct description (in terms of chain complexes) of the obstructions obtained in [Q1]. Unfortunately, this approach was never carried to fruition, although part of the projected CBMS volume appeared later as [Q3]. In fact, to the best of the present authors' knowledge,

(A) no one has ever established a strict link between the obstruction groups that appear in [Q1] and the groups defined in [Q3].

In particular, even though Quinn's obstruction groups do form a homology theory (as described in Section 8 of [Q1]),

(B) the well known folklore corollary to the effect that the inverse limit groups $Wh(B;p)_c$, $\tilde{K}_0(B;p)_c, \cdots$ are the groups in low degree of a homology theory in some great generality, appears unsubstantiated at the moment.

Also,

(C) no solid foundation for an interpretation of Quinn's obstructions in terms of chain complexes in some category of geometric modules seems to exist.

Along a similar vein, apparently

(D) nobody has taken time out to study in detail the expected isomorphism between Chapman's geometrically defined and Quinn's algebraically defined ϵ-controlled groups.

The present paper is related to all of these problems. Starting from behind, (D) should actually be very accessible now that the algebraic version of the groups involved is equally well understood (thanks to [RY] and the present paper) as is the original, geometric version in [Ch].

Since Chapman's obstructions are quite explicitly (and classically) given in [Ch], the precise form of a solution to (D) might actually be of value in trying to understand (C). Since the primary obstruction is in a group of type $\tilde{K}_0(B;p)_c$ and the secondary in one of type $\lim^1 Wh(B;p)_\epsilon$, Theorem 1.1 suggests a total obstruction in $Wh^{cc}(B \times (-\infty, 0], B \times 0; p \times id)$; see Remark **4)** following 6.1.

Recently, in [RY], Ranicki and Yamasaki have given a chain complex version of Quinn's ϵ-controlled groups from [Q3]. Their main goal is to finish the program that originally motivated Connell and Hollingsworth to invent the notion of a geometric group, see [CH], namely to give an algebraic proof of the invariance of Whitehead torsion, first proved by Chapman in [Ch1].

They also reprove other results, including the Borsuk conjecture (originally proved by West, in [We]). Their work includes "stably exact" Mayer-Vietoris sequences which are, of course, very much related to our problem (B), but they do not attack (B) directly. In [AM2], it is shown that the groups $K_*(B \times (-\infty, 0], B \times 0; p \times id)$ do form a homology theory in the (suitably restricted) variable $p : E \to B$. Thus, the controlled groups $K_{2-j}(B; p)_c$ have homology theory behaviour, provided the $\lim^1$-terms in Theorem 1.1 vanish. We briefly discuss the vanishing in Section 7.

Let $p : E \to B = |K|$ be a map which admits an iterated mapping cylinder structure[3] over the finite complex K and let $S : \mathcal{TOP} \to \mathcal{SPEC}$ be the functor which associates to any space X its (non connective, as in [PW]) K-theory spectrum. From Theorem 0.1 of [AM3], it is known that Quinn's homology groups $H_*(\Sigma K, v_+; \mathcal{S}(p))$, derived from this functor S, are isomorphic to the boundedly controlled K-theory groups $K_*(p)$ defined in [AM4]. On the other hand, in [AM2], the latter groups are shown to coincide with the *cc* K-theory groups occurring in Theorem 1.1. Thus for such maps $p : E \to |K|$ and this functor $S : \mathcal{TOP} \to \mathcal{SPEC}$, we have the following corollary.

Corollary 1.3 *For each $j \in \mathbb{N}$, there is a short exact sequence*

$$0 \to \lim^1 K_{2-j}(|K|; p) \to H_{2-j}(\Sigma|K|, v_+; \mathcal{S}(p)) \to K_{1-j}(B; p)_c \to 0.$$

In Section 7 we give our reasons for conjecturing that the $\lim^1$-term appearing here is actually zero.

1.3 Historical remarks and acknowledgements.

The exact sequences in Theorem 1.1 were conjectured to be true (with a *boundedly controlled* group instead of the *cc* group) a long time ago, by two of the authors (DRA and HJM). Actually, HJM's lecture at the Oberwolfach Topologie Tagung in 1987, contained an explicit version of this conjecture. However, one ingredient in the proof (injectivity of $\lim^1 K_1(B; p)_\epsilon \to K^{cc}(B \times (-\infty, 0], B \times 0; p \times 1))$ kept eluding DRA and HJM, so the project was essentially shelved until HJM came to University of Notre Dame on a sabbatical in 1992/93. There, discussions between him and the third author (FXC) about the groups $K^{cc}(\overline{X}, B)$ appearing in Theorem 1.2, renewed the interest and the missing part of the proof was finally found.

HJM wants to thank the Mathematics Department of University of Notre Dame for a pleasant (mathematical and otherwise) atmosphere during the above mentioned sabbatical. Special thanks go to Bruce Williams for his help in arranging the sabbatical and for his never ending willingness to discuss mathematics and to search through his extensive files of (p)reprints to find the one you want, or even the one you *ought* to want. HJM also thanks the Mathematical Sciences Research Institute, Berkeley, for its hospitality during the summer of 1993 when much of the actual writing was done.

2 Geometric modules and path matrix morphisms.

In this section we set up the general framework (a priori involving no control) for *geometric modules, path matrix morphisms, and path matrix homotopies* in any topological space E.

We recall that a *Moore path* in E is a pair (ω, l_ω) where $\omega : [0, \infty) \to E$ is continuous, and constant on the subray $[l_\omega, \infty)$. A homotopy between Moore paths is a pair (H, l_H) where $l_H : [0, 1] \to [0, \infty)$ and $H : [0, \infty) \times [0, 1] \to E$ are continuous, and $H(-, y)$ is constant on $[l_H(y), \infty)$ for each $y \in [0, 1]$. As usual, (H, l_H) is a homotopy *from* $\delta_1(H, l_H) = (H(-, 0), l_H(0))$ *to* $\delta_0(H, l_H) = (H(-, 1), l_H(1))$. Often l_ω and l_H will be suppressed from the notation, $\omega(l_\omega)$ will

[3] In [AM3] this is called a homotopy colimit structure.

be denoted $\omega(\infty)$, and $H(l_H(y), y)$ will be denoted $H(\infty, y)$. The set, $\mathcal{M}(E)$, of Moore paths in E will be viewed as a category whose objects are the points of E and where the morphisms from x to y are the Moore paths ω with $y = \omega(0)$ and $x = \omega(\infty)$ (so that the usual concatenation of paths defines the composition). Also, $\mathcal{H}(E)$ denotes the set of end point fixing homotopies, so δ_i is a map $\mathcal{H}(E) \to \mathcal{M}(E)$, for $i = 0, 1$.

A *geometric module* on E is a pair (S, σ) with S a discrete set and $\sigma : S \to E$ a function. We think of each $s \in S$ as a basis element which "sits at" the point $\sigma(s) \in E$. A *(path matrix) morphism* $\varphi : (T, \tau) \to (S, \sigma)$ is a function[4] $\varphi : S \times T \to R\mathcal{M}(E)$ into the free R-module generated by $\mathcal{M}(E)$. There are two requirements on $\varphi(s,t) = \Sigma_\omega r_{st}(\omega)\omega$, viz.

$$r_{st}(\omega) \neq 0 \Rightarrow [\omega(0) = \sigma(s) \text{ and } \omega(\infty) = \tau(t)], \tag{1}$$

and

$$\forall t \in T : \{(s, \omega) \in S \times \mathcal{M}(E) \,|\, r_{st}(\omega) \neq 0\} \text{ is finite.} \tag{2}$$

The former condition says that each Moore path is to be considered a morphism from its end point to its beginning point; general morphisms are then R-linear combinations of such "atomic" morphisms. The second condition guarantees that matrix multiplication and the composition in $\mathcal{M}(E)$ give a well defined composition of morphisms. The resulting category will be denoted $GM(E)$. It is additive, with addition of morphisms coming from the abelian group structure in $R\mathcal{M}(E)$ and with $(S_1, \sigma_1) \oplus (S_2, \sigma_2)$ given by the disjoint union $(S_1 \coprod S_2, \sigma_1 \coprod \sigma_2)$. The inclusions and projections

$$(S_i, \sigma_i) \xrightarrow{\iota_i} (S_1, \sigma_1) \oplus (S_2, \sigma_2) \xrightarrow{\pi_i} (S_i, \sigma_i),\ i = 1, 2,$$

involve only constant paths γ_x, $x \in E$. For example, $\iota_2(s, s_2) = \delta_{s,s_2}\gamma_{\sigma_2(s_2)}$ for $(s, s_2) \in (S_1 \coprod S_2) \times S_2$, where δ is the Kronecker delta. Incidentally, if $S_1 = \emptyset$, this defines the identity morphism on (S_2, σ_2).

We call a morphism $\varphi : M = (T, \tau) \to N = (S, \sigma)$ *geometric*[5] if it has the form

$$\varphi(t, s) = \delta_{t,\alpha(s)} u_s \omega_s, \tag{3}$$

where $\alpha : S \to T$ is a map; $u_s \in R$; and $\omega_s \in \mathcal{M}(E)$ has $\omega_s(\infty) = \tau\alpha(s), \omega_s(0) = \sigma(s)$[6]. We note that the above inclusions ι_i and projections π_i are examples of such geometric morphisms.

If α is a bijection, each u_s is a unit in R, and each ω_s is constant, then the corresponding φ is an isomorphism in $GM(E)$ with an inverse obtained by inverting α and each u_s. Clearly, the direct sum operation $\oplus$ is associative and commutative up to coherent, geometric isomorphisms of this sort (with all $u_* = 1$). By abuse of set theory, we shall actually proceed as if $\oplus$ is strictly associative, i.e., as if $\coprod$ is strictly associative in the category of sets.

A *(path matrix) homotopy* from (T, τ) to (S, σ) is a function $\Phi : S \times T \to R\mathcal{H}(E)$, say $\Phi(s, t) = \Sigma_H r_{st}(H)H$, which satisfies

$$r_{st}(H) \neq 0 \Rightarrow [H(0, y) = \sigma(s) \text{ and } H(\infty, y) = \tau(t), \forall y \in [0, 1]], \tag{4}$$

and

$$\forall t \in T : \{(s, H) \in S \times \mathcal{H}(E) \,|\, r_{st}(H) \neq 0\} \text{ is finite.} \tag{5}$$

[4]The indexing corresponds to classical matrix notation, i.e., the row index s corresponds to the range, and the column index t to the domain.

[5]In [AM1], such a morphism (or rather its homotopy class) is said to be "induced by a basis morphism". We have borrowed the present term from [RY] and [Q3]. However, we use the term "geometric" in connection with morphisms only in the context defined here, in contradistinction to [RY], where all path matrix morphisms are called "geometric morphisms"; but where "geometric" is also used in our, restricted sense.

[6]i.e., for given s, $\sigma(s)$ is the beginning point of at most one path having non zero coefficient in φ; this path ends at $\tau\alpha(s)$ and has multiplicity u_s.

The end point maps $\delta_i : \mathcal{H}(E) \to \mathcal{M}(E)$, $i = 0, 1$, extend by linearity to maps $\delta_i : R\mathcal{H}(E) \to R\mathcal{M}(E)$, and Φ is understood to be a homotopy from $\delta_1\Phi$ to $\delta_0\Phi$. We also write $\Phi : \delta_1\Phi \simeq \delta_0\Phi$. This homotopy notion behaves well viz-a-viz composition, sum and direct sums so that one has a corresponding, additive, homotopy category $\mathcal{GM}(E)$.

In this generality, $GM(E)$ and $\mathcal{GM}(E)$ are of little interest. To arrive at interesting constructions, one has to impose conditions on the objects, the morphisms, and the homotopies allowed: For objects one has some "finiteness" condition on $\sigma : S \to E$. For morphisms φ and homotopies Φ the conditions are expressed in terms of the following families of subsets of E (where $\varphi(s,t) = \Sigma r_{st}(\omega)\omega$ and $\Phi(s,t) = \Sigma r_{st}(H)H$).

$$\mathcal{F}(\varphi) = \{\omega([0,\infty)) \subseteq E \mid \exists (s,t) \in S \times T : r_{st}(\omega) \neq 0\}, \tag{6}$$

$$\mathcal{F}(\Phi) = \{H([0,\infty) \times [0,1]) \subseteq E \mid \exists (s,t) \in S \times T : r_{st}(H) \neq 0\}. \tag{7}$$

In Section 3 we study one such set of conditions ("ϵ-control") in some detail. Another set of conditions ("continuous control") is studied by two of the present authors in [AM2], in part in order to obtain the *cc* results needed in the present paper. Some definitions and results from [AM2] are summarized in Section 4 (which may undergo minor changes once [AM2] is finished).

3 Controlled K-theory.

The category $\mathcal{TOP}_c$ of *controlled spaces* has as its objects all pairs $(B;p)$ where $p : E \to B$ be a map from an arbitrary space E into a compact metric space B. A morphism $(f,g) : (B;p) \to (B';p')$ is defined to be a pair of continuous maps

$$(f : E \to E', g : B \to B') \text{ with } p'f = gp. \tag{8}$$

Let $j \in \mathbb{N}$ and $\epsilon > 0$. In this section we define and study the *ϵ-controlled groups* $K_{2-j}(B;p)_\epsilon$, (which are *not* functorial on $\mathcal{TOP}_c$) as well as the *controlled groups* $K_{2-j}(B;p)_c$ ($= \lim K_{2-j}(B;p)_\epsilon$), and the *$lim^1$-groups* $\lim^1 K_{2-j}(B;p)_\epsilon$ (which *are* functorial on $\mathcal{TOP}_c$).

The first subsection deals with the cases $j = 1$ and $j = 2$, i.e., with K_1- and K_0-groups. Reduced versions of these are defined in Section 3.2, and compared to the similar groups defined by Ranicki and Yamasaki ([RY]) in Section 3.3. Finally, in Section 3.4, results from [RY] and Section 3.3 are utilized to define the *lower* groups $K_{2-j}(B;p)_\epsilon$, $K_{2-j}(B;p)_c$ and $\lim^1 K_{2-j}(B;p)_\epsilon$ for $j > 2$. This uses a geometric form of Bass's contracted functor viewpoint, cf. [Ba].

First, however, we must discuss the relevant conditions on modules, morphisms and homotopies. All geometric modules $\sigma : S \to E$ will have S *finite*. They will often be denoted by $M, N, L, \cdots$ instead of $(S,\sigma), (T,\tau), (R,\rho), \cdots$. The families $\mathcal{F}(\varphi)$ and $\mathcal{F}(\Phi)$ associated to a morphism φ or a homotopy Φ in $GM(E)$ (see (6) and (7)) are *end pointed* in the sense that each $F \in \mathcal{F}$ ($= \mathcal{F}(\varphi)$ or $\mathcal{F}(\Phi)$) comes equipped with an ordered pair of points (b_F, e_F). Such an end pointed family is said to be *bounded by* $\epsilon \geq 0$, if

$$\forall F \in \mathcal{F} : p(F) \subseteq B(p(b_F),\epsilon) \cap B(p(e_F),\epsilon), \tag{9}$$

where $B(x,\epsilon)$ denotes the closed ball of radius ϵ and with center $x \in B$. An *ϵ-morphism* $\varphi : N \to M$ between geometric modules on E is a morphism for which $\mathcal{F}(\varphi)$ is bounded by ϵ. Similarly, one defines an ϵ-homotopy between ϵ-morphisms, written $\varphi \simeq_\epsilon \psi$. The collection $GM_\epsilon(B;p)$ of all (finite) geometric modules on E and all ϵ-morphisms between them, is not a subcategory of $GM(E)$. The best one can say is that the composition maps $GM_\delta(B;p)(M,L) \times GM_\epsilon(B;p)(N,M)$ into $GM_{\delta+\epsilon}(B;p)(N,L)$, and it is compatible with the inclusions $GM_\epsilon(B;p) \subseteq GM_{\epsilon'}(B;p)$ for $\epsilon \leq \epsilon'$. On the other hand, ϵ-homotopy *is* an equivalence relation in the collection $GM_\epsilon(B;p)$, so one does have the corresponding quotient "non-categories" $\mathcal{GM}_\epsilon(B;p)$ with compositions

$$\mathcal{GM}_\delta(B;p)(M,L) \times \mathcal{GM}_\epsilon(B;p)(N,M) \to \mathcal{GM}_{\delta+\epsilon}(B;p)(N,L).$$

Also, ϵ-homotopies can be composed with δ-morhisms or δ-homotopies on either side to give $(\epsilon+\delta)$-homotopies. We leave it to the reader to keep track of the obvious formal properties of such compositions, cf. Section 2 of [RY].

An ϵ-morphism $\xi : N \to N$ is called an *ϵ-idempotent*, if $\xi^2 \simeq_{2\epsilon} \xi$. If ξ is an ϵ-idempotent, then so is $1_N - \xi$. A *geometric ϵ-idempotent* is an ϵ-morphism which is geometric in the sense of (3), and for which $\alpha = 1_T : S = T \to T$ and each u_s is an idempotent in R.

An ϵ-morphism $\varphi : N \to M$ is called an *ϵ-isomorphism*, if there exists an ϵ-morphism $\varphi' : M \to N$ such that $\varphi'\varphi \simeq_{2\epsilon} 1_N$ and $\varphi\varphi' \simeq_{2\epsilon} 1_M$. We also call φ' *an ϵ-inverse* for φ. We note that an ϵ-morphism, φ, which is geometric and for which the map $\alpha : S \to T$ is a bijection and each u_s is a unit in R, is an ϵ-isomorphism, with a preferred, geometric ϵ-inverse, φ^{-1}, obtained by inverting α, reversing all the paths involved, and inverting all the units u_s. We shall refer to such a morphism as a *geometric ϵ-isomorphism.*

Clearly, any geometric module $M = (S, \sigma)$ admits a geometric 0-isomorphism to a module (T, τ) with $T \subset E \times \mathbb{N}$ and $\tau = proj_1|T : T \to E$. In fact, one may take each path involved in such an isomorphism to be constant. Therefore, up to conjugation by geometric 0-isomorphisms, there is only a *set* of ϵ-automorphisms and a *set* of ϵ-idempotents in GM_ϵ. This is important for set theoretic reasons in the following section.

3.1 Definition of $K_i(B;p)_\epsilon$ for $i = 0, 1$.

The group $K_0(B;p)_\epsilon$ will be defined in terms of the class of ϵ-idempotents in $GM_\epsilon(B;p)$. Thus, let $\xi : M \to M$ and $\eta : N \to N$ be ϵ-idempotents. We write $\xi \sim_\epsilon \eta$, if there exist a geometric module L and ϵ-morphisms $\varphi : M \oplus L \to N \oplus L$, $\psi : N \oplus L \to M \oplus L$ such that

(10) $$\psi\varphi \simeq_{2\epsilon} \xi \oplus 1_L \text{ and } \varphi\psi \simeq_{2\epsilon} \eta \oplus 1_L;$$

and

(11) $$\psi \simeq_{2\epsilon} (\xi \oplus 1_L)\psi \simeq_{2\epsilon} \psi(\eta \oplus 1_L) \text{ and } \varphi \simeq_{2\epsilon} (\eta \oplus 1_L)\varphi \simeq_{2\epsilon} \varphi(\xi \oplus 1_L).$$

The idea here is that up to homotopy φ is a morphism $(M \oplus L, \xi \oplus 1_L) \to (N \oplus L, \eta \oplus 1_L)$ between objects in the idempotent completion of $GM_\epsilon(B;p)$ and that ψ is its inverse. We note that $\sim_\epsilon$ is not an equivalence relation, but it is symmetric and reflexive, and it does have the following properties

(12) $$[\xi \sim_\epsilon \eta \text{ and } \eta \sim_\epsilon \zeta] \Rightarrow \xi \sim_{5\epsilon} \zeta,$$

(13) $$\forall \epsilon\text{-idempotent } \xi : M \to M : \xi \oplus (1_M - \xi) \sim_\epsilon 1_M,$$

(14) $$[\xi_i \sim_\epsilon \eta_i, i = 1, 2] \Rightarrow [\xi_1 \oplus \xi_2 \sim_\epsilon \eta_1 \oplus \eta_2],$$

and

(15) $$[\xi \sim_\delta \eta \text{ and } \delta < \epsilon] \Rightarrow [\xi \sim_\epsilon \eta].$$

Let $\approx_\epsilon$ denote the smallest equivalence relation in the class of all ϵ-idempotents, containing $\sim_\epsilon$. Since ϵ-idempotents which are conjugate by means of a geometric 0-isomorphism, are $\sim_\epsilon$-equivalent, the $\approx_\epsilon$-equivalence classes form a set. Since $\oplus$ is associative and commutative up to geometric 0-isomorphisms, this set is an abelian semigroup under a sum induced by $\oplus$. We define $K_0(B;p)_\epsilon$ to be the corresponding abelian group. Thus, any $x \in K_0(B;p)_\epsilon$ has the form $x = [\xi] - [\eta]$ with ξ and η ϵ-idempotents, and $[\xi] - [\eta] = [\xi'] - [\eta']$ if and only if there exist ϵ-idempotents $\xi_1, \xi_2, \cdots, \xi_{m+1}$, and ζ, such that

(16) $$\xi \oplus \eta' \oplus \zeta = \xi_1 \sim_\epsilon \xi_2 \sim_\epsilon \xi_3 \sim_\epsilon \cdots \sim_\epsilon \xi_{m+1} = \xi' \oplus \eta \oplus \zeta.$$

Since there is no bound on m, it seems that there is no control on the relation between $\xi \oplus \eta'$ and $\xi' \oplus \eta$. However, by using (12)–(15), one can imitate Theorem 3.5 of [Ch] (see also Lemma 4.4 of [Q1]) to get the following equivalences (where $\bar{\xi}$ is short for $1-\xi$)

$$\begin{aligned}
&\xi_1 \oplus 1 \oplus \cdots \oplus 1 \oplus 1 \sim_\epsilon \\
&\xi_1 \oplus \bar{\xi}_2 \oplus \xi_2 \oplus \cdots \oplus \bar{\xi}_{m-1} \oplus \xi_{m-1} \oplus \bar{\xi}_m \oplus \xi_m \sim_\epsilon \\
&\xi_2 \oplus \bar{\xi}_2 \oplus \xi_3 \oplus \cdots \oplus \bar{\xi}_{m-1} \oplus \xi_m \oplus \bar{\xi}_m \oplus \xi_{m+1} \sim_\epsilon \\
&1 \oplus 1 \oplus \cdots \oplus 1 \oplus 1 \oplus \xi_{m+1} \sim_0 \xi_{m+1} \oplus 1 \oplus 1 \oplus \cdots \oplus 1 \oplus 1.
\end{aligned}$$

In view of (12) and (13), this proves the following proposition.

Proposition 3.1 *There is a universal constant $k \in \mathbb{N}$ such that whenever ξ, η, ξ', and η' are ϵ-idempotents with $[\xi] - [\eta] = [\xi'] - [\eta'] \in K_0(B;p)_\epsilon$, then $\xi \oplus \eta' \sim_{k\epsilon} \xi' \oplus \eta'$.*

The inclusions $GM_\delta \subseteq GM_\epsilon$, $\delta < \epsilon$, give rise to "relax control" homomorphisms $K_0(B;p)_\delta \to K_0(B;p)_\epsilon$. We define the *controlled group* $K_0(B;p)_c$ to be the (inverse) limit of the resulting inverse system indexed on all $\epsilon > 0$. We also need the first derived, $\lim^1 K_0(B;p)_\epsilon$.

We note that $K_0(B;p)_\epsilon$ is *not functorial* on the category $\mathcal{TOP}_c$. However, since B is assumed compact, the inverse system $\{K_0(B;p)_\epsilon\}_\epsilon$, when viewed as an object in the usual pro category (cf. the Appendix in [AM]), *is* functorial. Hence the controlled groups $K_0(B;p)_c$ and the groups $\lim^1 K_0(B;p)_\epsilon$ are also functorial.

The groups $K_1(B;p)_\epsilon$ and $K_1(B;p)_c$ are constructed in a similar way, starting from the class of ϵ-automorphisms in $GM_\epsilon(B;p)$. An ϵ-automorphism $\gamma : N \to N$ is called ϵ-*elementary*, if N admits a decomposition[7] $N = N_1 \oplus N_2$ so that γ is of the form

$$\gamma = \begin{pmatrix} 1 & \gamma' \\ 0 & 1 \end{pmatrix} : N_1 \oplus N_2 \to N_1 \oplus N_2 \tag{17}$$

for some ϵ-morphism $\gamma' : N_2 \to N_1$. Such a γ has a preferred ϵ-inverse

$$\gamma^{-1} = \begin{pmatrix} 1 & -\gamma' \\ 0 & 1 \end{pmatrix} : N_1 \oplus N_2 \to N_1 \oplus N_2. \tag{18}$$

which is itself ϵ-elementary and has $\gamma\gamma^{-1} = \gamma^{-1}\gamma = 1$. Let $\varphi : M \to M$ and $\psi : N \to N$ be ϵ-automorphisms. We write $\varphi \sim_\epsilon \psi$, if there exist geometric modules L', L''; a geometric 0-isomorphism $\nu : M \oplus L' \to N \oplus L''$; an ϵ-elementary automorphism $\gamma : N \oplus L'' \to N \oplus L''$; and a 2ϵ-homotopy

$$\nu(\varphi \oplus 1_{L'})\nu^{-1} \simeq_{2\epsilon} (\psi \oplus 1_{L''})\gamma. \tag{19}$$

The relation thus defined is not an equivalence relation, but it does satisfy

$$\varphi \sim_\epsilon \psi \Rightarrow \psi \sim_{3\epsilon/2} \varphi, \tag{20}$$

$$[\varphi_i \sim_\epsilon \psi_i, i = 1, 2] \Rightarrow \varphi_1 \oplus \varphi_2 \sim_\epsilon \psi_1 \oplus \psi_2, \tag{21}$$

and

$$[\varphi \sim_\delta \psi \text{ and } \delta < \epsilon] \Rightarrow \varphi \sim_\epsilon \psi. \tag{22}$$

Let $\approx_\epsilon$ be the smallest equivalence relation in the class of all ϵ-automorphisms, containing $\sim_\epsilon$. The equivalence classes form (a set, and) an abelian semigroup under a sum induced by $\oplus$. The zero element is the class of 1_M for any M.

If φ has φ' as an ϵ-inverse, then successive multiplication on the right by the following six ϵ-elementary automorphisms

[7] Recall that if $N = (T, \tau)$, then this means that $N_i = (T_i, \tau | T_i)$ where $T = T_1 \coprod T_2$.

$$(23)\qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\varphi & 1 \end{pmatrix}, \begin{pmatrix} 1 & \varphi' \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\varphi & 1 \end{pmatrix}$$

gives a string of six $\sim_\epsilon$-equivalences

$$(24)\qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim_\epsilon \cdots \sim_\epsilon \begin{pmatrix} \varphi & 0 \\ 0 & \varphi' \end{pmatrix}.$$

This establishes the existence of inverses, and the group so defined will be called $K_1(B;p)_\epsilon$.

The following proposition is essentially a version of Lemma 4.4 of [Q1] or Theorem 3.5 of [Ch]

Proposition 3.2 *There is a universal constant $k \in \mathbb{N}$ such that whenever two ϵ-automorphisms $\varphi : M \to M$ and $\psi : N \to N$ represent the same element in $K_1(B;p)_\epsilon$, then there exist a geometric module L and thirteen ϵ-elementary automorphisms $\Gamma^i : M \oplus L \oplus N \to M \oplus L \oplus N$, $i = 1, 2, \cdots, 13$, such that*

$$\varphi \oplus 1_L \oplus 1_N \simeq_{k\epsilon} (1_M \oplus 1_L \oplus \psi)\Gamma^1\Gamma^2 \cdots \Gamma^{13}.$$

Proof: Since conjugation by geometric 0-isomorphisms preserves ϵ-elementary automorphisms, we do not have to worry about associativity and commutativity isomorphisms in the proof. Using (20), we see that there is an integer m and a sequence of 3ϵ-homotopies $\nu_i(\varphi_i \oplus 1_{L'_i})\nu_i^{-1} \simeq_{3\epsilon} (\varphi_{i+1} \oplus 1_{L''_i})\gamma^i, i = 1, 2, \cdots, m-1$, where each φ_i is an ϵ-automorphism, each ν_i a geometric 0-isomorphism, and each γ^i an ϵ-elementary[8] automorphism. Moreover, $\varphi_1 = \varphi$ and $\varphi_m = \psi$. After replacing each φ_i by $1_{L''_1 \oplus \cdots \oplus L''_{i-1}} \oplus \varphi_i \oplus 1_{L'_i \oplus \cdots \oplus L'_{m-1}}$ (and adjusting each ν_i and γ^i correspondingly), we may assume that $L'_i = L''_i = 0$ for all i. This, of course, also changes φ and ψ, but the conclusion for the new version does imply that for the old one. Next, we can replace each (new) φ_i by $\nu_{[i]}\varphi_1\nu_{[i]}^{-1}$, where $\nu_{[i]} = \nu_{m-1}\nu_{m-2}\cdots\nu_i$, and each ν_i by 1. Again, this changes φ and ψ, but the conclusion for the new pair implies that for the old. We now have $\varphi_i \simeq_{3\epsilon} \varphi_{i+1}\gamma^i, i = 1, 2, \cdots m-1$, from which we conclude that

$$(25)\qquad \varphi_1 \oplus [\oplus_{i=2}^{m-1}(\varphi'_i \oplus \varphi_i)] \simeq_{3\epsilon} [\oplus_{i=2}^{m-1}(\varphi_i \oplus \varphi'_i) \oplus \varphi_m]\delta,$$

where φ'_i is an ϵ-inverse of φ_i and $\delta = \oplus_{i=1}^{m-2}(\gamma^i \oplus 1) \oplus \gamma^{m-1}$ is an ϵ-elementary automorphism. For any ϵ-morphism φ with ϵ-inverse φ', the product of the six factors in (23) is 3ϵ-homotopic to $\varphi \oplus \varphi'$. Therefore, there are twelve ϵ-elementary automorphisms δ^i such that

$$(26)\qquad 1_M \oplus [\oplus_{i=2}^{m-1}(\varphi'_i \oplus \varphi_i] \simeq_{3\epsilon} \prod_{i=1}^{6} \delta^i, \text{ and } [\oplus_{i=2}^{m-1}\varphi_i \oplus \varphi'_i] \oplus 1_N \simeq_{3\epsilon} \prod_{i=7}^{12} \delta^i.$$

The desired conclusion follows easily from (25) and (26). □

If $\epsilon \leq \delta$, then by (22), the inclusions $GM_\epsilon \subseteq GM_\delta$ give rise to "relax control" homomorphisms $K_1(B;p)_\epsilon \to K_1(B;p)_\delta$. We define the *controlled group* $K_1(B;p)_c$ to be the (inverse) limit of the resulting inverse system indexed on all $\epsilon > 0$. We also need the first derived, $\lim^1 K_1(B;p)_\epsilon$.

As for K_0, the individual groups $K_1(B;p)_\epsilon$ are *not* functorial, but the inverse system $\{K_1(B;p)_\epsilon\}_\epsilon$ and the groups $K_1(B;p)_c$ and $\lim^1 K_1(B;p)_\epsilon$ *are* functorial on $\mathcal{TOP}_c$.

[8] The wording of (20) would give the bound $3\epsilon/2$ only, but the easy proof of (20) does give ϵ here.

3.2 The reduced groups for $i = 0, 1$.

We recall the notion of a *geometric* morphism, defined in (3), and denote by $GK_i(B;p)_\epsilon$ the subgroup of $K_i(B;p)_\epsilon$ generated by all $[\alpha]$ where α is a geometric ϵ-idempotent with each $u_s \in \{0,1\}$ (if $i = 0$) or a geometric ϵ-automorphism with each $u_s \in \{\pm 1\}$ (if $i = 1$). The reduced groups $\tilde{K}_i(B;p)_\epsilon$, are defined to fit into the short exact sequences.

$$0 \to GK_i(B;p)_\epsilon \to K_i(B;p)_\epsilon \to \tilde{K}_i(B;p)_\epsilon \to 0, \tag{27}$$

$i = 0, 1$. For $i = 1$, we also write $Wh(B;p)_\epsilon$ for the reduced group. In order to study the behavior of the reduced groups under lim and $\lim^1$, we need two lemmas.

Lemma 3.3 *If two geometric modules are ϵ-isomorphic, then there is a geometric ϵ-isomorphism between them.*

Proof: One can easily adapt the proof of Theorem 2.6, pp. 140ff. of [AM1] to the present situation. □

Lemma 3.4 *For any geometric ϵ-isomorphism $\nu : M \to N$ and any $\delta > 0$, there exist an $m \in \mathbb{N}$; geometric modules $M_1 = M, M_2, \cdots, M_m = N$; and geometric δ-isomorphisms $\nu_i : M_i \to M_{i+1}, i = 1, 2, \cdots, m-1$, such that $\nu = \nu_{m-1} \cdots \nu_2 \nu_1$.*

Proof: One uses a straigthforward subdivision of paths (cf. Section 4 of [An]). It is, of course, essential that B is compact. □

Proposition 3.5 *The inverse system $\{GK_0(B;p)_\epsilon\}_\epsilon$ is stably isomorphic to the constant system $H_0(E,\mathbb{Z})$, where H_0 denotes singular homology. Consequently, one has the following natural isomorphisms and natural short exact sequence*

$$\lim^1 GK_0(B;p)_\epsilon = 0 \ , \ \lim^1 K_0(B;p)_\epsilon \cong \lim^1 \tilde{K}_0(B;p)_\epsilon,$$

$$0 \to H_0(E;\mathbb{Z}) \to K_0(B;p)_c \to \tilde{K}_0(B;p)_c \to 0.$$

The inverse system $\{GK_1(B;p)_\epsilon\}_\epsilon$ satisfies the Mittag-Leffler condition. Consequently, one has the following natural isomorphisms and natural short exact sequence.

$$\lim^1 GK_1(B;p)_\epsilon = 0 \ , \ \lim^1 K_1(B;p)_\epsilon \cong \lim^1 Wh(B;p)_\epsilon,$$

$$0 \to \lim GK_1(B;p)_\epsilon \to K_1(B;p)_c \to Wh(B;p)_c \to 0.$$

Proof: Any geometric ϵ-idempotent $\xi : M \to M$ is $\sim_\epsilon$-equivalent to a geometric ϵ-idempotent $\xi' : M' \to M'$ which is an ϵ-isomorphism (i.e., the corresponding α' is the identity). In fact, if $M = (S, \sigma)$, one can take $\xi' = \xi | M'$ where $M' = (S', \sigma | S')$ and $S' = \{s \in S | u_s = 1\}$. Thus we need only consider geometric ϵ-idempotents $\xi : M \to M$ which have ϵ-inverses (denoted ξ'). Then $\xi \simeq_{3\epsilon} \xi\xi\xi' \simeq_{3\epsilon} \xi\xi' \simeq_{2\epsilon} 1$, so $\xi \simeq_{3\epsilon} 1_M$. Hence, if two ϵ-invertible ϵ-idempotents, $\xi : M \to M$ and $\eta : N \to N$, have $\xi \sim_\epsilon \eta$ so that (10) and (11) hold, then ψ and φ are $\frac{3\epsilon}{2}$-inverses of one another. Then by Lemma 3.3, there is a geometric $\frac{3\epsilon}{2}$-isomorphism $M \oplus L \to N \oplus L$. Therefore, there is a homomorphism $\rho_\epsilon : GK_0(B;p)_\epsilon \to H_0(E,\mathbb{Z})$ mapping $[\xi]$ to $\Sigma n_\beta [e_\beta]$ where $M = (S, \sigma)$, $[e_\beta]$ is the canonical, zero dimensional homology class in the path component $E_\beta \subseteq E$, and n_β is the cardinality of $\sigma^{-1}(E_\beta)$. Conversely, if we fix the points $\{e_\beta\}$, there is an obvious homomorphism $\rho'_\epsilon : H_0(E;\mathbb{Z}) \to GK_0(B;p)_\epsilon$ with $\rho'_\epsilon([e_\beta]) = 1_{(T,\tau)}$ where T is a one point set which maps to e_β under τ. Clearly $\rho_* = \{\rho_\epsilon\}_\epsilon$ and $\rho'_* = \{\rho'_\epsilon\}_\epsilon$ are maps of the inverse systems involved and $\rho_*\rho'_* = 1$. On the other hand, it is easily seen that $\rho'_\epsilon\rho_\epsilon([1_M]) = [1_M]$ for any geometric module M. Since (as remarked above) any $[\xi] \in GK_0(B;p)_\epsilon$ maps to an element of the form $[1_M]$ when control is relaxed to 3ϵ, it follows that $\rho'_\epsilon\rho_\epsilon$ and 1 are coequalized by the map $GK_0(B;p)_\epsilon \to GK_0(B;p)_{3\epsilon}$. Thus ρ_* is a stable isomorphism of inverse systems. Therefore, $\{GK_0(B;p)_\epsilon\}_\epsilon$ satisfies the Mittag-Leffler condition, so its $\lim^1$ vanishes, and the long exact $\lim^*$-sequence induced by (27) proves the rest of the results claimed for K_0.

The second part of the proposition follows in the same way once we have established the Mittag-Leffler condition for the system $\{GK_1(B;p)_\epsilon\}$. To do so, it suffices to show that $[\nu] \in \mathrm{Im}[GK_1(B;p)_\delta \to GK_1(B;p)_{2\epsilon}]$ for every geometric ϵ-automorphism $\nu : M \to M$, every $\epsilon > 0$, and every positive $\delta < \epsilon$. By Lemma 3.4, we can write ν as a composition, $\nu = \nu_{m-1}\cdots\nu_2\nu_1$, where ν_i is a geometric δ-isomorphism $M_i \to M_{i+1}, i = 1, 2, \cdots, m-1$, and $M_1 = M_m = M$. It follows easily that each sub-composition $\nu_j\nu_{j-1}\cdots\nu_i$ with $1 \le i \le j \le m-1$ is a geometric 2ϵ-isomorphism. Therefore, it suffices to prove the following lemma.

Lemma 3.6 *Let $\varphi_i : M_i \to M_{i+1}$ be a δ-isomorphism for $i = 1, 2, \cdots m-1$. Assume that $\epsilon \ge \delta$ and that each composition $\varphi_{i,j} = \varphi_{j-1}\varphi_{j-2}\cdots\varphi_i : M_i \to M_j$ is a 2ϵ-isomorphism. Let $\varphi = \varphi_{1,m} : M_1 \to M_m = M_1$. Then the δ-automorphism Φ defined below has $[\varphi] = [\Phi] \in K_1(B;p)_{2\epsilon}$. In particular, $[\varphi] \in Im[K_1(B;p)_\delta \to K_1(B;p)_{2\epsilon}]$.*

Proof: Let

$$\Phi = \begin{pmatrix} 0 & \varphi_{m-1} \\ d & 0 \end{pmatrix} : [M_1 \oplus \cdots] \oplus M_{m-1} \to M_1 \oplus [\cdots \oplus M_{m-1}], \tag{28}$$

where d is the diagonal morphism

$$\mathrm{diag}(-\varphi_1, \cdots, -\varphi_{m-2}) : M_1 \oplus \cdots \oplus M_{m-2} \to M_2 \oplus \cdots \oplus M_{m-1}. \tag{29}$$

If each φ_i were an honest isomorphism (in some category), then a trivial exercise would show that Φ could be reduced to the form

$$\varphi \oplus 1 : M_1 \oplus [M_2 \oplus \cdots \oplus M_{m-1}] \to M_1 \oplus [M_2 \oplus \cdots \oplus M_{m-1}], \tag{30}$$

by elementary column operations, i.e., by right multiplication by elementary matrices. To do the actual proof, one has to write out such a reduction; to observe that each of the elementary matrices will represent a 2ϵ-elementary automorphism; and to check that the homotopies which occur because we deal with 2ϵ-isomorphisms rather than real isomorphisms are 4ϵ-homotopies. We leave the details to the reader. $\square$

3.3 Ranicki and Yamasaki's ϵ-controlled K-theory.

In [RY], two inverse systems of groups, $\{\tilde{K}_0(B, p, 0, \epsilon)\}_\epsilon$ and $\{Wh(B, p, 1, \epsilon)\}_\epsilon$, are introduced. In this section we compare these systems to those defined by us. The following proposition is essential for our treatment of the lower groups in the next section.

Proposition 3.7 *The inverse systems $\{\tilde{K}_0(B;p)_\epsilon\}_\epsilon$ and $\{Wh(B;p)_\epsilon\}_\epsilon$ are stably isomorphic to $\{\tilde{K}_0(B, p, 0, \epsilon)\}_\epsilon$ and $\{Wh(B, p, 1, \epsilon)\}_\epsilon$, respectively.*

Proof: We first note that our *geometric modules*, *ϵ-morphisms* and *ϵ-homotopies* coincide with the *geometric modules*, the *geometric morphisms of radius* ϵ and the *ϵ homotopies* in [RY][9]. In [RY], an *ϵ-projection* $\xi : M \to M$ is required to have bound ϵ and satisfy $\xi^2 \simeq_\epsilon \xi$ while we have

[9]Actually, here we are slightly inaccurate, as Ranicki and Yamasaki's geometric morphisms have $3(s,\omega,t) + 2(s,\omega,t) \neq 5(s,\omega,t)$. Since we always pass to some homotopy classes, the inaccuracy is OK.

required an ϵ-*idempotent* to have bound ϵ and satisfy $\xi^2 \simeq_{2\epsilon} \xi$. Thus, each ϵ-idempotent is a 2ϵ-projection. As such, it can be viewed as a 0-dimensional 2ϵ projective chain complex (M, ξ) in the sense of the Definition following Proposition 2.2 in [RY]. Moreover, if $\xi \sim_\epsilon \eta$ so that (10) and (11) hold, then ψ and φ show that (M, ξ) and (N, η) are 0-stable 2ϵ chain equivalent (in the sense of the definitions preceeding Proposition 3.1 in [RY]). Thus they represent the same element of $\tilde{K}_0(B, p, 0, 2\epsilon)$, so there is a homomorphism $\rho_\epsilon : K_0(B;p)_\epsilon \to \tilde{K}_0(B, p, 0, 2\epsilon)$ with $\rho_\epsilon[\xi] = [M, \xi]$. The collection $\{\rho_\epsilon\}_\epsilon$ is compatible with the relaxation of control, so one has a short exact sequence of inverse systems

$$0 \to \{\mathrm{Ker}\rho_\epsilon\}_\epsilon \to \{K_0(B;p)_\epsilon\}_\epsilon \to \{\mathrm{Im}\rho_\epsilon\}_\epsilon \to 0. \tag{31}$$

If $\xi : M \to M$ is a geometric ϵ-idempotent, then (cf. the proof of Proposition 3.5) ξ can be assumed $\simeq_{3\epsilon} 1_M$, so (M, ξ) represents 0 in $\tilde{K}_0(B, p, 0, 3\epsilon)$. Therefore,

$$\mathrm{Im}[GK_0(B;p)_\epsilon \subseteq K_0(B;p)_\epsilon \to K_0(B;p)_{\frac{3\epsilon}{2}}] \subseteq \mathrm{Ker}\rho_{\frac{3\epsilon}{2}}. \tag{32}$$

Conversely, if $\xi : M \to M$ has $[\xi] \in \mathrm{Ker}\rho_\epsilon$, then, by Proposition 3.1 of [RY], $(M, \xi) \oplus (E, 1)$ is 3ϵ chain equivalent to $(F, 1)$ for suitable modules E and F. As noticed in [RY] (immediately before Proposition 2.3), this implies that $(F, 1)$ and $(M \oplus E, \xi \oplus 1)$ are 3ϵ isomorphic. But then $[\xi] = [1_F] - [1_E] \in K_0(B;p)_{3\epsilon}$, so $[\xi] \in GK_0(B;p)_{3\epsilon}$, and we have shown that

$$\mathrm{Im}[\mathrm{Ker}\rho_\epsilon \subseteq K_0(B;p)_\epsilon \to K_0(B;p)_{3\epsilon}] \subseteq GK_0(B;p)_{3\epsilon}. \tag{33}$$

The equations (32) and (33) imply that $\{\mathrm{Ker}\rho_\epsilon\}_\epsilon$ and $\{GK_0(B;p)_\epsilon\}_\epsilon$ are stably isomorphic subsystems of $\{K_0(B;p)_\epsilon\}_\epsilon$. It follows that the inverse systems $\{\tilde{K}_0(B;p)_\epsilon\}_\epsilon$ and $\{\mathrm{Im}\rho_\epsilon\}_\epsilon$ are also stably isomorphic, and we can finish our study of the K_0-case by showing that $\{\mathrm{Im}\rho_\epsilon\}_\epsilon$ is stably isomorphic to $\{\tilde{K}_0(B, p, 0, \epsilon)\}_\epsilon$. A map from the former to the latter is given by the inclusions $\mathrm{Im}\rho_\epsilon \subseteq \tilde{K}_0(B, p, 0, 2\epsilon)$, and this is a stable isomorphism of inverse systems because of the inclusion $\mathrm{Im}[\tilde{K}_0(B, p, 0, 2\epsilon) \to \tilde{K}_0(B, p, 0, 4\epsilon)] \subseteq \mathrm{Im}\rho_{2\epsilon}$.

We next turn to the K_1 case. Any ϵ-automorphism $\varphi : M \to M$ can be viewed as a 1-dimensional, free ϵ chain complex, to be denoted $(\varphi : M \to M)$, in the sense of the Definition following Proposition 2.2 in [RY]. As such it is strongly[10] ϵ chain contractible, so it represents an element $[\varphi : M \to M] \in Wh(X, p, 1, \epsilon)$, cf. the definition preceding Proposition 4.1 in [RY]. If $\psi : N \to N$ is another ϵ-automorphism, and (19) holds, then the pair $(\gamma\nu, \nu)$ describes a 1-stable 2ϵ-simple[11] chain equivalence between $(\varphi : M \to M)$ and $(\psi : N \to N)$. Since the equivalence relation in the above mentioned definition in [RY] is generated by 1-stable 40ϵ-simple chain equivalence, there results a homomorphism $\rho_\epsilon : K_1(B;p)_\epsilon \to Wh(B, p, 1, \epsilon)$ given by $\rho_\epsilon([\varphi]) = [\varphi : M \to M]$, and a short exact sequence of inverse systems

$$0 \to \{\mathrm{Ker}\rho_\epsilon\}_\epsilon \to \{K_1(B;p)_\epsilon\}_\epsilon \to \{\mathrm{Im}\rho_\epsilon\}_\epsilon \to 0. \tag{34}$$

If $\varphi : M \to M$ is an ϵ-automorphism with $\rho_\epsilon[\varphi] = 0$, then by Proposition 4.1 of [RY], the ϵ chain complex $(\varphi : M \to M)$ is 1-stable 86ϵ-simple chain equivalent to the 0 complex, i.e. there are free modules T and T' and two 86ϵ deformations, say $D_1 = f_m, f_{m-1}, \cdots, f_1$ and $D_0 = g_n, g_{n-1}, \cdots, g_1$, such that the compositions $f = f_m f_{m-1} \cdots f_1$ and $g = g_n g_{n-1} \cdots g_1$ and their inverses define 86ϵ chain maps $(f, g) : (\varphi \oplus 1 : M \oplus T \to M \oplus T) \to (1 : T' \to T')$, respectively, $(f^{-1}, g^{-1}) : (1 : T' \to T') \to (\varphi \oplus 1 : M \oplus T \to M \oplus T)$. It follows that $\varphi \oplus 1_T \simeq_{172\epsilon}$

[10] Actually , "strongly" is vacuous for contractions of 1-dimensional complexes.

[11] We get 2ϵ rather than ϵ, because [RY] insists that an ϵ chain map between free ϵ chain complexes commutes with the boundary maps up to ϵ homotopy.

$g_1^{-1} \cdots g_m^{-1} f_m \cdots f_1$. Since each of the factors is either a 172ϵ-elementary automorphism or a geometric 172ϵ-isomorphism, it follows from Lemma 3.6 that

$$\text{(35)} \qquad \mathrm{Im}[\mathrm{Ker}\rho_\epsilon \subseteq K_1(B;p)_\epsilon \to K_1(B;p)_{344\epsilon}] \subseteq GK_1(B;p)_{344\epsilon},$$

Conversely, it is easily seen that any geometric ϵ-isomorphism $\xi : M \to M$ when viewed as chain complex $\xi : M \to M$ is 1-stable ϵ-simple equivalent to the 0 chain complex. Therefore,

$$\text{(36)} \qquad GK_0(B;p)_\epsilon \subseteq \mathrm{Ker}\rho_\epsilon$$

The equations (36) and (35) imply that $\{\mathrm{Ker}\rho_\epsilon\}_\epsilon$ and $\{GK_1(B;p)\}_\epsilon$ are stably isomorphic subsystems of $\{K_1(B;p)\}_\epsilon$.

As in the proof of the K_0 case, it now suffices to show that the system of inclusions $\{\mathrm{Im}\rho_\epsilon\}_\epsilon \subseteq \{Wh(B,p,1,\epsilon)\}_\epsilon$ is a stable isomorphism. To that end, let $[\delta : C_1 \to C_0]$ be a typical generator for $Wh(B,p,1,\epsilon)$, i.e., $\delta : C_1 \to C_0$ is a free ϵ chain complex which is ϵ chain contractible. Since an ϵ chain contraction is precisely the same as an ϵ-inverse for δ, Lemma 3.3 supplies a geometric ϵ-isomorphism $\nu : C_1 \to C_0$. Since $[\delta : C_1 \to C_0] = [\delta\nu^{-1} : C_0 \to C_0] \in Wh(B,p,1,2\epsilon)$, it follows that

$$\mathrm{Im}[Wh(B,p,1,\epsilon) \to Wh(B,p,1,\epsilon] \subseteq \mathrm{Im}\rho_{2\epsilon},$$

and the proof is complete.

3.4 Lower controlled K-theory.

In the Appendix of [RY], Ranicki and Yamasaki introduce the *lower, reduced* K-theory groups $\tilde{K}_{1-j}(B,p,n-1,\epsilon)$ and $Wh_{2-j}(B,p,n,\epsilon)$ for all $\epsilon > 0$ and all $j,n \in \mathbb{N}$. They show that the inverse systems $\{Wh_{1-j}(B,p,n,\epsilon)\}_\epsilon$ and $\{\tilde{K}_{1-j}(B,p,n,\epsilon)\}_\epsilon$ are stably isomorphic for all $j,n \in \mathbb{N}$. Moreover, they (essentially) give the following inductive characterization of the lower groups in terms of the maps induced by the standard inclusion $i : (B;p) \to (B \times S^1; p \times id_{S^1})$.

Proposition 3.8 *For each $j \in \mathbb{N}$ and each $n \in \mathbb{N}$, there is a natural stable isomorphism of inverse systems*

$$\{Coker[i_* : \tilde{K}_{2-j}(B,p,n,\epsilon) \to \tilde{K}_{2-j}(B,p,n,\epsilon)]\}_\epsilon \to \{\tilde{K}_{1-j}(B,p,n,\epsilon)\}_\epsilon.$$

Proof: (Sketch) Of course, $\tilde{K}_{2-j}$ has to be interpreted as Wh when $j = 1$. In that case, the result then follows from the stably exact sequence

$$0 \to Wh(B,p,n,\epsilon) \to Wh(B \times S^1, p \times id, n, \epsilon) \to \tilde{K}_0(B,p,n,\epsilon) \to 0,$$

which is established immediately after Corollary 7.3 in [RY] as a formal consequence of the Mayer-Vietoris sequence (Theorem 6.2 of [RY]) for the triad $B \times (S^1; S^1_+, S^1_-)$ and Lemma 7.2 of [RY]. In the Appendix of [RY] it is shown that Theorem 6.2 generalizes to lower, reduced K-theory, and it is easy to check a generalized version of Lemma 7.2. Therefore, the whole proof goes through for all $j \in \mathbb{N}$. □

Definition 3.9 *For any $j \in \mathbb{N}$, let $K_{-j}(B;p)_\epsilon$ be defined to coincide with the group $\tilde{K}_{-j}(B,p,1,\epsilon)$ of [RY].*

Since Ranicki and Yamasaki's groups are independent of the choise of n up to stable isomorphism (see [RY], immediately before Corollary 3.5 and immediately before Proposition 4.8), and since the inclusion $i : (B;p) \to (B \times S^1, p \times id)$ is easily seen to induce a stable isomorphism on the inverse systems $\{GK_0(-;-)_\epsilon\}_\epsilon$, it is not hard to derive the following inductive characterization of the inverse systems of unreduced groups.

Proposition 3.10 *For each $j \in \mathbb{N}$, there is a natural stable isomorphism of inverse systems*

$$\{Coker[i_* : K_{1-j}(B;p)_\epsilon) \to K_{1-j}(B \times S^1; p \times id)_\epsilon)]\}_\epsilon \to \{K_{-j}(B;p)_\epsilon)\}_\epsilon.$$

The morphisms i_* in the above propositions have natural splittings induced by the projection $(B \times S^1; p \times id) \to (B;p)$. Therefore, one gets the following corollary, where naturality refers to morphisms $(B;p) \to (B';p')$ in the sense of (8). Note that the first of the three sequences involves degrees[12] $1-j<1$ while the other two hold in degrees $2-j \le 1$. Also note that the third sequence coincides with its own reduced version because of Proposition 3.5.

Corollary 3.11 *For any $j \in \mathbb{N}$, there are natural, naturally split, exact sequences*

$$0 \to K_{1-j}(B;p)_c \xrightarrow{i_*} K_{1-j}(B \times S^1; p \times 1)_c \to K_{-j}(B;p)_c \to 0,$$

$$0 \to \tilde{K}_{2-j}(B;p)_c \xrightarrow{i_*} \tilde{K}_{2-j}(B \times S^1; p \times 1)_c \to \tilde{K}_{1-j}(B;p)_c \to 0,$$

and

$$0 \to \lim{}^1 K_{2-j}(B;p)_\epsilon \xrightarrow{i_*} \lim{}^1 K_{2-j}(B \times S^1; p \times 1)_\epsilon \to \lim{}^1 K_{1-j}(B;p)_\epsilon \to 0.$$

This corollary will be used in Section 5 in order to extend the basic exact sequence to lower K-theory.

4 K-theory with continuous control.

In this section we state some definitions and results from [AM2]. Slight changes may be needed once [AM2] is finished. The category $\mathcal{TOP}^{cc}$ of *continuously controlled spaces* has as its objects all triples $(\overline{X}, B; p)$ where $(\overline{X}, B)$ is a pair of locally compact, Hausdorff spaces with B compact, and $p : E \to X$ is a continuous map from some space E into $X = \overline{X} - B$. We give $\overline{E} = E \coprod B$ the smallest topology for which $E \subseteq \overline{E}$ is an open embedding, and $\overline{p} = p \coprod 1_B : \overline{E} \to \overline{X}$ is continuous. A morphism $f : (\overline{X}, B; p) \to (\overline{X'}, B'; p')$ is then a continuous map $f : E \to E'$ which admits a continuous extension $\overline{f} : (\overline{E}, B) \to (\overline{E'}, B')$ and for which the relation $r = p' f p^{-1} : X \rightharpoonup X'$ is relatively proper[13].

We associate to each $(\overline{X}, B; p)$ an additive subcategory $GM^{cc}(\overline{X}, B; p) \subseteq GM(E)$ (and a corresponding additive homotopy category $\mathcal{GM}^{cc}(\overline{X}, B; p)$) by imposing the following conditions on objects, morphisms and homotopies in $GM(E)$. Each object (S, σ) must have $p\sigma : S \to B$ proper. For each morphism φ, the family $\mathcal{F}(\varphi)$ defined in (6) must have $p_*\mathcal{F}(\varphi)$ *cc* (in the sense of [AM2]). Similarly, $p_*\mathcal{F}(\Phi)$ must be *cc* for each homotopy Φ. If φ is a morphism in $GM^{cc}(\overline{X}, B; p)$, then its image in $\mathcal{GM}^{cc}(\overline{X}, B; p)$ will be denoted $cls(\varphi)$.

The groups $K_*^{cc}(\overline{X}, B; p)$ are defined to be the algebraic K-theoretic groups of the additive category $\mathcal{GM}^{cc}(\overline{X}, B; p)$ in the sense of [PW], i.e., including the lower K-theory. The notation indicates that we view them functors of $(\overline{X}, B)$ "with coefficients in the map p". If $E = X$ and $p = 1_X$, then we write[14] $K_*^{cc}(\overline{X}, B)$ instead of $K_*^{cc}(\overline{X}, B; p)$ and similarly for GM^{cc} and $\mathcal{GM}^{cc}$. There are also reduced versions of these groups, $\tilde{K}^{cc}(\overline{X}, B; p)$, defined for all $j \in \mathbb{N}$ and coinciding with the unreduced groups for $j \ge 3$.

The following results are proved in [AM2]. In the first one, we use the standard inclusion $i : E \times (-\infty, 0) \to E \times S^1 \times (-\infty, 0)$ and the functor $(B;p) \mapsto (B \times S^1; p \times 1)$ from $\mathcal{TOP}_c$ to $\mathcal{TOP}^{cc}$. Naturality refers to morphisms $(B;p) \to (B';p')$ in $\mathcal{TOP}_c$ as in (8).

[12]Actually, one *can* analyze the map $i_* : GK_1(B;p) \to GK_1(B \times S^1; p \times id)$ and prove exactness of the first sequence also for $1-j=1$, but we do not need this fact here.

[13]i.e., if $K' \subseteq X'$ is compact, then $r^{-1}(K')$ is relatively compact in X. If $\overline{X}$ is compact, then the relative propriety of r is implied by the continuity of the extension $\bar{f}$. Therefore, this condition is not present in [ACFP].

[14]cf. the functor $\mathcal{TOP}^2 \to \mathcal{TOP}^{cc}$ defined at the beginning of Section 6.

Proposition 4.1 *For each $i \in \mathbb{Z}$, there is a natural, naturally split, exact sequence*

$$\begin{array}{rl} 0 \quad \to & K_i^{cc}(B \times (-\infty, 0], B \times 0; p \times id) \\ \xrightarrow{i_*} & K_i^{cc}(B \times S^1 \times (-\infty, 0], B \times S^1 \times 0; p \times id \times id) \\ \to & K_{i-1}^{cc}(B \times (-\infty, 0], B \times 0; p \times id) \to 0. \end{array}$$

A Karoubi fibration of additive categories is a sequence (of additive categories and additive functors) $\mathcal{A} \hookrightarrow \mathcal{U} \to \mathcal{U}/\mathcal{A}$ where $\hookrightarrow$ is full and faithful, $\mathcal{U}$ is $\mathcal{A}$-filtered (in the sense of [Ka]), and $\mathcal{U}/\mathcal{A}$ is the (Karoubi) quotient. The second result from [AM2] which is needed here is the following generalization of Corollary 5.7 of [PW]. We note that with the definition of K-theory given in [AM2], cf. also [CP], $K_0(\mathcal{A})$ is Grothendieck group of the *idempotent completion* of $\mathcal{A}$. This is the "price" one has to pay to make idempotent completions disappear in the following theorem (and various other results).

Proposition 4.2 *Any Karoubi fibration gives rise to a long exact sequence in K-theory*

$$\cdots \to K_i(\mathcal{A}) \to K_i(\mathcal{U}) \to K_i(\mathcal{U}/\mathcal{A}) \to K_{i-1}(\mathcal{A}) \to \cdots \ \textit{for } i \in \mathbb{Z}.$$

This sequence is natural with respect to functors between Karoubi fibrations.

For any space E, let $c_E : E \to c$ denote the unique map to the one point space c. Then for each object $(\overline{X}, B; p)$ as above, 1_E is a morphism $(c, \emptyset; c_E) \to (\overline{X}, B; p)$, and the induced functor $\mathcal{GM}^{cc}(c, \emptyset; c_E)$ is full and faithful. Indeed, $\mathcal{GM}^{cc}(\overline{X}, B; p)$ is $\mathcal{GM}^{cc}(c, \emptyset; c_E)$-filtered. We denote the Karoubi quotient by $\mathcal{GM}^{cc,\infty}(\overline{X}, B; p)$ and its K-theory by $K^{cc,\infty}(\overline{X}, B; p)$. The category $\mathcal{GM}^{cc}(c, \emptyset; c_E)$ consists of all finite geometric modules in E and all unrestricted-homotopy classes of path matrix morphisms betwen them. Thus its K-theory is simply $K_*(E)$, and the long exact sequence takes the form

$$\cdots \to K_i(E) \to K_i^{cc}(\overline{X}, B; p) \to K_i^{cc,\infty}(\overline{X}, B; p) \to K_{i-1}(E) \to \cdots . \tag{37}$$

5 The basic exact sequence.

This section is devoted to the proof of our first main theorem. In it, $p : E \to B$ is a map from some topological space E to a compact metric space B, and naturality refers to morphisms $(B; p) \to (B'; p')$ as in(8.

Theorem 5.1 *For each $j \in \mathbb{N}$, there is a natural short exact sequence*

$$0 \to \lim{}^1 K_{2-j}(B; p)_\epsilon \to K_{2-j}^{cc}(B \times (-\infty, 0], B \times 0; p \times id) \to K_{1-j}(B; p)_c \to 0.$$

Remark: This is the unreduced version of Theorem 1.1. We leave it to the reader to deduce the reduced version. □

Proof: We first note that it suffices to prove this for $j = 1$. In fact, the inclusion $i : \{*\} \to S^1$ induces a split monomorphism from the exact sequence for $j = 1$ and $(B; p)$ to that for $j = 1$ and $(B \times S^1; p \times id)$. In view of Corollary 3.11 and Proposition 4.1, the resulting exact sequence of cokernels gives the desired exact sequence for $j = 2$ and $(B; p)$. One proceeds in the same way (inductively) to get the sequence for all $j \in \mathbb{N}$.

Before we turn to the proof for $j = 1$, we introduce some abbreviations and fix some notation.

$$\begin{array}{rclcrcl} GM_\epsilon & = & GM_\epsilon(B;p) & \text{and} & \mathcal{GM}_\epsilon & = & \mathcal{GM}_\epsilon(B;p); \\ \lim^1 K_1 & = & \lim^1 K_1(B;p)_\epsilon & \text{and} & \lim K_0 & = & \lim K_0(B;p)_\epsilon; \\ GM^{cc} & = & GM^{cc}(B\times(-\infty,0], B\times 0; p\times id); \\ \mathcal{GM}^{cc} & = & \mathcal{GM}^{cc}(B\times(-\infty,0], B\times 0; p\times id); \\ K_1^{cc} & = & K_1^{cc}(B\times(-\infty,0], B\times 0; p\times id). \end{array}$$

If $M = (S,\sigma)$, where $\sigma : S \to E \times (-\infty,0)$, is a geometric module on $E\times(-\infty,0)$, and $U \subseteq B\times(-\infty,0]$, we denote by $j_U : M|U \to M$ the inclusion of the direct summand $M|U = (\sigma^{-1}(p\times id)^{-1}(U), \sigma|)$ into M; by $q_U : M \to M|U$ the corresponding projection; and by π_U the idempotent $\pi_U = j_U q_U : M \to M$. Note that j_U, q_U, π_U are considered morphisms in GM rather than $\mathcal{GM}$.

The necessary connection between geometric modules on $E\times(-\infty,0)$ and E will be given by the homotopy preserving functors

$$\Sigma : GM(E\times(-\infty,0)) \to GM(E), \tag{38}$$

$$i_{n*} : GM(E) \to GM(E\times(-\infty,0)),\ n\in\mathbb{Z}. \tag{39}$$

The ("summation") functor Σ is induced by the projection $E\times(-\infty,0)\to E$, and $i_n : E \to E\times(-\infty,0)$ has $i_n(e) = (e,-2^{-n})$. In general, if $\varphi : M \to N$ in GM^{cc}, then $\Sigma(\varphi) : \Sigma(M)\to\Sigma(N)$ will not be in GM_ϵ, even for $\epsilon = \infty$ (in fact, $\Sigma(M)$ will not be finite). However, for judiciously chosen subsets $U, V \subseteq B\times(-\infty,0)$ and a suitable ϵ, one may have $\Sigma(q_V\varphi j_U) : \Sigma(M|U) \to \Sigma(N|V)$ in GM_ϵ. On the other hand, i_{n*} does map GM_ϵ into GM^{cc}, but we will often have to consider infinite direct sums $\oplus_n i_{n*}$ in which case one has to be careful that the sums make sense in GM^{cc}.

The composite functors satisfy

$$\Sigma i_{n*} = 1_{GM(E)} \text{ and } i_{n*}\Sigma \simeq 1_{GM(E\times(-\infty,0))}, \tag{40}$$

where the natural homotopy consists of paths in the $(-\infty,0)$-direction. Such paths also define natural transformations

$$\nu_n : i_{n*} \to i_{(n+1)*} \text{ and } \mu_n : i_{(n+1)*} \to i_{n*}, \tag{41}$$

and the composite transformations are homotopy equivalent to the identity transformations, i.e.,

$$\nu_n\mu_n \simeq 1 \text{ and } \mu_n\nu_n \simeq 1. \tag{42}$$

Also in this case, when infinite direct sums of the transformations ν_n or μ_n are formed, attention has to be paid to the cc condition, and the homotopy equivalence from $i_{n*}\Sigma$ to the identity is in GM^{cc} only for suitable objects.

In our dealings with lim and $\lim^1$ we shall actually work over the cofinal subset $\{2^{-n}|n\in\mathbb{Z}\}$, so that an element $x \in \lim^1 K_1$ is represented by a sequence $\{[\varphi_n]\}_n \in \prod_n K_1(B;p)_{2^{-n}}$ where $\varphi_n : M_n \to M_n$ in $GM_{2^{-n}}$, $n\in\mathbb{Z}$, and there exist $\psi_n : M_n \to M_n$ in $GM_{2^{-n}}$ and homotopies $\Phi_n : \varphi_n\psi_n \simeq 1$, $\Psi_n : \psi_n\varphi_n \simeq 1$ in $GM_{2^{1-n}}$. It is easily seen that

$$\varphi_\oplus = \bigoplus_{n\in\mathbb{Z}} i_{n*}(\varphi_n) : M_\oplus = \bigoplus_{n\in\mathbb{Z}} i_{n*}(M_n) \to M_\oplus \tag{43}$$

defines a morphism in GM^{cc}. Similarly, one has $\psi_\oplus$ and homotopies $\Phi_\oplus : \varphi_\oplus\psi_\oplus \simeq 1$, $\Psi_\oplus : \psi_\oplus\varphi_\oplus \simeq 1$ in GM^{cc}. Thus, $\varphi_\oplus$ represents an automorphism $cls(\varphi_\oplus) \in \mathcal{GM}^{cc}$, and we claim that

$$i(x) = [cls(\varphi_\oplus)] \text{ defines a monomorphism } i : \lim^1 \to K_1^{cc}. \tag{44}$$

First note *that* $cls(\varphi_\oplus) \in \mathcal{GM}^{cc}$ is unchanged if the sequence φ_n is replaced by a sequence φ'_n with $\varphi_n \simeq \varphi'_n$ in $GM_{2^{-n}}$; *that* $cls(\varphi_\oplus)$ is replaced by $cls(\varphi_\oplus)cls(\gamma_\oplus)$ with $cls(\gamma_\oplus)$ elementary in $\mathcal{GM}^{cc}$, if the sequence φ_n is replaced by a sequence $\varphi_n\gamma_n$ $[\in GM_{2^{1-n}}]$ with γ_n elementary in $GM_{2^{-n}}$; *that* a stabilization of each φ_n (to $\varphi_n \oplus 1_{N_n}$) leads to a stabilization of $\varphi_\oplus$ (to $\varphi_\oplus \oplus 1_{N_\oplus}$); and *that* the operation preserves (direct) sums. Therefore, $\{[\varphi_n]\}_n \to [cls(\varphi_\oplus)]$ defines a homomorphism $\prod_n K_1(B;p)_{2^{-n}} \to K_1^{cc}$.

We must show that this homomorphism factors over $\lim^1 K_1$, so we suppose that $\{[\varphi_n]\}_n$ represents 0 in $\lim^1 K_1$. This means that we have an element $\{[\sigma_n]\}_n \in \prod_n K_1(B;p)_{2^{-n}}$ so that $[\varphi_n] = [\sigma_n] - [\sigma_{n+1}] \in K_1(B;p)_{2^{-n}}$, $n \in \mathbb{Z}$. Let $\sigma_n : N_n \to N_n$ have θ_n as a 2^{-n}-inverse. Then $-[\sigma_{n+1}] = [\theta_{n+1}]$, so we may assume that $M_n = N_n \oplus N_{n+1}$, and $\varphi_n = \sigma_n \oplus \theta_{n+1} : N_n \oplus N_{n+1} \to N_n \oplus N_{n+1}$, $n \in \mathbb{Z}$. We note that $M_\oplus \neq N_\oplus \oplus N_\oplus$, but the natural transformations ν_n, μ_n of (41) give rise to an isomorphism (in $\mathcal{GM}^{cc}$) $M_\oplus \to N_\oplus \oplus N_\oplus$ which conjugates $cls(\varphi_\oplus)$ into $cls(\sigma_\oplus) \oplus cls(\theta_\oplus)$. Since $cls(\sigma_\oplus)$ and $cls(\theta_\oplus)$ are inverses in $\mathcal{GM}^{cc}$, it follows that $[cls(\varphi_\oplus)] = 0 \in K_1^{cc}$, so we do have the desired factorization.

To show that i is monic, we need the following lemma, the straightforward proof of which will be left to the reader.

Lemma 5.2 *Let $\mathcal{F}$ be a cc family of subsets of $B \times (-\infty, 0]$ and let $U_0 = B \times (-\infty, 0]$. There is a strictly increasing function $\lambda : \mathbb{N} \to \mathbb{Z}$ such that any $F \in \mathcal{F}$ which meets $U_n = B \times [-2^{-\lambda(n)}, 0]$ is completely contained in U_{n-1}, and has diameter $diam(F) \leq 2^{-n}$, $n \in \mathbb{N}$.*

If this lemma is applied to $\mathcal{F}(\varphi)$ for some morphism $\varphi : M \to N$, then for any $n < m \in \mathbb{N}\cup\infty$, φ maps $M|(U_n - U_m)$ into $N|(U_{n-1} - U_{m+1})$, and the resulting morphism $\Sigma(M|(U_n - U_m)) \to \Sigma(N|(U_{n-1} - U_{m+1}))$, has bound 2^{-n}. A similar remark holds for homotopies.

Returning to the proof of the injectivity of i, we assume that $x = \{[\varphi_n]\}_n$ has $[cls(\varphi_\oplus)] = 0 \in K_1^{cc}$. After a stabilization (which can easily be absorbed into the given morphisms φ_n), then in GM^{cc} we have thirteen[15] elementary automorphisms $\gamma^i : M_\oplus \to M_\oplus$, $i = 1, 2, \cdots, 13$ and a homotopy $\Gamma : \varphi_\oplus \simeq \gamma = \gamma^1\gamma^2\cdots\gamma^{13}$. We apply Lemma 5.2 to $\mathcal{F} = \mathcal{F}(\Gamma) \cup \bigcup_i \mathcal{F}(\gamma^i)$ to determine $\lambda : \mathbb{N} \to \mathbb{Z}$ and the neighborhoods U_n of B, $n \in \mathbb{N}$.

For each pair $(n_1, n_2) \in (\mathbb{N} \cup \{\infty\})^2$ with $n_1 < n_2$, let $\mathbf{n} = [n_1, n_2)$ be the corresponding half open interval. Let $M_{\mathbf{n}} = M_\oplus|(U_{n_1} - U_{n_2})$ with *complement* $M_{\mathbf{n}}^\perp$ $(= M_\oplus|(U_0 - U_{n_1}) \oplus M_\oplus|U_{n_2})$; *inclusion* $j_{\mathbf{n}} : M_{\mathbf{n}} \to M_\oplus$; *projection* $q_{\mathbf{n}} : M_\oplus \to M_{\mathbf{n}}$; and *idempotent* $\pi_{\mathbf{n}} = j_{\mathbf{n}}q_{\mathbf{n}} : M_\oplus \to M_\oplus$. Also let $\varphi_{\mathbf{n}} = q_{\mathbf{n}}\varphi_\oplus j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$ (so $\Sigma\varphi_{\mathbf{n}} = \oplus_k \varphi_k$ with $\lambda(n_1) \leq k < \lambda(n_2)$). Finally, for the above $\mathbf{n} = [n_1, n_2)$, let

$$\mathbf{n}'' = [n_1, n_2 + 1) \ ; \ \mathbf{n}' = [n_1 - 1, n_2), \text{ if } n_1 > 1; \text{ and} \tag{45}$$

$$\mathbf{n} + k\mathbf{e} = [n_1 - k, n_2 + k), \text{ if } k \in \mathbb{Z} \text{ and } 0 < n_1 - k < n_2 + k. \tag{46}$$

For any $\mathbf{n}$ there is the elementary automorphism (of bound 2^{-n_1}) $\gamma^i_{\mathbf{n}} = 1 + (\gamma^i - 1)\pi_{\mathbf{n}} : M_\oplus \to M_\oplus$. It maps $M_{\mathbf{n}+\mathbf{e}}$ into itself, is the identity on $M^\perp_{\mathbf{n}+\mathbf{e}}$, and agrees with the restriction of γ^i on the submodule $M_{\mathbf{n}} \subseteq M_\oplus$. We think of $\gamma^i_{\mathbf{n}}$ as a "concentration of γ^i on $M_{\mathbf{n}}$" although this is not literally true. If $n_2 > n_1 + 26$, then the composition

$$\gamma_{\mathbf{n}} = \gamma^1_{\mathbf{n}-\mathbf{e}}\gamma^2_{\mathbf{n}-2\mathbf{e}}\cdots\gamma^{13}_{\mathbf{n}-13\mathbf{e}} : M_\oplus \to M_\oplus \tag{47}$$

has bound $2^{-n_1-13}(1+2+4+\cdots+2^{12}) < 2^{-n_1}$. Moreover, each factor maps $M_{\mathbf{n}}$ into itself and is the identity on $M^\perp_{\mathbf{n}}$. Hence $q_{\mathbf{n}}\gamma_{\mathbf{n}}j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$ is a product of the 13 elementary automorphisms $q_{\mathbf{n}}\gamma^i_{\mathbf{n}-i\mathbf{e}}j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$, so

$$[\Sigma(q_{\mathbf{n}}\gamma_{\mathbf{n}}j_{\mathbf{n}})] = 0 \in K_1(B;p)_{2^{-n_1}}, \text{ if } n_2 > n_1 + 26. \tag{48}$$

[15] In [AM1] (see pp. 145-46), two of the present authors observed that 16 elementary automorphisms will do. The reduction to 13 factors can be found in [FP], or by contemplating the proof of Proposition 3.2.

We now assume that $n_2 - n_1 \geq 39$, and write $M_{\mathbf{n}} = M_{\mathbf{l}} \oplus M_{\mathbf{m}} \oplus M_{\mathbf{r}}$ where $\mathbf{l} = [n_1, n_1 + 13)$, $\mathbf{m} = \mathbf{n} - 13\mathbf{e}$, and $\mathbf{r} = [n_2 - 13, n_2)$. The indexing is supposed to convey the **l**eft hand, the **m**iddle, and the **r**ight hand summand of $M_{\mathbf{n}}$, respectively. We note that $M_{\mathbf{r}} = 0$ when $n_2 = \infty$. Since the length of the middle interval $\mathbf{m}$ is at least 13, $\gamma_{\mathbf{n}}$ maps the left summand $M_{\mathbf{l}}$ into $M_{\mathbf{l}} \oplus M_{\mathbf{m}}$ and the right summand $M_{\mathbf{r}}$ into $M_{\mathbf{m}} \oplus M_{\mathbf{r}}$. Also, $\gamma_{\mathbf{n}}$ coincides with γ on $M_{\mathbf{m}}$, so

$$q_{\mathbf{n}}\gamma_{\mathbf{n}}j_{\mathbf{m}} = q_{\mathbf{n}}\gamma j_{\mathbf{m}} \simeq q_{\mathbf{n}}\varphi_{\oplus}j_{\mathbf{m}} : M_{\mathbf{m}} \to M_{\mathbf{n}}$$

(the homotopy is the reverse of $q_{\mathbf{n}}\Gamma j_{\mathbf{m}}$ and has bound 2^{-n_1}). In terms of matrices w.r.t. the decomposition $M_{\mathbf{n}} = M_{\mathbf{l}} \oplus M_{\mathbf{m}} \oplus M_{\mathbf{r}}$ this gives us the equation

$$(49) \qquad q_{\mathbf{n}}\gamma_{\mathbf{n}}j_{\mathbf{n}} = \begin{pmatrix} \gamma_{\mathbf{n}}^{(l)} & q_{\mathbf{l}}\gamma_{\mathbf{n}}j_{\mathbf{m}} & 0 \\ * & q_{\mathbf{m}}\gamma_{\mathbf{n}}j_{\mathbf{m}} & * \\ 0 & q_{\mathbf{r}}\gamma_{\mathbf{n}}j_{\mathbf{m}} & \gamma_{\mathbf{n}}^{(r)} \end{pmatrix} \simeq \begin{pmatrix} \gamma_{\mathbf{n}}^{(l)} & 0 & 0 \\ * & \varphi_{\mathbf{m}} & * \\ 0 & 0 & \gamma_{\mathbf{n}}^{(r)} \end{pmatrix}$$

in $GM(E \times (-\infty, 0); p \times id)_{2^{-n_1}}$ (note that the center columns represent $q_{\mathbf{n}}\gamma j_{\mathbf{m}}$ and $q_{\mathbf{n}}\varphi_{\oplus}j_{\mathbf{m}}$, respectively, and that the equation *defines* $\gamma_{\mathbf{n}}^{(l)}$ and $\gamma_{\mathbf{n}}^{(r)}$).

Since the left hand side and $\varphi_{\mathbf{m}}$ are 2^{-n_1}-isomorphisms, it easily follows that $\gamma_{\mathbf{n}}^{(l)}$ and $\gamma_{\mathbf{n}}^{(r)}$ are 2^{2-n_1}-isomorphisms. Clearly, (48) and (49) imply that[16]

$$(50) \qquad [\Sigma\gamma_{\mathbf{n}}^{(l)}] + [\Sigma\varphi_{\mathbf{m}}] + [\Sigma\gamma_{\mathbf{n}}^{(r)}] = 0 \in K_1(B; p)_{2^{3-n_1}}, \text{ if } 39 \leq n_2 - n_1 < \infty.$$

It is easily seen that the upper left hand corner of the above matrix is unchanged if one augments n_2, keeping n_1 fixed, i.e., $\gamma_{\mathbf{n}}^{(l)} = \gamma_{\mathbf{n}''}^{(l)}$ (and, similarly, $\gamma_{\mathbf{n}}^{(r)} = \gamma_{\mathbf{n}'}^{(r)}$). Hence the definition $b_{n_1-3} = [\Sigma(\gamma_{\mathbf{n}}^{(l)})] \in K_1(B; p)_{2^{3-n_1}}$ gives an element which is independent of the choice of $n_2 \in [n_1 + 39, \infty)$. For any $n \in \mathbb{N}$, we can apply (50) to $\mathbf{n} = [n + 4, n + 43)$ and $\mathbf{n}' = [n + 3, n + 43)$ to get

$$\begin{aligned} b_{n+1} + \sum_{k=\lambda(n+17)}^{\lambda(n+30)-1} [\varphi_k] + [\Sigma(\gamma_{\mathbf{n}}^{(r)})] &= 0 \in K_1(B; p)_{2^{-n-1}}, \text{ and} \\ b_n + \sum_{k=\lambda(n+16)}^{\lambda(n+30)-1} [\varphi_k] + [\Sigma(\gamma_{\mathbf{n}'}^{(r)})] &= 0 \in K_1(B; p)_{2^{-n}}. \end{aligned}$$

If we let $c_n = -b_n + \sum_{k=n}^{\lambda(n+16)-1} [\varphi_k] \in K_1(B; p)_{2^{-n}}$, then the above formulae immediately imply that $c_n - r_{n,n+1}(c_{n+1}) = [\varphi_n] \in K_1(B; p)_{2^{-n}}$ for all $n \in \mathbb{N}$. This shows that $x = \{[\varphi_k]\}_k = 0 \in \lim^1 K_1$ and finishes the proof of the injectivity of i.

We go on to define the map $\partial : K_1^{cc} \to \lim K_0$. Let $y = [cls\varphi] \in K_1^{cc}$. Thus $\varphi : M \to M$ in GM^{cc} and there exist $\psi : M \to M$ and homotopies $\Phi : \varphi\psi \simeq 1$ and $\Psi : \psi\varphi \simeq 1$ in GM^{cc}. We use Lemma 5.2 with $\mathcal{F} = \mathcal{F}(\varphi) \cup \mathcal{F}(\psi) \cup \mathcal{F}(\Phi) \cup \mathcal{F}(\Psi)$ and continue to denote the resulting neighborhoods by U_n. Also, the notation $j_{\mathbf{n}}, q_{\mathbf{n}}, \pi_{\mathbf{n}}, M_{\mathbf{n}}$, introduced in relation to $M_{\oplus}$ immediately before (45), will now be used for M. Thus

$$\pi_{\mathbf{n}}\varphi j_{\mathbf{k}} = \varphi j_{\mathbf{k}} \text{ whenever } \mathbf{n} \supseteq \mathbf{k} + \mathbf{e},$$

and analogous identities hold for ψ, Φ, and Ψ. For any $\mathbf{n} = [n_1, n_2)$ with $n_2 - n_1 \geq 3$, we let

$$(51) \qquad p_{\mathbf{n}} = p_{\mathbf{n}}(\varphi, \psi) = q_{\mathbf{n}}\varphi\pi_{\mathbf{n}-\mathbf{e}}\psi j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$$

[16]The loss of a factor 2 in control happens because one must get rid of the off diagonal terms by an elementary operation.

Since $j_{\mathbf{n}} q_{\mathbf{n}} \varphi \pi_{\mathbf{n-e}} = \varphi \pi_{\mathbf{n-e}}$, the homotopy $q_{\mathbf{n}} \varphi \pi_{\mathbf{n-e}} \Psi \pi_{\mathbf{n-e}} \psi j_{\mathbf{n}}$ shows that $p_{\mathbf{n}}$ is a homotopy idempotent of bound 2^{-n_1} on the module $M_{\mathbf{n}}$.

If we assume that $n_2 - n_1 \geq 6$, then we get a decomposition into a left hand, a middle, and a right hand summand as before, $M_{\mathbf{n}} = M_{\mathbf{l}} \oplus M_{\mathbf{m}} \oplus M_{\mathbf{r}}$, but this time $\mathbf{l} = [n_1, n_1+2)$, $\mathbf{m} = \mathbf{n} - 2\mathbf{e}$, $\mathbf{r} = [n_2 - 2, n_2)$. As before, the right hand summand vanishes if $n_2 = \infty$. Since $\mathbf{m}$ has length at least two, $p_{\mathbf{n}}$ maps $M_{\mathbf{l}}$ into $M_{\mathbf{l}} \oplus M_{\mathbf{m}}$ and $M_{\mathbf{r}}$ into $M_{\mathbf{m}} \oplus M_{\mathbf{r}}$. Also,

$$p_{\mathbf{n}} | M_{\mathbf{m}} = q_{\mathbf{n}} \varphi \pi_{\mathbf{n-e}} \psi j_{\mathbf{m}} = q_{\mathbf{n}} \varphi \psi j_{\mathbf{m}} \simeq q_{\mathbf{n}} j_{\mathbf{m}} = incl : M_{\mathbf{m}} \to M_{\mathbf{n}}$$

by the homotopy $q_{\mathbf{n}} \Phi j_{\mathbf{m}}$ of bound $2^{-m_1} = 2^{-(n_1+2)}$. In matrix language relative to the decomposition $M_{\mathbf{n}} = M_{\mathbf{l}} \oplus M_{\mathbf{m}} \oplus M_{\mathbf{r}}$, this means that we have a homotopy of bound 2^{-m_1}

$$(52) \qquad p_{\mathbf{n}} = \begin{pmatrix} p_{\mathbf{n}}^{(l)} & q_{\mathbf{l}} \varphi \psi j_{\mathbf{m}} & 0 \\ * & q_{\mathbf{m}} \varphi \psi j_{\mathbf{m}} & * \\ 0 & q_{\mathbf{r}} \varphi \psi j_{\mathbf{m}} & p_{\mathbf{n}}^{(r)} \end{pmatrix} \simeq \begin{pmatrix} p_{\mathbf{n}}^{(l)} & 0 & 0 \\ * & 1 & * \\ 0 & 0 & p_{\mathbf{n}}^{(r)} \end{pmatrix}$$

in $GM(E \times (-\infty, 0); p \times id)$. It follows easily that the right hand matrix is a homotopy idempotent of bound 2^{1-n_1}. Hence $p_{\mathbf{n}}^{(l)}$ and $p_{\mathbf{n}}^{(r)}$ (which are *defined* by the equation) are homotopy idempotents of bound 2^{1-n_1}. Moreover, the left hand idempotent $p_{\mathbf{n}}^{(l)}$ is unchanged if one augments n_2, keeping n_1 fixed (and dually for the right hand idempotent $p_{\mathbf{n}}^{(r)}$), i.e., with notation from (45),

$$(53) \qquad p_{\mathbf{n}'}^{(r)} = p_{\mathbf{n}}^{(r)} \text{ and } p_{\mathbf{n}''}^{(l)} = p_{\mathbf{n}}^{(l)}.$$

We define $\partial : K_1^{cc} \to \lim K_0(B;p)_{2^{-n}}$ by

$$(54) \qquad \partial[cls\varphi] = \{\partial_n(\varphi, \psi)\}_n,$$

where

$$(55) \qquad \partial_{n-1}(\varphi, \psi) = [\Sigma p_{[n,n+k)}^{(l)}(\varphi, \psi)] - [\Sigma p_{[n,n+k)}^{(l)}(1_M, 1_M)], 6 \leq k \leq \infty.$$

To check that this actually gives an element of $\lim K_0$ which is independent of the various choices we need the following identity in $K_0(B;p)_{2^{1-n_1}}$

$$(56) \qquad [\Sigma p_{\mathbf{n}}(1_M, 1_M)] = [\Sigma p_{\mathbf{n}}(\varphi, \psi)] = [\Sigma p_{\mathbf{n}}^{(l)}(\varphi, \psi)] + [1_{\Sigma M_{\mathbf{m}}}] + [\Sigma p_{\mathbf{n}}^{(r)}(\varphi, \psi)].$$

The second part of (56) follows from (52). To prove the first part, let $f = q_{\mathbf{n}} \varphi \pi_{\mathbf{n-e}} j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$ and $g = q_{\mathbf{n}} \pi_{\mathbf{n-e}} \psi j_{\mathbf{n}} : M_{\mathbf{n}} \to M_{\mathbf{n}}$. Clearly $fg = p_{\mathbf{n}}(\varphi, \psi)$, and, since $\pi_{\mathbf{n}} \varphi \pi_{\mathbf{n-e}} = \varphi \pi_{\mathbf{n-e}}$, $gf \simeq p_{\mathbf{n}}(1_M, 1_M)$ with bound 2^{-1-n_1}.

The identity shows that $\partial_{n-1}(\varphi, \psi)$ is also given by

$$(57) \qquad \partial_{n-1}(\varphi, \psi) = -[\Sigma p_{[n,n+k)}^{(r)}(\varphi, \psi)] + [\Sigma p_{[n,n+k)}^{(r)}(1_M, 1_M)], 6 \leq k < \infty.$$

It follows from (57) and (53) that $\{\partial_n(\varphi, \psi)\}_n \in \lim K_0(B;p)_{2^{-n}}$. It also follows that the choice of the neighborhoods U_n satisfying Lemma 5.2 is immaterial. Indeed, if $\{U_n'\}_n$ is a different choice, then $\{U_n \cap U_n'\}_n$ is also a possible choice, so in a comparison we may assume that $U_n' \subseteq U_n$. But then for any n_0, the system $\{V_n^{(n_0)}\}_n$ which coincides with U_n for $n < n_0$ and with U_n' for $n \geq n_0$ is allowed and one only has to use (57) and (53) to the system $\{V_n^{(n_0)}\}_n$ for a suitable choice of n_0 and k to finish the proof. Similarly, a modification of φ (or ψ) by a homotopy or by right multiplication by an elementary γ can be decomposed into two consecutive modifications where one does not change anything above some arbitrarily chosen U_{n_0} while the other one leaves everything intact above the complement of U_{n_0+k} for suitable k. Therefore, the above argument easily adapts to show that, indeed, (54) and (55) give a well defined homomorphism.

To show that ∂ is epic, let $z = \{[\alpha_n] - [\beta_n]\}_n \in \lim K_0(B;p)_{2^{-n}}$. This means that $\alpha_n : A_n \to A_n$ and $\beta_n : B_n \to B_n$ are homotopy idempotents of bound 2^{-n}, and in $GM_{2^{-n}}$ there exist geometric modules S_n and morphisms $\psi_n : A_n \oplus B_{n+1} \oplus S_n \to A_{n+1} \oplus B_n \oplus S_n$ and $\varphi_n : A_{n+1} \oplus B_n \oplus S_n \to A_n \oplus B_{n+1} \oplus S_n$ such that we have the following homotopies of bound 2^{1-n}

$$\alpha_n^2 \simeq \alpha_n, \qquad \text{and } \beta_n^2 \simeq \beta_n; \tag{58}$$

$$\varphi_n\psi_n \simeq \alpha_n \oplus \beta_{n+1} \oplus 1_{S_n}, \text{ and } \psi_n\varphi_n \simeq \alpha_{n+1} \oplus \beta_n \oplus 1_{S_n}; \tag{59}$$

$$\varphi_n \simeq \varphi_n(\alpha_{n+1} \oplus \beta_n \oplus 1_{S_n}) \simeq (\alpha_n \oplus \beta_{n+1} \oplus 1_{S_n})\varphi_n; \tag{60}$$

$$\psi_n \simeq \psi_n(\alpha_n \oplus \beta_{n+1} \oplus 1_{S_n}) \simeq (\alpha_{n+1} \oplus \beta_n \oplus 1_{S_n})\psi_n. \tag{61}$$

We now set $T = \oplus_n i_{n*}(A_n \oplus B_n \oplus S_n)$, $T' = \oplus_n i_{n*}(A_{n+1} \oplus B_n \oplus S_n)$, and $T'' = \oplus_n i_{n*}(A_n \oplus B_{n+1} \oplus S_n)$, where n ranges over $\mathbb{Z}$. These three geometric modules in $GM(E \times (-\infty, 0))$ are connected by natural transformations (cf. (41)) $\nu' : T' \to T$, $\mu' : T \to T'$, $\nu'' : T'' \to T$, $\mu'' : T \to T''$ which are pairwise inverses up to natural, cc homotopies. Put $\Phi = \oplus_n i_{n*}(\varphi_n) : T' \to T''$, $\Psi = \oplus_n i_{n*}(\psi_n) : T'' \to T'$, and $\Gamma = \oplus_n i_{n*}((1-\alpha_n) \oplus (1-\beta_n) \oplus 0) : T \to T$. Elementary computations using the homotopies from (58) – (61) show that

$$\tilde{\Phi}\Gamma \simeq 0, \Gamma\tilde{\Phi} \simeq 0, \tilde{\Psi}\Gamma \simeq 0, \Gamma\tilde{\Psi} \simeq 0, \Gamma^2 \simeq \Gamma, \tilde{\Psi}\tilde{\Phi} \simeq 1 - \Gamma, \text{ and } \tilde{\Phi}\tilde{\Psi} \simeq 1 - \Gamma,$$

where $\tilde{\Psi} = \nu'\Psi\mu'' : T \to T$, and $\tilde{\Phi} = \nu''\Phi\mu' : T \to T$. Moreover, all of these maps and homotopies are in GM^{cc}, so $\psi = \tilde{\Psi} + \Gamma : T \to T$ and $\varphi = \tilde{\Phi} + \Gamma : T \to T$ are cc homotopy inverses. We claim that $\partial[cls(\varphi)] = z$.

To compute $\partial_n(\varphi, \psi)$, we chose $U_n = B \times [-2^{-n}, 0]$ in Lemma 5.2 and $k = 6$ in (55). The homotopy idempotent $\Sigma p^{(l)}_{[n,n+6)}(\varphi, \psi)$ then lives on the module $A_n \oplus B_n \oplus S_n \oplus A_{n+1} \oplus B_{n+1} \oplus S_{n+1}$, and a simple computation gives us the following matrix form up to 2^{1-n}-homotopies coming from (58) – (61). Here a superscript like ^{AB} denotes the component of a morphism which maps the relevant B-summand to the relevant A-summand.

$$\begin{pmatrix}
\varphi_n^{AA}\psi_n^{AA} & 0 & \varphi_n^{AA}\psi_n^{AS} & 0 & \varphi_n^{AA}\psi_n^{AB} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\varphi_n^{SA}\psi_n^{AA} & 0 & \varphi_n^{SA}\psi_n^{AS} & 0 & \varphi_n^{SA}\psi_n^{AB} & 0 \\
0 & 0 & 0 & 1_{A_{n+1}} & 0 & 0 \\
\varphi_n^{BA}\psi_n^{AA} & 0 & \varphi_n^{BA}\psi_n^{AS} & 0 & \varphi_n^{BA}\psi_n^{AB} + (1-\beta_{n+1}) & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{S_{n+1}}
\end{pmatrix}$$

This is the sum of two homotopy orthogonal homotopy idempotents, namely the diagonal matrix $\mathrm{diag}(0, 0, 0, 1_{A_{n+1}}, 1 - \beta_{n+1}, 1_{S_{n+1}})$ and the product X^tY where $X = (\varphi_n^{AA}, 0, \varphi_n^{SA}, 0, \varphi_n^{BA}, 0)$, $Y = (\psi_n^{AA}, 0, \psi_n^{AS}, 0, \psi_n^{AB}, 0)$ and ^{t} denotes a transposition. These terms represent $[1_{A_{n+1}}] + [1 - \beta_{n+1}] + [1_{S_{n+1}}]$, respectively, $[X^tY] = [YX^t] = [(\psi_n\varphi_n)^{AA}] = [\alpha_{n+1}]$. Since $[p^{(l)}_{[n,n+6)}(1_T, 1_T)] = [1_{A_{n+1} \oplus B_{n+1} \oplus S_{n+1}}]$, it follows that $\partial_{n-1}(\varphi, \psi) = [\alpha_{n+1}] - [\beta_{n+1}] \in K_0(B;p)_{2^{1-n}}$, so $\partial[cls\varphi] = z$ as claimed.

We leave to the reader the easy proof that $\partial i = 0$ and go on to prove that $\mathrm{Ker}(\partial) \subseteq \mathrm{Im}(i)$. We keep the notation from the last few lines preceding (51) and assume that $y = [cls\varphi] \in \mathrm{Ker}(\partial)$, i.e., for each $n \in \mathbb{N}$, the element $\partial_{n-1}(\varphi, \psi)$ defined in (55) vanishes in $K_0(B;p)_{2^{1-n}}$. The basic lemma is the following one.

Lemma 5.3 *There exists a universal constant $c \in \mathbb{N}$ for which the following holds. If $\partial_{n-1}(\varphi, \psi) = 0 \in K_0(B;p)_{2^{1-n}}$, then there exist a (finite) geometric module L_{n+1} in $GM(B;p)$, seven 2^{c-n}-elementary automorphisms $\gamma^i : M' = M \oplus i_{n+1*}L_{n+1} \to M \oplus i_{n+1*}L_{n+1}$, and a morphism $\varphi' \simeq \gamma^7\gamma^6 \cdots \gamma^1\varphi : M' \to M'$ such that φ' is diagonal w.r.t. the decomposition $M' = M'_{(-\infty,n+1)} \oplus M'_{[n+1,\infty)}$. Moreover, each γ^i is the identity on $(M'_{[n,n+3)})^\perp$. Finally, the homotopy is bounded by 2^{c-n} measured in B and it is stationary on $(M'_{[n,n+3)})^\perp$.*

Proof: Throughout the proof, the constant c will be increased a few times without our changing its name. We take $k = \infty$ in the definition (55) and write $\mathbf{n} = [n, \infty)$. Then the right hand summand in the decomposition used in (52) vanishes, so $M = M_{\mathbf{n}}^{\perp} \oplus M_{\mathbf{l}} \oplus M_{\mathbf{m}}$ where $\mathbf{l} = [n, n+2)$ and $\mathbf{m} = [n+2, \infty)$. Up to 2^{c-n}-homotopy, one has the following matrix presentation relative to this decomposition (compare (52) and (51)).

$$\varphi \pi_{\mathbf{n-e}} \psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_{\mathbf{n}}^{(l)}(\varphi, \psi) & 0 \\ 0 & x & 1 \end{pmatrix} \tag{62}$$

Since $\varphi \pi_{\mathbf{n-e}} \psi$ is a 2^{c-n}-idempotent, $x p_{\mathbf{n}}^{(l)}(\varphi, \psi) \simeq 0$ (with bound 2^{c+1-n}). We let $\gamma^1 = e_{32}(x)$ where the classical index-notation for the elementary automorphism refers to the decomposition of M used in (62). The effect of replacing (φ, ψ) by $(\gamma^1 \varphi, \psi(\gamma^1)^{-1})$ then is to replace x by 0 in (62) (up to 2^{1+c-n}-homotopy). Thus we shall proceed with $x = 0$ in (62). Note that this first elementary γ^1 is supported on $M_{[n,n+3)}$ as claimed. The remaining six elementary automorphisms will actually be supported on $M'_{\mathbf{l}}$ (recall that $\mathbf{l} = [n, n+2)$).

Since $p_{\mathbf{n}}^{(l)}(1_M, 1_M) : M_{\mathbf{l}} \to M_{\mathbf{l}}$ is the canonical projection onto $M_{[n+1,n+2)}$, Proposition 3.1 gives the following translation of the vanishing of $\partial_{n-1}(\varphi, \psi)$. For some constant $c \in \mathbb{N}$ we have a geometric module $L'_n \in GM_0$; two 2^{c-n}-morphisms

$$\alpha_n : \Sigma M_{\mathbf{l}} \oplus L'_n \to \Sigma M_{[n+1,n+2)} \oplus L'_n \text{ and} \tag{63}$$

$$\beta_n : \Sigma M_{[n+1,n+2)} \oplus L'_n \to \Sigma M_{\mathbf{l}} \oplus L'_n; \tag{64}$$

and four 2^{c+1-n}-homotopies (note that two of the conditions in (11) are vacuous, because the second idempotent involved here is the identity)

$$\beta_n \alpha_n \simeq \Sigma p_{\mathbf{n}}^{(l)}(\varphi, \psi) \oplus 1_{L'_n} \text{ and } \alpha_n \beta_n \simeq 1_{\Sigma M_{[n+1,n+2)} \oplus L'_n}; \text{ and} \tag{65}$$

$$\alpha_n(\Sigma p_{\mathbf{n}}^{(l)}(\varphi, \psi) \oplus 1_{L'_n}) \simeq \alpha_n \text{ and } \beta_n \simeq (\Sigma p_{\mathbf{n}}^{(l)}(\varphi, \psi) \oplus 1_{L'_n})\beta_n. \tag{66}$$

If we stabilize φ and ψ by adding $1_{i_{n+1*}L'_n}$, the summands L'_n will appear in the original data, and no stabilization will be needed. Hence, we proceed under the assumption that $L'_n = 0$.

We now let $L_{n+1} = \Sigma M_{[n+1,n+2)}$. Then $\alpha_n : \Sigma M_{\mathbf{l}} \to L_{n+1}$, $\beta_n : L_{n+1} \to \Sigma M_{\mathbf{l}}$, and we also have the inclusion $j : L_{n+1} \to \Sigma M_{\mathbf{l}}$ and the projection $q : \Sigma M_{\mathbf{l}} \to L_{n+1}$. The following product of six 2^{c-n}-elementary automorphisms in the module $\Sigma M_{\mathbf{l}} \oplus L_{n+1}$

$$\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_n & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_n & 1 \end{pmatrix} \tag{67}$$

is easily seen to conjugate[17] the diagonal matrix $\mathrm{diag}(\beta_n \alpha_n, 0)$ into $\mathrm{diag}(jq, 0)$ (up to 2^{c-n}-homotopy). Now we get the elementaries γ^i, $i = 2, 3, \cdots 7$, by applying i_{n+1*} to each of the elementaries above, using (40) to identify $i_{n+1*}\Sigma M_{\mathbf{l}}$ with $M_{\mathbf{l}}$, and extending the resulting elementaries to be the identity on $(M'_{\mathbf{l}})^{\perp}$. With the identification mentioned, $i_{n+1*}(jq) : M_{\mathbf{l}} \to M_{\mathbf{l}}$ is the standard projection onto $M_{[n+1,n+2)}$, so the resulting $\varphi' = \gamma^7 \gamma^6 \cdots \gamma^2 \varphi : M' \to M'$ conjugates $\pi_{[n+1,\infty)} : M' \to M'$ into itself (up to a 2^{c-n}-homotopy supported on $M'_{\mathbf{l}}$), and the proof is complete. □

[17] This is based on the following (well known) fact: If, in some additive category, we have morphisms $\xi : X \to Y$ and $\eta : Y \to X$ satisfying $\xi \eta \xi = \xi$ and $\eta \xi \eta = \eta$, then the product

$$\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \tag{68}$$

conjugates $diag(\eta \xi, 0)$ into $diag(0, \xi \eta)$.

We can now finish the proof of Theorem 5.1 as follows. Recall that we have assumed that $y \in K_1^{cc}$ has $\partial(y) = 0$. Using the lemma, we can first write a representative of y as $\varphi' \oplus \varphi : M' \oplus M \to M' \oplus M$ with $M' = M'_{[0,4)}$ and $M = M_{[4,\infty)}$. A standard Eilenberg swindle on the ray $(-\infty, 0]$ shows that $[cls\varphi'] = 0 \in K_1^{cc}$. Repeated applications of the lemma let us split φ as an infinite direct sum

$$\varphi = \bigoplus_{k=1}^{\infty} \varphi_{4k} : \bigoplus M_{[4k,4k+4)} \to \bigoplus M_{[4k,4k+4)}$$

Indeed, the resulting (seven) infinite families of elementary automorphisms assemble to seven *cc* elementary automorphisms on the direct sum, because of the conditions in Lemma 5.3, and a similar remark holds for the infinite family of stabilizations, and the infinite family of homotopies. The family $\{\Sigma\varphi_{4k}\}_k$ defines an element $z \in \lim^1 K_1$ and clearly $i(z) = y$.

6 Computation of $K_{2-j}^{cc}(\overline{X}, B)$ for $j \in \mathbb{N}$.

In this section we work with a category called $\mathcal{TOP}^2$. Its objects are pairs, $(\overline{X}, B)$, of *compact, metric* spaces with B *tame* in $\overline{X}$ in the sense of [Q2]. The morphisms are *strict* maps $(\overline{X}, B) \to (\overline{X'}, B')$. The obvious functor[18] $(\overline{X}, B) \mapsto (\overline{X}, B; 1_X)$ from $\mathcal{TOP}^2$ to $\mathcal{TOP}^{cc}$ will be used to consider K_*^{cc} as a functor on $\mathcal{TOP}^2$. We recall that the *holink diagram* for $(\overline{X}, B)$ is

$$X = \overline{X} - B \xleftarrow{q} H \xrightarrow{p} B, \tag{69}$$

where $H = \text{Holink}(\overline{X}, B)$ is the space of strict maps $f : ([0,1], \{0\}) \to (\overline{X}, B)$, and q and p are given by evaluation at 1 and 0, respectively. Since the holink diagram is functorial on $\mathcal{TOP}^2$, so are the groups $K_*(X)$, and $K_{2-j}(B;p)_c$ as well as $\lim^1 K_{2-j}(B;p)_\epsilon$ for each $j \in \mathbb{N}$. We shall fit these groups into a chain complex which is natural in $\overline{X}, B) \in \mathcal{TOP}^2$,

$$K_2(X) \xrightarrow{i_2} K_2^{cc}(\overline{X}, B) \xrightarrow{\Delta_2} K_1(B;p)_c \xrightarrow{\ell_1} K_1(X) \xrightarrow{i_1} \cdots \tag{70}$$

and prove

Theorem 6.1 *Let $j \in \mathbb{N}$. The above chain complex is exact at $K_{1-j}(B;p)_c$ and $K_{1-j}(X)$, and the homology group at $K_{2-j}^{cc}(\overline{X}, B)$ is naturally isomorphic to $\lim^1 K_{2-j}(B;p)_\epsilon$.*

Remarks: 1) This is the unreduced version of Theorem 1.2. We leave it to the reader to derive the reduced version.
2) We note that the theorem as stated contains no claims about the homology at $K_2^{cc}(\overline{X}, B)$ or exactness at $K_1(X)$ and $K_1(B;p)_c$. In fact, two of the present authors (FXC and HJM) have worked out a definiton of ϵ-controlled groups $K_2(B;p)_\epsilon$ which appear to fit into an exact sequence

$$0 \to \lim^1 K_2(B;p)_\epsilon \to K_2^{cc}(B \times (-\infty, 0], B \times 0; p \times 1) \to K_1(B;p)_c \to 0.$$

If this holds up, then the following proof will cover also the three cases currently excluded.
3) If we let[19] $BK_*^{cc}(\overline{X}, B) = \text{Ker}\Delta_*$, then the theorem translates into two exact sequences for each $j \in \mathbb{N}$:

$$0 \to BK_{2-j}^{cc}(\overline{X}, B) \to K_{2-j}^{cc}(\overline{X}, B) \to K_{1-j}(B;p)_c \to K_{1-j}(X),$$

and

$$K_{1-j}(B;p)_c \to K_{1-j}(X) \to BK_{1-j}^{cc}(\overline{X}, B) \to \lim^1 K_{1-j}(B;p)_\epsilon \to 0.$$

[18]This functor is full; if we had required that $\overline{X} - B$ be dense in $\overline{X}$, then it would also be faithful.
[19]B stands for "bumpy", cf. [Si].

They are analogous to the sequences in Siebenmann's Theorems I and II in [Si]. In fact, if X is locally compact, and one takes $\overline{X}$ to be the one point compactification (so that $B = *$), then the connection can be made more than an analogy; but we shall not pursue this here. Note, though, that the present version must be extended to cover the case $j = 0$ in order to recover Siebenmann's result in the correct degree.

4) We recall that Chapman, see Theorems 13.1 and 13.2 of [Ch], studies the controlled boundary problem for a manifold M^m parametrized over a metric space B via a map[20] $p' : M \to B$. Assuming $m \geq 6$, and suitable "movability"and "tameness" conditions, he obtains a solution in terms of a primary obstruction $\sigma_\infty^{p'}(M) \in \mathrm{lim} K_0(M - C_i)_{1/i}$ and a secondary obstruction $\tau_\infty^{p'}(M) \in \lim^1 Wh(M - C_i)_{1/i}$. Here $\{C_i | i = 1, 2, 3 \cdots\}$ is an expanding and exhaustive sequence of compacta in M. For any space Y with a map $p' : Y \to B$, the groups $Wh^(Y)_\epsilon$, are defined geometrically in terms of ϵ-controlled (in B) strong deformation retraction pairs, and the group $K_0^p(Y)_\epsilon$ is given as the transfer invariant subgroup of $Wh(Y \times S^1)_\epsilon$. Presumably,

$$\mathrm{lim} K_0(M - C_i)_{1/i} \cong \tilde{K}_0(B; p), \tag{71}$$

and

$$\lim^1 Wh(M - C_i)_{1/i} \cong \lim^1 Wh(B; p), \tag{72}$$

where p is the holink map for a suitable pair ($\overline{M}$,B) constructed from the given data. If so, then it ought to be possible to give a direct definiton of a total obstruction $\rho(M; p) \in Wh(\overline{M}; B)$ such that $\Delta_1(\rho(M; p'))$ correponds to $\sigma^{p'}(M)$ and such that the homology class represented by $\rho(M; p')$ corresponds to $\tau_\infty^{p'}(M)$ when $\sigma_\infty^{p'}(M)$ vanishes. Since we have not (yet) come up with such a definition, we have not pursued the conjectured isomorphisms in (71) and (72). □

Proof: The structure of the proof is very simple: Suppose that we are given a long exact sequence

$$\cdots \xrightarrow{\delta_{k+1}} A_k \xrightarrow{\alpha_k} B_k \xrightarrow{\beta_k} D_k \xrightarrow{\delta_k} A_{k-1} \xrightarrow{\alpha_{k-1}} \cdots \text{ for } k \in \mathbb{Z}, \tag{73}$$

and short exact sequences

$$0 \to C'_k \xrightarrow{\gamma'_k} C_k \xrightarrow{\gamma''_k} C''_{k-1} \to 0 \text{ for } k \leq n. \tag{74}$$

Suppose further that we have isomorphisms

$$\gamma_k : D_k \xrightarrow{\cong} C_k \text{ for } k \leq n, \tag{75}$$

and that

$$\delta_k \gamma_k^{-1} \gamma'_k = 0 : C'_k \to A_{k-1} \text{ for } k \leq n. \tag{76}$$

Under these assumptions, a trivial diagram chase shows that the chain complex

$$A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\gamma''_n \gamma_n \beta_n} C''_{n-1} \xrightarrow{\delta''_{n-1}} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \tag{77}$$

(where δ''_{k-1} has $\delta''_{k-1} \gamma''_k \gamma_k = \delta_k$ for $k \leq n$) is exact at C''_{k-1} and A_{k-1} and has homology group equal to C'_k at B_k (for all $k \leq n$).

Except for the first 3 terms[21], the proof is completed by feeding relevant data into this construction. For (73), we use (37) with $E = X$ and $p = 1_X$, so here is the list of characters in (73)

$$A_k = K_k(X),\ B_k = K_k^{cc}(\overline{X}, B),\ D_k = K_k^{cc,\infty}(\overline{X}, B), \text{ for } k \in \mathbb{Z}. \tag{78}$$

[20] Chapman calls this map p, but we have already used the letter p for the holink situation.

[21] which we included only to be able to make the remark **1)** after the theorem and which the reader can easily add on for herself.

For (74) we use the sequences from Theorem 5.1, so

$$C'_k = \lim{}^1 K_k(B;p)_\epsilon,\ C''_k = K_k(B;p)_c, \text{ and } C_k = K^{cc}_k(B \times (-\infty, 0], B \times 0; p \times id), \tag{79}$$

for $k \leq n = 1$.

We go on to produce the isomorphisms needed in (75). This involves the use of Quinn's double cylinder $P = cyl(q,p)$ for the holink diagram (69). We recall from [Q2] that $P = X \coprod H \times (-1,0) \coprod B$ with the smallest topology which makes the inclusion $H \times (-1,0) \subseteq P$ an open embedding, and for which the obvious functions $\bar{q} = 1_X \cup (p \times id) : P - B \to X \times [-1,0)$ and $\bar{p} = (p \times id) \cup 1_B : P - X \to B \times (-1,0]$ are continuous. By Lemma 2.4 of [Q2], there is a strict homotopy equivalence[22] (relative to B) $\overline{\alpha} : (P,B) \to (\overline{X}, B)$. We define an object $(cB, B; \pi)$ in $\mathcal{TOP}^{cc}$ as follows. The cone on B is $cB = B \times [-1,0]/B \times \{-1\}$, with $B = B \times 0 \subseteq cB$. The map $\pi : P - B \to cB - B$ has $\pi | X = c_X$ (the constant map to $\{c\}$) and $\pi | H \times (-1,0) = p \times id$. Since $\bar{\alpha}$ is a strict homotopy equivalence, it is easily verified that $\alpha = \bar{\alpha} | P - B : P - B \to X$ is a homotopy equivalence $(cB, B; \pi) \xrightarrow{\simeq} (\overline{X}, B; 1_X)$ in the category $\mathcal{TOP}^{cc}$. Therefore, by [AM2], there is a natural isomorphism $K^{cc}_*(\overline{X}, B) \longleftarrow K^{cc}_*(cB, B; \pi)$. It is induced by the functor $\alpha_* : \mathcal{GM}^{cc}(cB, B; \pi) \to \mathcal{GM}^{cc}(\overline{X}, B)$ which fixes the subcategory $\mathcal{GM}^{cc}(c, \emptyset; c_X)$. Therefore, the induced functor on Karoubi quotients also induces an isomorphism (cf. Section 4, just before (37))

$$D_* = K^{cc,\infty}_*(\overline{X}, B) \overset{a_1 \cong}{\longleftarrow} K^{cc,\infty}_*(cB, B; \pi). \tag{80}$$

We may consider $K^{cc}_*(cB, B; \pi)$ as a functor defined on the category of diagrams of the form $X \xleftarrow{q} H \xrightarrow{p} B$ (morphims are natural transformations, as usual). In this category, the obvious morphism from $H \xleftarrow{1} H \xrightarrow{p} B$ to the holink diagram (69) is easily seen to induce an equivalence on the functor $\mathcal{GM}^{cc,\infty}(c(B), B; -)$ (when one passes from the original categories to their Karoubi quotients all influence from the subspaces H or X disappear - see [AM2] for details). Hence one also has the natural isomorphism

$$K^{cc,\infty}_*(cB, B; \pi) \overset{a_2 \cong}{\longleftarrow} K^{cc,\infty}_*(cB, B; \pi_0), \tag{81}$$

where $\pi_0 : H \times [-1,0) \to cB - B$ maps $H \times \{-1\}$ to c, and $\pi_0 = p \times id$ on $H \times (-1,0)$. A similar argument shows that the morphism $1_{H \times [-1,0)} : (B \times [-1,0], B \times 0; p \times id) \to (cB, B; \pi_0)$ induces a natural isomorphism

$$K^{cc,\infty}_*(cB, B; \pi_0) \overset{a_3 \cong}{\longleftarrow} K^{cc,\infty}_*(B \times [-1,0], B \times 0; p \times id). \tag{82}$$

We finally observe that the inclusion $i : H \times [-1,0) \subseteq H \times (-\infty, 0)$ induces an (additive) functor from the Karoubi fibration

$$\mathcal{GM}^{cc}(B \times \{-1\}, \emptyset; p \times id)) \hookrightarrow \mathcal{GM}^{cc}(B \times [-1,0], B \times 0; p \times id) \longrightarrow \mathcal{Q}_1$$

to the Karoubi fibration

$$\mathcal{GM}^{cc}(B \times (-\infty, -1], \emptyset; p \times id)) \hookrightarrow \mathcal{GM}^{cc}(B \times (-\infty, 0], B \times 0; p \times id) \to \mathcal{Q}_2.$$

Since the functor $\mathcal{Q}_1 \to \mathcal{Q}_2$ induced on the Karoubi quotients is clearly a natural equivalence of categories, it induces a natural isomorphism

$$K^{cc,\infty}_*(B \times [-1,0], B \times 0; p \times id) \overset{a_4 \cong}{\longrightarrow} K^{cc,\infty}_*(B \times (-\infty, 0], B \times 0; p \times id). \tag{83}$$

[22] Actually, the map $P \to \overline{X}$ given in [Q2] is not always continuous, but this malady can be repaired by a reference to [No]. See also [HTWW] and [Hu].

But also, $\mathcal{GM}^{cc}(B \times (-\infty,-1], \emptyset; p \times id)$ is flasque (as in [PW] or [AM2]), so[23] $K_*^{cc}(B \times (-\infty,-1], \emptyset; p \times id) = 0$. The long exact K-theory sequence for the second Karoubi fibration then gives a natural isomorphism

$$K_*^{cc,\infty}(B \times (-\infty,0], B \times 0; p \times id) \xleftarrow{a_5 \cong} K_*^{cc}(B \times (-\infty,0], B \times 0; p \times id) = C_*. \tag{84}$$

We let $\gamma_* = a_5^{-1} a_4 (a_1 a_2 a_3)^{-1}$, and continue with the proof of (76). By the naturality in (37), there is a commutative square

$$\begin{array}{ccc} D_{2-j} = K_{2-j}^{cc,\infty}(\overline{X}, B) & \xrightarrow{\delta_{2-j}} & K_{1-j}(X) = A_{1-j} \\ {\scriptstyle a_1a_2a_3}\uparrow\cong & & \uparrow{\scriptstyle q_*} \\ K_{2-j}^{cc,\infty}(B \times [-1,0], B \times 0; p \times id) & \xrightarrow{\delta_{2-j}} & K_{1-j}(H) \end{array}$$

Therefore, it suffices to show that $\delta_{2-j} a_4^{-1} a_5 \gamma'_{2-j} = 0$, or equivalently, that

$$\begin{aligned} \mathrm{Im}[a_4^{-1} a_5 \gamma'_{2-j} : \lim{}^1 K_{2-j}(B;p)_\epsilon \to K_{2-j}^{cc,\infty}(B \times [-1,0], B \times 0; p \times 1)] \quad \subseteq \\ \mathrm{Im}[K_{2-j}^{cc}(B \times [-1,0], B \times 0; p \times 1) \to K_{2-j}^{cc,\infty}(B \times [-1,0], B \times 0; p \times 1)]. \end{aligned}$$

By naturality, and because of Corollary 3.11 and Proposition 4.1, it suffices to establish this inclusion for $j = 1$. In this case, γ'_1 equals the map i defined in (44), and any $x \in \lim^1 K_1(B;p)_\epsilon$ is represented by some sequence $\{[\varphi]_n\}_n$ as in (44). We may assume that $\varphi_n : M_n \to M_n$ has $M_n = 0$ for $n \leq 0$. Then the automorphism $cls(\varphi_\oplus) : M_\oplus \to M_\oplus$ defined in (43) is in the image of the functor

$$\mathcal{GM}^{cc}(B \times [-1,0], B \times 0; p \times 1) \to \mathcal{GM}^{cc}(B \times (-\infty,0], B \times 0; p \times 1),$$

so the proof is complete.

7 Vanishing results for $\lim^1 K_{2-j}(B;p)_\epsilon$.

Let $p : E \to B$ be in $\mathcal{TOP}_c$. We say that p *allows squeezing of automorphisms* if

$$\begin{aligned} &\forall \epsilon > 0 \ \exists \delta \in (0,\epsilon) \ \forall \mu \in (0,\delta) : \\ &\mathrm{Im}[K_1(B;p)_\delta \to K_1(B;p)_\epsilon] \subseteq \mathrm{Im}[K_1(B;p)\_mu \to K_1(B;p)_\epsilon], \end{aligned}$$

i.e., if the inverse system $\{K_1(B;p)_epsilon\}_\epsilon$ satisfies the Mittag Leffler condition. As is well known, this condition implies that $lim K_1(B;p)_\epsilon = 0$. Thus to prove the following proposition, we only need to show that whenever p allows squeezing, then so does $p \times id : E \times S^1 \to B \times S^1$.

Proposition 7.1 *If $p : E \to B$ allows squeezing of automorphisms, then the groups* $\lim^1 K_{2-j}(B;p)_\epsilon$ *vanish for all $j \in \mathbb{N}$*

Conjecture 7.2 *If $p : E \to B = |K|$ has an iterated mapping cylinder stucture w.r.t. to a triangulation K of B, then p allows squeezing of automorphisms. This applies in particular, if p is a* pl *map between polyhedra.*

Our background for this conjecture is the fact that DRA and HJM went through Ferry and Pedersen's proof of squeezing (in [FP]) for the constant coefficient case, and it seems to carry over with little trouble. However, a "final" version of [FP] has never appeared, and we have yet to write out our more general version. As mentioned in the introduction, we plan to do so in the final version of the present paper.

[23]Some readers may prefer to argue for this vanishing result with the magic wand "an Eilenberg swindle type argument shows that $\cdots$."

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