AN EXPLICIT PROJECTION

by Andrew Ranicki

Abstract *The Wall finiteness obstruction is the principal application of the projective class group* $K_0(\Lambda)$ *to topology, with* $\Lambda = \mathbb{Z}[G]$ *the group ring of the fundamental group* G*. The finiteness obstruction is an element of the reduced projective class group*

$$\widetilde{K}_0(\Lambda) \ = \ \mathrm{coker}(K_0(\mathbb{Z}) \longrightarrow K_0(\Lambda)) \ .$$

This paper uses the Rim cartesian square and the Milnor Mayer-Vietoris exact sequence to provide an explicit construction of a f.g. projective $\mathbb{Z}[G]$*-module* $P = \mathrm{im}(p)$ *such that*

$$[P] \neq 0 \in \widetilde{K}_0(\mathbb{Z}[G]) \ ,$$

with $G = Q(8)$ *the quaternion group of order 8 and* $p = p^2$ *a* 2×2 *projection matrix with entries in* $\mathbb{Z}[Q(8)]$*. This example is well-known to the experts, but an explicit construction of* p *might be of interest to students of algebraic* K*-theory.*

A module P over a ring A is f.g. (finitely generated) projective if it is isomorphic to the image $\mathrm{im}(p : A^n \longrightarrow A^n)$ of a projection $p = p^2 : A^n \longrightarrow A^n$ of a f.g. free module A^n. Projective modules and the projective class groups $K_0(A)$, $\widetilde{K}_0(A)$ entered topology via the work of Swan [9] on finite group actions on homotopy spheres, and more generally via the finiteness obstruction theory of Wall [11]. In various papers (Munkholm and Ranicki [5], Ranicki [7], Lück [1], Pedersen and Weibel [6], Lück and Ranicki [2]) it has actually been found more convenient to work with the projections rather than the modules.

In this note an explicit projection is obtained for the f.g. projective A-module constructed by the standard Mayer-Vietoris procedure (Milnor [4,§2]) from an automorphism of a f.g. free A'-module, with A and A' related by a cartesian square of rings

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B' & \xrightarrow{\ g'\ } & A' \end{array}$$

with $f' : A \longrightarrow B'$ and $g : B \longrightarrow A'$ onto. In view of the theorem of Swan that the map $\widetilde{K}_0(\mathbb{Z}[G]) \longrightarrow \widetilde{K}_0(\mathbb{Q}[G])$ is trivial for finite groups G, this is the generic construction of f.g. projective $\mathbb{Z}[G]$-modules for finite G. By way of an example an explicit projection is constructed for a generator of $\widetilde{K}_0(\mathbb{Z}[Q(8)]) = \mathbb{Z}_2$, with $Q(8)$ the quaternion group $Q(8)$ of order 8. This is the simplest example of a group G with $\widetilde{K}_0(\mathbb{Z}[G]) \neq 0$.

A commutative square of rings (as above) is **cartesian** if the sequence of additive groups

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} B \oplus B' \xrightarrow{(g \quad -g')} A' \longrightarrow 0$$

is exact.

Given an automorphism $\alpha' : A'^n \longrightarrow A'^n$ of a f.g. free A'-module define the pullback f.g. projective A-module

$$P(\alpha') \;=\; \{(x, x') \in B^n \oplus B'^n \,|\, \alpha'(g(x)) = g'(x') \in A'^n\}$$

which fits into an exact sequence of additive groups

$$0 \longrightarrow P(\alpha') \longrightarrow B^n \oplus B'^n \xrightarrow{(\alpha' g \quad -g')} A'^n \longrightarrow 0$$

with A acting by

$$A \times P(\alpha') \longrightarrow P(\alpha') \;;\; (a, (x, x')) \longrightarrow (f(a)x, f'(a)x') \;.$$

The construction is used to define the connecting map ∂ in the Mayer-Vietoris exact sequence (Milnor [4,§4]) of algebraic K-groups

$$K_1(A) \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} K_1(B) \oplus K_1(B') \xrightarrow{(g \quad -g')} K_1(A') \xrightarrow{\partial}$$
$$K_0(A) \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} K_0(B) \oplus K_0(B') \xrightarrow{(g \quad -g')} K_0(A') \;,$$

with

$$\partial \;:\; K_1(A') \longrightarrow K_0(A) \;;\; \tau(\alpha' : A'^n \longrightarrow A'^n) \longrightarrow [P(\alpha')] - [A^n] \;.$$

Given A'-module automorphisms $\alpha' : A'^n \longrightarrow A'^n$, $\alpha'' : A'^m \longrightarrow A'^m$, and also a B-module morphism $\beta : B^n \longrightarrow B^m$ and a B'-module morphism $\beta' : B'^n \longrightarrow B'^m$ such that the square

$$\begin{array}{ccc} A'^n & \xrightarrow{\;g(\beta)\;} & A'^m \\ {\scriptstyle \alpha'} \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha''} \\ A'^n & \xrightarrow{\;g'(\beta')\;} & A'^m \end{array}$$

commutes let

$$(\beta,\beta') \ : \ P(\alpha') \longrightarrow P(\alpha'') \ ; \ (x,x') \longrightarrow (\beta(x),\beta'(x'))$$

be the pullback A-module morphism.

Proposition *Given an A'-module automorphism $\alpha' : A'^n \longrightarrow A'^n$ and any lifts of α', α'^{-1} to B-module endomorphisms $\beta,\gamma : B^n \longrightarrow B^n$ there is defined an A-module projection*

$$p(\alpha') \ = \ \begin{pmatrix} ((2-\beta\gamma)\beta\gamma,1) & ((2-\beta\gamma)(1-\beta\gamma)\beta,0) \\ (\gamma(1-\beta\gamma),0) & ((1-\gamma\beta)^2,0) \end{pmatrix} \ : \ A^n \oplus A^n \longrightarrow A^n \oplus A^n$$

such that up to isomorphism

$$P(\alpha') \ = \ \mathrm{im}(p(\alpha')) \ .$$

Proof: Lift the Whitehead lemma identity of A'-module automorphisms

$$\begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha'^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$: \ A'^n \oplus A'^n \longrightarrow A'^n \oplus A'^n$$

to define a B-module automorphism

$$\phi \ = \ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ = \ \begin{pmatrix} (2-\beta\gamma)\beta & \beta\gamma-1 \\ 1-\gamma\beta & \gamma \end{pmatrix}$$

$$: \ B^n \oplus B^n \longrightarrow B^n \oplus B^n$$

with inverse

$$\phi^{-1} \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \ = \ \begin{pmatrix} \gamma & 1-\gamma\beta \\ \beta\gamma-1 & (2-\beta\gamma)\beta \end{pmatrix}$$

$$: \ B^n \oplus B^n \longrightarrow B^n \oplus B^n \ .$$

Identifying $A^n \oplus A^n = A^{2n}$ define an A-module isomorphism

$$h \ = \ (\phi,1) \ : \ P(\alpha') \oplus P(\alpha'^{-1}) \longrightarrow P(1 : A'^{2n} \longrightarrow A'^{2n}) \ = \ A^{2n}$$

with inverse

$$h^{-1} \ = \ (\phi^{-1},1) \ : \ P(1 : A'^{2n} \longrightarrow A'^{2n}) \ = \ A^{2n} \longrightarrow P(\alpha') \oplus P(\alpha'^{-1}) \ .$$

It is now immediate from the identity

$$p(\alpha') \ = \ h(1\oplus 0)h^{-1} \ :$$

$$A^{2n} \xrightarrow{h^{-1}} P(\alpha') \oplus P(\alpha'^{-1}) \xrightarrow{1\oplus 0} P(\alpha') \oplus P(\alpha'^{-1}) \xrightarrow{h} A^{2n}$$

that $p(\alpha') : A^{2n} \longrightarrow A^{2n}$ is a projection with image isomorphic to $P(\alpha')$. Explicitly, the restriction of h defines an A-module isomorphism

$$P(\alpha') \longrightarrow \mathrm{im}(p(\alpha')) \ ; \ (x, x') \longrightarrow ((2-\beta\gamma)\beta(x), x') \oplus ((1-\gamma\beta)(x), 0) \ .$$

□

Example Given a finite group G consider the Rim cartesian square of rings

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G]/N \\ {\scriptstyle \epsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle \epsilon} \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/|G| \end{array}$$

in which all the morphisms are onto, with

$$N = \sum_{g \in G} g \in \mathbb{Z}[G] \ , \ \epsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z} \ ; \ g \longrightarrow 1 \ .$$

The canonical isomorphism of rings $\mathbb{Z}[G] \longrightarrow (\mathbb{Z}[G]/N, 1, \mathbb{Z})$ has inverse

$$(\mathbb{Z}[G]/N, 1, \mathbb{Z}) \longrightarrow \mathbb{Z}[G] \ ; \ (b, b') \longrightarrow a + (b' - \epsilon(a))(N/|G|)$$

with $a \in \mathbb{Z}[G]$ any lift of $b \in \mathbb{Z}[G]/N$ (so that $\epsilon(a) \equiv b' (\mathrm{mod}\ |G|)$). In this case the boundary map in the Mayer-Vietoris sequence is given by

$$\begin{aligned} \partial : K_1(\mathbb{Z}/|G|) = (\mathbb{Z}/|G|)^{\times} &\longrightarrow K_0(\mathbb{Z}[G]) \ ; \\ \tau(\alpha') &\longrightarrow [\mathrm{im}(p(\alpha'))] - [\mathbb{Z}[G]^2] \end{aligned}$$

for any unit $\alpha' \in (\mathbb{Z}/|G|)^{\times}$, with $\beta, \gamma \in \mathbb{Z}$ such that $[\beta] = \alpha'$, $[\gamma] = \alpha'^{-1} \in \mathbb{Z}/|G|$, and $p(\alpha')$ the $\mathbb{Z}[G]$-module projection

$$p(\alpha') = \begin{pmatrix} 1 - (1-\beta\gamma)^2(N/|G|) & (2-\beta\gamma)(1-\beta\gamma)\beta(N/|G|) \\ \gamma(1-\beta\gamma)(N/|G|) & (1-\gamma\beta)^2(N/|G|) \end{pmatrix}$$
$$: \mathbb{Z}[G] \oplus \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G] \oplus \mathbb{Z}[G] \ .$$

□

Example For the quaternion group of order 8

$$G = Q(8) = \{\pm 1, \pm i, \pm j, \pm k\}$$

and the unit $\alpha' = 3 \in (\mathbb{Z}/8)^{\times}$ take $\beta = \gamma = 3 \in \mathbb{Z}$ in the previous Example. By the

Proposition the corresponding projection

$$p(\alpha') = \begin{pmatrix} 1-8N & 21N \\ & \\ -3N & 8N \end{pmatrix} : \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \longrightarrow \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)]$$

is such that $P(\alpha') \cong \mathrm{im}(p(\alpha'))$ is a f.g. projective $\mathbb{Z}[Q(8)]$-module isomorphic to the two-sided ideal

$$\langle 3, N\rangle = \mathrm{im}((3\ N) : \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \longrightarrow \mathbb{Z}[Q(8)]) \subset \mathbb{Z}[Q(8)]$$

of the type considered by Swan [9,§6], with an isomorphism

$$\langle 3, N\rangle \longrightarrow P(\alpha') \; ; \; 3x + Ny \longrightarrow x(1,3) + y(0,8) \quad (x, y \in \mathbb{Z}[Q(8)]) \; .$$

The reduced projective class

$$\partial\tau(3) = [P(\alpha')] \in \widetilde{K}_0(\mathbb{Z}[Q(8)]) = \mathbb{Z}/2$$

represents the generator (Martinet [3], Reiner and Ullom [8, §9]). As noted in [3] $P(\alpha')$ is isomorphic to the f.g. projective $\mathbb{Z}[Q(8)]$-module P_3 defined by the f.g. free $\mathbb{Z}$-module $\mathbb{Z}^8$ on 8 generators $\{e_0\} \cup \{e_s | s \in Q(8), s \neq 1\}$, with $Q(8)$ acting by

$$se_0 = e_0 \quad , \quad se_{s^{-1}} = 3e_0 - \sum_{t\neq 1} e_t \quad (s \in Q(8)) \; ,$$

$$se_t = e_{st} \quad (t \neq 1, s^{-1}) \; .$$

The element defined by

$$e_1 = 3e_0 - \sum_{t\neq 1} e_t \in P_3$$

is such that

$$se_1 = e_s \in P_3 \quad (s \neq 1) \; .$$

Thus

$$Ne_1 = e_1 + \sum_{t\neq 1} e_t = 3e_0 \in P_3 \; ,$$

and there is defined a $\mathbb{Z}[Q(8)]$-module isomorphism

$$\langle 3, N\rangle \longrightarrow P_3 \; ; \; 3x + Ny \longrightarrow xe_1 + ye_0 \; .$$

□

Given a ring A and a multiplicative subset $S \subset A$ of central non-zero divisors there is defined a cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ \big\downarrow & & \big\downarrow \\ \widehat{A} & \longrightarrow & S^{-1}\widehat{A} \end{array}$$

with $S^{-1}A$ the localization of A inverting S, and

$$\widehat{A} \ = \ \varprojlim_{s \in S} A/sA$$

the S-adic completion of A. The algebraic K-theory Mayer-Vietoris exact sequence determined by such a square

$$K_1(A) \longrightarrow K_1(S^{-1}A) \oplus K_1(\widehat{A}) \longrightarrow K_1(S^{-1}\widehat{A}) \xrightarrow{\ \partial\ }$$
$$K_0(A) \longrightarrow K_0(S^{-1}A) \oplus K_0(\widehat{A}) \longrightarrow K_0(S^{-1}\widehat{A})$$

is widely used in the computations of the K-groups of the group rings $A = \mathbb{Z}[G]$ of finite groups G, with $S = \mathbb{Z}$-$\{0\}$, $S^{-1}A = \mathbb{Q}[G]$. Again, the connecting map ∂ is defined by the pullback construction: if $\alpha : S^{-1}\widehat{A}^n \longrightarrow S^{-1}\widehat{A}^n$ is an automorphism of a f.g. free $S^{-1}\widehat{A}$-module then the pullback

$$P(\alpha) \ = \ \{(x,y) \in S^{-1}A^n \oplus \widehat{A}^n \,|\, \alpha(x) = y \in S^{-1}\widehat{A}^n\}$$

is a f.g. projective A-module, and

$$\partial \ : \ K_1(S^{-1}\widehat{A}) \longrightarrow K_0(A) \ ; \ \tau(\alpha : S^{-1}\widehat{A}^n \longrightarrow S^{-1}\widehat{A}^n) \longrightarrow [P(\alpha)] - [A^n] \ .$$

It is possible to obtain an explicit projection for $P(\alpha)$ from the material in Appendix A of Swan [10], but the actual formula is much more complicated than in the cartesian case. (I am grateful to Jim Davis for the reference to [10]).

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