Finite domination and Novikov rings

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Introduction

In order to distinguish between the combinatorial properties of finite simplicial complexes and the topology of compact polyhedra and compact manifolds it is necessary to consider infinite simplicial complexes, non-compact polyhedra, open manifolds, and algebraic K- and L-theory. The classic cases are the Milnor Hauptvermutung counterexamples of non-combinatorial homeomorphisms of compact polyhedra, the proof by Novikov of the topological invariance of the rational Pontrjagin classes, and the structure theory of Kirby and Siebenmann for high-dimensional compact topological manifolds. The open manifolds arise geometrically as tame ends: in the applications it is necessary to close them. The obstruction theory for closing tame ends of open manifolds is also the obstruction theory for deciding if a finitely dominated space is homotopy equivalent to a finite CW complex.

DEFINITION A topological space X is *finitely dominated* if it is a homotopy retract of a finite CW complex, i.e. if there exist a finite CW complex K, maps $f: X \longrightarrow K$, $g: K \longrightarrow X$ and a homotopy $gf \simeq 1: X \longrightarrow X$.

□

The Wall [16] finiteness obstruction $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of a finitely dominated space X is such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex. A tame end ϵ of an open n-dimensional manifold W has a neighbourhood $\overline{V} \subset W$ which is a finitely dominated infinite cyclic cover of a compact n-dimensional manifold V with $\pi_1(\overline{V}) = \pi_1(\epsilon)$. It is possible to express $\overline{V}$ as a union $\overline{V}^+ \cup \overline{V}^-$ with $\overline{V}^+ \cap \overline{V}^-$ a compact $(n-1)$-dimensional manifold and $\overline{V}^+, \overline{V}^-$ finitely dominated, with $\pi_1(\overline{V}^+) = \pi_1(\overline{V}^-) = \pi_1(\epsilon)$. The end obstruction of Siebenmann [11] is the finiteness obstruction $[\epsilon] = [\overline{V}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$, with $[\epsilon] = 0$ if (and for $n \geq 6$ only if) the tame end can be closed.

In this paper the Novikov rings of formal power series will be used to obtain a homological characterization of finite domination for an infinite cyclic cover of a finite CW complex. In Ranicki [10] this characterization will be applied to the study of fibre bundles over S^1, fibred knots, the bordism of diffeomorphisms and open book decompositions.

DEFINITION The *Novikov rings* of a ring A are the completions $A((z))$, $A((z^{-1}))$ of the Laurent polynomial extension $A[z,z^{-1}]$ given by

$$A((z)) \;=\; \{\sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \leq 0 \,|\, a_j \neq 0 \in A\} \text{ finite}\}\;,$$

$$A((z^{-1})) \;=\; \{\sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \geq 0 \,|\, a_j \neq 0 \in A\} \text{ finite}\}$$

with intersection

$$A((z)) \cap A((z^{-1})) \;=\; A[z,z^{-1}] \;=\; \{\sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \in \mathbb{Z} \,|\, a_j \neq 0 \in A\} \text{ finite}\}\;.$$

□

Traditional Morse theory deals with $\mathbb{R}$-valued functions on compact manifolds. Novikov [6] suggested the use of these rings in counting the critical points of S^1-valued Morse functions on compact manifolds M, initially in the case $\pi_1(M) = \mathbb{Z}$, $A = \mathbb{Z}$. Pazhitnov [7] showed how to apply the Novikov rings with $A = \mathbb{Z}[\pi]$ to the Morse theory of S^1-valued Morse functions on arbitrary compact manifolds M with $\pi_1(M) = \pi \times \mathbb{Z}$. The Novikov rings also appear in the Morse-theoretic chain complex construction of Floer homology by Hofer and Salamon [2].

DEFINITION The Λ-*coefficient homology* of a connected CW complex X is

$$H_*(X;\Lambda) \;=\; H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X}))$$

with Λ a $\mathbb{Z}[\pi_1(X)]$-module and $C(\widetilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex of the universal cover $\widetilde{X}$.

□

Algebraic K-theory decides if a finitely dominated space is homotopy equivalent to a finite CW complex. Homology with coefficients in the Novikov rings decides if a space is finitely dominated:

THEOREM *Let X be a finite CW complex with universal cover $\widetilde{X}$ and fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$, so that $\mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi][z,z^{-1}]$. The infinite cyclic cover $\overline{X} = \widetilde{X}/\pi$ is finitely dominated if and only if X is $\mathbb{Z}[\pi]((z))$- and $\mathbb{Z}[\pi]((z^{-1}))$-acyclic*

$$H_*(X;\mathbb{Z}[\pi]((z))) \;=\; H_*(X;\mathbb{Z}[\pi]((z^{-1}))) \;=\; 0\;.$$

□

The Theorem is proved in §5 by an application to the cellular chain complex $C(\widetilde{X})$

and the group ring $A = \mathbb{Z}[\pi]$ of the corresponding characterization of finite domination for a finite f.g. (= finitely generated) free $A[z, z^{-1}]$-module chain complex C with arbitrary A. By definition, C is finitely dominated if and only if it is A-module chain equivalent to a finite f.g. projective A-module chain complex. It is known from the work of Wall [16] that $\overline{X}$ is finitely dominated if and only if $C(\widetilde{X})$ is finitely dominated. It is proved in §5 that C is finitely dominated if and only if

$$H_*(A((z)) \otimes_{A[z,z^{-1}]} C) \ = \ H_*(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) \ = \ 0 \ .$$

However, the proof is best understood in terms of the topology of infinite cyclic covers of compact spaces. The two conditions in the Theorem arise as follows: by the CW analogue of manifold transversality it is possible to decompose the infinite cyclic cover $\overline{X}$ of X as a union $\overline{X} = \overline{X}^+ \cup \overline{X}^-$ of infinite subcomplexes $\overline{X}^+, \overline{X}^- \subset \overline{X}$ with $\overline{X}^+ \cap \overline{X}^-$ finite and $\pi_1(\overline{X}^+) = \pi_1(\overline{X}^-) = \pi_1(\overline{X}) = \pi$. The cover $\overline{X}$ is finitely dominated if and only if both $\overline{X}^+$ and $\overline{X}^-$ are finitely dominated. It turns out that $H_*(X; \mathbb{Z}[\pi]((z))) = 0$ if and only if $\overline{X}^-$ is finitely dominated, and $H_*(X; \mathbb{Z}[\pi]((z^{-1}))) = 0$ if and only if $\overline{X}^+$ is finitely dominated. See §6 for the connection with the algebraic chain complex theory of tame ends developed in Hughes and Ranicki [3].

Example The universal cover of S^1 is $\overline{S}^1 = \mathbb{R}$ with

$$C(\mathbb{R}) \ : \ \dots \longrightarrow 0 \longrightarrow \mathbb{Z}[z, z^{-1}] \xrightarrow{1-z} \mathbb{Z}[z, z^{-1}]$$

and

$$\begin{aligned} H_*(S^1; \mathbb{Z}[z, z^{-1}]) \ &= \ H_*(1 - z : \mathbb{Z}[z, z^{-1}] \longrightarrow \mathbb{Z}[z, z^{-1}]) \ = \ \mathbb{Z} \ , \\ H_*(S^1; \mathbb{Z}((z))) \ &= \ H_*(1 - z : \mathbb{Z}((z)) \longrightarrow \mathbb{Z}((z))) \ = \ 0 \ , \\ H_*(S^1; \mathbb{Z}((z^{-1}))) \ &= \ H_*(1 - z : \mathbb{Z}((z^{-1})) \longrightarrow \mathbb{Z}((z^{-1}))) \ = \ 0 \ , \end{aligned}$$

since $1 - z \in \mathbb{Z}[z, z^{-1}]$ becomes a unit in $\mathbb{Z}((z))$ and $\mathbb{Z}((z^{-1}))$ with

$$(1 - z)^{-1} \ = \ \sum_{i=0}^{\infty} z^i \in \mathbb{Z}((z)) \ \ , \ \ (1 - z)^{-1} \ = \ - \sum_{i=-\infty}^{-1} z^i \in \mathbb{Z}((z^{-1})) \ .$$

□

Pazhitnov [7] asked if for a compact manifold M with universal cover $\widetilde{M}$ and fundamental group $\pi_1(M) = \pi \times \mathbb{Z}$ finite domination of the infinite cyclic cover $\overline{M} = \widetilde{M}/\pi$ is equivalent to $H_*(M; \mathbb{Z}[\pi]((z))) = 0$. The Theorem shows that such is indeed the case, since for a manifold $H_*(M; \mathbb{Z}[\pi]((z))) = 0$ if and only if $H_*(M; \mathbb{Z}[\pi]((z^{-1}))) = 0$ by Poincaré duality.

A compact manifold M with a finitely dominated infinite cyclic cover $\overline{M}$ is a potential fibre bundle over S^1. Farrell [1] and Siebenmann [14] formulated the obstruction to M being a fibre bundle over S^1 as a Whitehead torsion in $Wh(\pi_1(M))$. Pazhitnov [7] used the S^1-valued Morse theory of Novikov [6] to formulate the fibering obstruction as an element in the group $Wh((\pi_1(M))$ defined below. The following corollary clarifies the precise relationship between the two fibering obstructions. It will be proved in detail in Ranicki [10].

COROLLARY *Let M be a compact n-dimensional manifold with universal cover $\widetilde{M}$ and fundamental group $\pi_1(M) = \pi \times \mathbb{Z}$. Assume that the infinite cyclic cover $\overline{M} = \widetilde{M}/\pi$ is finitely dominated, so that $H_*(M; \mathbb{Z}[\pi]((z))) = 0$. For $n \geq 6$ the manifold M fibres over S^1 if and only if the $\mathbb{Z}[\pi]((z))$-coefficient Reidemeister torsion*

$$\begin{aligned}\tau(M; \mathbb{Z}[\pi]((z))) \ &= \ \tau(\mathbb{Z}[\pi]((z)) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\widetilde{M})) \\ &\in Wh((\pi \times \mathbb{Z})) \ = \ K_1(\mathbb{Z}[\pi]((z)))/\{\pm \pi \times \mathbb{Z}\}\end{aligned}$$

is such that

$$\tau(M; \mathbb{Z}[\pi]((z))) \ = \ 0 \in Wh((\pi \times \mathbb{Z}))/\{1 + z\mathbb{Z}[\pi][[z]]\} \ .$$

□

The fundamental group of a connected infinite cyclic cover $\overline{X}$ of a connected space X is the α-twisted extension of $\pi_1(\overline{X})$ by the infinite cyclic group $\mathbb{Z}$

$$\pi_1(X) \ = \ \pi_1(\overline{X}) \times_\alpha \mathbb{Z} \ ,$$

with $\alpha = \zeta_* : \pi_1(\overline{X}) \longrightarrow \pi_1(\overline{X})$ the automorphism induced (up to inner automorphisms) by a generating covering translation $\zeta : \overline{X} \longrightarrow \overline{X}$. For the sake of simplicity we shall only be concerned with the untwisted case $\alpha = 1$, $\pi_1(X) = \pi_1(\overline{X}) \times \mathbb{Z}$, but all the results obtained here extend to the general case with arbitrary α and with the additional hypothesis that $\pi_1(\overline{X})$ is finitely presented.

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§1. **Transversality**

The homological characterization of finite domination is an application of an algebraic theory of transversality for chain complexes over polynomial extensions, which mimics the existence of compact fundamental domains for infinite cyclic covers of compact manifolds.

The infinite cyclic covers $\overline{X}$ of a space X are classified by the homotopy classes of maps $c : X \longrightarrow S^1$ with

$$\overline{X} \;=\; \{(x,t) \in X \times \mathbb{R} \,|\, c(x) = [t] \in S^1 = \mathbb{R}/\mathbb{Z}\} \;.$$

The connected covers $\overline{X}$ of a connected space X correspond to maps c with $c_* : \pi_1(X) \longrightarrow \pi_1(S^1) = \mathbb{Z}$ onto, in which case the fundamental group exact sequence of the fibration $\overline{X} \longrightarrow X \xrightarrow{c} S^1$ expresses $\pi_1(X)$ as an extension of $\pi_1(\overline{X})$ by $\mathbb{Z}$

$$\{1\} \longrightarrow \pi_1(\overline{X}) \longrightarrow \pi_1(X) \longrightarrow \mathbb{Z} \longrightarrow \{1\} \;.$$

The connected covers $\overline{X}$ are thus classified by the normal subgroups $\pi \subset \pi_1(X)$ with infinite cyclic quotient $\pi_1(X)/\pi = \mathbb{Z}$, such that

$$\overline{X} \;=\; \widetilde{X}/\pi \;\;,\;\; \pi_1(\overline{X}) \;=\; \pi \;,$$

with $\widetilde{X}$ the universal cover of X. As stipulated in the Introduction it will be assumed that the generating covering translation

$$\zeta \;:\; \overline{X} \longrightarrow \overline{X} \;;\; x \longrightarrow zx$$

induces an inner automorphism $\zeta_* : \pi_1(\overline{X}) = \pi \longrightarrow \pi$, so that

$$\pi_1(X) \;=\; \pi \times \mathbb{Z} \;\;,\;\; \mathbb{Z}[\pi_1(X)] \;=\; \mathbb{Z}[\pi][z,z^{-1}] \;.$$

Manifold transversality *Let M be a compact n-dimensional manifold with $\pi_1(M) = \pi \times \mathbb{Z}$ and infinite cyclic cover $\overline{M} = \widetilde{M}/\pi$. The classifying map $c : M \longrightarrow S^1$ can be made transverse regular at a point $* \in S^1$, and cutting M along a codimension 1 submanifold $N = c^{-1}(*) \subset M$ gives a compact fundamental domain $(M_N; N, \zeta N)$ with*

$$M \;=\; M_N \cup N \times [0,1] \;\;,\;\; \overline{M} \;=\; \bigcup_{j=-\infty}^{\infty} \zeta^j M_N \;\;,\;\; \pi_1(M_N) \;=\; \pi_1(N) \;=\; \pi \;.$$

The two inclusions $f, g : N \longrightarrow M_N$ determine an exact sequence of $\mathbb{Z}[\pi][z,z^{-1}]$-module chain complexes

$$0 \longrightarrow C(\widetilde{N})[z,z^{-1}] \xrightarrow{f-zg} C(\widetilde{M}_N)[z,z^{-1}] \longrightarrow C(\widetilde{M}) \longrightarrow 0$$

with $\widetilde{M}$, $\widetilde{N}$, $\widetilde{M}_N$ the universal covers of M, N, M_N.

□

Combinatorial transversality *A finite CW complex X with $\pi_1(X) = \pi \times \mathbb{Z}$ and infinite cyclic cover $\overline{X} = \widetilde{X}/\pi$ is simple homotopy equivalent to an identification space of the type*

$$X' \;=\; X_Y \cup Y \times [0,1]$$

for a finite CW triad $(X_Y;Y,\zeta Y)$ *such that*

$$\overline{X} \simeq \overline{X}' = \bigcup_{j=-\infty}^{\infty} \zeta^j X_Y \quad , \quad \pi_1(X_Y) = \pi_1(Y) = \pi \ .$$

□

Combinatorial transversality can be obtained from manifold transversality by using a regular neighbourhood of X in some high-dimensional Euclidean space. Alternatively, there is a CW analogue of the following algebraic transversality:

Algebraic transversality (Waldhausen [15], Ranicki [9, 8.12]) *Every finite based f.g. free* $A[z,z^{-1}]$*-module chain complex* C *is such that there is defined a simple exact sequence*

$$0 \longrightarrow D[z,z^{-1}] \xrightarrow{f-zg} E[z,z^{-1}] \longrightarrow C \longrightarrow 0$$

for some finite based f.g. free A*-module chain complexes* D,E *and* A*-module chain maps* $f,g : D \longrightarrow E$*. In fact,* E *can be chosen to be an* A*-module subcomplex of* C *with base elements of the type* $z^j b$ *for the base elements* b *of* C*, and*

$$f : D = E \cap z^{-1}E \longrightarrow E \ ; \ x \longrightarrow zx \ ,$$
$$g : D = E \cap z^{-1}E \longrightarrow E \ ; \ x \longrightarrow x \ .$$

□

§2. <u>**The mapping torus**</u>

The mapping torus is a homotopy model for a space with an infinite cyclic cover.

Definition The *mapping torus* of a self-map $f : X \longrightarrow X$ is the identification space

$$T(f) = X \times [0,1]/\{(x,0) = (f(x),1) \,|\, x \in X\}$$

with canonical infinite cyclic cover the two-sided mapping telescope

$$\overline{T}(f) = \Big(\coprod_{n=-\infty}^{\infty} X \times [0,1] \times \{n\}\Big)/\{(x,0,n) = (f(x),1,n+1) \,|\, x \in X\} \ .$$

□

An infinite cyclic cover of a space X corresponds to a mapping torus in the homotopy type of X:

Proposition *If* $p : \overline{X} \longrightarrow X$ *is the projection of an infinite cyclic cover with covering* *translation* $\zeta : \overline{X} \longrightarrow \overline{X}$ *then*

$$q \ : \ T(\zeta) \longrightarrow X \ ; \ (\overline{x}, t) \longrightarrow p(\overline{x})$$

is a homotopy equivalence.

□

'WHITEHEAD LEMMA' FOR MAPPING TORI (Mather [4]) *For any maps* $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ *there are defined inverse homotopy equivalences*

$$T(gf) \longrightarrow T(fg) \ ; \ (x, s) \longrightarrow (f(x), s) \ ,$$
$$T(fg) \longrightarrow T(gf) \ ; \ (y, t) \longrightarrow (g(y), t) \ .$$

□

§3. **Homotopy finiteness**

Finitely dominated spaces are topological analogues of f.g. projective modules, which are the direct summands of f.g. free modules. The difference between the homotopy types of finite and finitely dominated CW complexes is precisely the difference between f.g. projective and f.g. free modules.

An infinite cyclic cover $\overline{X}$ of a finite CW complex X is an infinite CW complex: in general it is not even homotopy equivalent to a finite CW complex.

A finite f.g. free $A[z, z^{-1}]$-module chain complex C is not finitely generated over A, and is not in general chain equivalent to a finite f.g. projective A-module chain complex.

DEFINITION A topological space X is *homotopy finite* if it is homotopy equivalent to a finite CW complex.

□

EXAMPLE If $X = K \times S^1$ for a finite CW complex K then the infinite cyclic cover $\overline{X} = K \times \mathbb{R} \simeq K$ is homotopy finite.

□

PROPOSITION *A topological space* X *is finitely dominated if and only if* $X \times S^1$ *is homotopy finite.*
PROOF Given a finite domination $(K, f : X \longrightarrow K, g : K \longrightarrow X, gf \simeq 1 : X \longrightarrow X)$

apply the Mather lemma to obtain a homotopy equivalence

$$X \times S^1 \simeq T(gf) \simeq T(fg) = \text{finite } CW \text{ complex} .$$

The converse is trivial.

□

In the simply-connected case $\pi_1(X) = \{1\}$ there is already a homological criterion for finite domination:

PROPOSITION (Milnor) *A simply-connected CW complex X is finitely dominated if and only if the homology $H_*(X)$ is a f.g. $\mathbb{Z}$-module, in which case X is homotopy finite.*

□

EXAMPLE If $X = S^1 \vee S^2$ then $\overline{X} = \mathbb{R} \cup \mathbb{Z} \times S^2$ is not finitely dominated, since $H_2(\overline{X}) = \mathbb{Z}[z, z^{-1}]$ is not a f.g. $\mathbb{Z}$-module. The cellular $\mathbb{Z}[z, z^{-1}]$-module chain complex is

$$C(\overline{X}) : \ldots \longrightarrow 0 \longrightarrow \mathbb{Z}[z, z^{-1}] \xrightarrow{0} \mathbb{Z}[z, z^{-1}] \xrightarrow{1-z} \mathbb{Z}[z, z^{-1}]$$

and

$$H_2(X; \mathbb{Z}((z))) = \mathbb{Z}((z)) , \quad H_2(X; \mathbb{Z}((z^{-1}))) = \mathbb{Z}((z^{-1})) .$$

□

In the non-simply-connected case $\pi_1(X) \neq \{1\}$ finite domination and homotopy finiteness are detected on the chain level by:

WALL FINITENESS OBSTRUCTION [16] *Let X be a connected CW complex with universal cover $\widetilde{X}$, and let $C(\widetilde{X})$ be the cellular free $\mathbb{Z}[\pi_1(X)]$-module chain complex. X is finitely dominated (resp. homotopy finite) if and only if $\pi_1(X)$ is finitely presented and $C(\widetilde{X})$ is chain equivalent to a finite f.g. projective (resp. free) $\mathbb{Z}[\pi_1(X)]$-module chain complex. If X is finitely dominated the reduced projective class of any finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex P in the chain homotopy type of $C(\widetilde{X})$*

$$\begin{aligned} [X] &= \sum_{r=0}^{\infty} (-)^r [P_r] \\ &\in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) = \frac{\{\text{f.g. projective } \mathbb{Z}[\pi_1(X)]\text{-modules}\}}{\{\text{f.g. free } \mathbb{Z}[\pi_1(X)]\text{-modules}\}} \end{aligned}$$

is the finiteness obstruction, such that X is homotopy finite if and only if $[X] = 0$.

□

SIEBENMANN END OBSTRUCTION [11] *Let $n \geq 6$. An open n-dimensional manifold W with one tame end ϵ is the interior of a compact n-dimensional manifold if and only if there exists a finitely dominated cocompact closed neighbourhood of the end $V \subseteq W$ with $\pi_1(V) = \pi_1(\epsilon)$ and $[V] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$.*

□

§4. **Bands**

DEFINITION (Siebenmann [13]) A *band* is a finite CW complex X with a finitely dominated infinite cyclic cover $\overline{X}$.

□

PROPOSITION *Every finitely dominated CW complex X is homotopy equivalent to the infinite cyclic cover $\overline{Y}$ of a band Y.*

PROOF For any finite domination $(K, f : X \longrightarrow K, g : K \longrightarrow X, gf \simeq 1 : X \longrightarrow X)$ there is defined a homotopy equivalence

$$X \times S^1 \ \simeq\ T(gf)\ \simeq\ T(fg)\ =\ \text{finite } CW \text{ complex}\ =\ Y$$

with $X \simeq X \times \mathbb{R} \simeq \overline{T}(gf) \simeq \overline{T}(fg) \simeq \overline{Y}$. Y is a band.

□

EXAMPLE (Siebenmann [12]) A non-compact n-dimensional manifold W with one tame end ϵ has an open neighbourhood $\overline{X} \subseteq W$ which is the infinite cyclic cover of a compact n-dimensional manifold band X, the *wrapping up* of the end, with $\overline{X} = \overline{X}^+ \cup \overline{X}^-$ such that

$$\pi_1(\epsilon)\ =\ \pi_1(\overline{X}^+)\ =\ \pi_1(\overline{X})\ ,$$
$$[\epsilon]\ =\ [\overline{X}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])\ .$$

The idea is to use tameness and a proper Morse function $W \longrightarrow \mathbb{R}^+$ to lift $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$; $x \longrightarrow x+1$ to a shift map $T : W \longrightarrow W$ which is the covering translation on an open neighbourhood of the end.

□

§5. **Algebraic finite domination**

DEFINITION A finite f.g. free $A[z, z^{-1}]$-module chain complex C is *A-finitely dominated* if it is A-module chain equivalent to a finite chain complex of f.g. projective A-modules, in which case it is a *chain complex band*.

□

Example (Cayley-Hamilton, Milnor [5]) If F is a field then a finite f.g. free $F[z,z^{-1}]$-module chain complex C is F-finitely dominated if and only if

$$H_*(F(z)\otimes_{F[z,z^{-1}]}C) \;=\; 0$$

with $F(z)$ the function field of F (= the quotient field of $F[z]$).
Proof If C is F-finitely dominated the homology $H_*(C)$ is a finite-dimensional F-vector space, and the characteristic polynomial of the 'monodromy'

$$\zeta \;:\; H_*(C) \longrightarrow H_*(C) \;;\; x \longrightarrow zx$$

is a non-zero polynomial

$$p(z) \;=\; \det(z-\zeta : H_*(C)[z]{\longrightarrow}H_*(C)[z]) \in F[z]$$

such that $p(z)H_*(C)=0$.
Conversely, if $H_*(F(z)\otimes_{F[z,z^{-1}]}C)=0$ there exists a non-zero polynomial $p(z)\in F[z]$ such that $p(z)H_*(C)=0$. The ring $F[z,z^{-1}]$ is noetherian, so that $H_*(C)$ is a f.g. $F[z,z^{-1}]$-module, hence a f.g. $F[z,z^{-1}]/(p(z))$-module, and hence a f.g. F-module.

□

A connected finite CW complex X with $\pi_1(X)=\pi\times\mathbb{Z}$ is a band if and only if the cellular $\mathbb{Z}[\pi][z,z^{-1}]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is a chain complex band, by Wall's chain level criterion for finite domination.

Theorem *A finite f.g. free $A[z,z^{-1}]$-module chain complex C is A-finitely dominated if and only if*

$$H_*(A((z))\otimes_{A[z,z^{-1}]}C) \;=\; H_*(A((z^{-1}))\otimes_{A[z,z^{-1}]}C) \;=\; 0 \;.$$

Proof Let $i : A{\longrightarrow}A[z,z^{-1}]$ be the inclusion, and let

$$\begin{aligned} i_! &= \text{induction} \;:\; \{\,A\text{-modules}\} \longrightarrow \{\,A[z,z^{-1}]\text{-modules}\} \;, \\ i^! &= \text{restriction} \;:\; \{\,A[z,z^{-1}]\text{-modules}\} \longrightarrow \{\,A\text{-modules}\} \;. \end{aligned}$$

Given an A-module P let

$$P[z,z^{-1}] \;=\; i_!P \;=\; A[z,z^{-1}]\otimes_A P \quad , \quad P((z)) \;=\; A((z))\otimes_A P$$

be the induced $A[z,z^{-1}]$- and $A((z))$-modules. For any A-modules P, Q there is a natural injection

$$\mathrm{Hom}_{A((z))}(P((z)),Q((z))) \longrightarrow \mathrm{Hom}_A(P,Q)((z)) \;.$$

For f.g. projective P this is an isomorphism, with an identification

$$\mathrm{Hom}_{A((z))}(P((z)),Q((z))) \;=\; \mathrm{Hom}_A(P,Q)((z)) \;.$$

If $h : P{\longrightarrow}P$ is an automorphism of a f.g. projective A-module P then $z-h$:

$P((z)) \longrightarrow P((z))$ is an automorphism, with inverse

$$(z-h)^{-1} \;=\; (-h)^{-1}\sum_{j=0}^{\infty}(h^{-1}z)^j \;:\; P((z)) \longrightarrow P((z)) \;.$$

This is definitely false if P is not f.g. projective, e.g. if

$$h \;:\; P \;=\; i^!A[z,z^{-1}] \longrightarrow P \;;\; \sum_{j=-\infty}^{\infty} a_j z^j \longrightarrow \sum_{j=-\infty}^{\infty} a_j z^{j+1} \;.$$

A finite f.g. free $A[z,z^{-1}]$-module chain complex C is chain equivalent to the algebraic mapping torus

$$T(\zeta) \;=\; \mathcal{C}(z-\zeta : i_!i^!C \longrightarrow i_!i^!C)$$

of the A-module automorphism

$$\zeta \;:\; i^!C \longrightarrow i^!C \;;\; x \longrightarrow zx$$

of the infinitely generated free A-module chain complex $i^!C$. If C is A-finitely dominated then $i^!C$ is A-module chain equivalent to a finite f.g. projective A-module chain complex P with an A-module chain equivalence $h : P \longrightarrow P$ such that C is $A[z,z^{-1}]$-module chain equivalent to $T(h)$. Since P is f.g. projective it is possible to identify

$$\mathrm{Hom}_{A((z))}(P((z)),P((z))) \;=\; \mathrm{Hom}_A(P,P)((z)) \;.$$

The $A[z,z^{-1}]$-module chain map $z-h : P[z,z^{-1}] \longrightarrow P[z,z^{-1}]$ induces an $A((z))$-module chain equivalence $z-h : P((z)) \longrightarrow P((z))$, so that

$$A((z)) \otimes_{A[z,z^{-1}]} C \;\simeq\; A((z)) \otimes_{A[z,z^{-1}]} T(h) \;\simeq\; 0 \;.$$

Similarly for $A((z^{-1}))$.

For the converse use the cartesian squares of rings

$$\begin{array}{ccc} A[z] & \longrightarrow & A[z,z^{-1}] \\ \big| & & \big| \\ A[[z]] & \longrightarrow & A((z)) \end{array} \qquad \begin{array}{ccc} A[z^{-1}] & \longrightarrow & A[z,z^{-1}] \\ \big| & & \big| \\ A[[z^{-1}]] & \longrightarrow & A((z^{-1})) \end{array} \;.$$

By algebraic transversality, every finite f.g. free $A[z]$-module chain complex C is isomorphic to one of the type

$$C \;=\; A[z,z^{-1}] \otimes_{A[z]} C^+ \;=\; A[z,z^{-1}] \otimes_{A[z^{-1}]} C^-$$

for a finite f.g. free $A[z]$-module chain complex C^+ and a finite f.g. free $A[z^{-1}]$-module chain complex C^-, with a Mayer-Vietoris exact sequence

$$0 \longrightarrow C^+ \cap C^- \longrightarrow C^+ \oplus C^- \longrightarrow C \longrightarrow 0$$

such that $C^+ \cap C^-$ is a finite f.g. free A-module chain complex. The chain complex C is A-finitely dominated if and only if C^+ and C^- are A-finitely dominated. We shall show that $A((z^{-1})) \otimes_{A[z,z^{-1}]} C \simeq 0$ implies that C^+ is A-finitely dominated. The f.g. free $A[z^{-1}]$-module chain complex C^- fits into an exact sequence of $A[z^{-1}]$-module chain complexes

$$0 \longrightarrow C^- \xrightarrow{\ i\ } C^-[z,z^{-1}] \oplus C^-[[z^{-1}]] \longrightarrow C^-((z^{-1})) \longrightarrow 0 \ ,$$

with

$$\begin{aligned} C^-[z,z^{-1}] &= A[z,z^{-1}] \otimes_{A[z^{-1}]} C^- \ = \ C \ , \\ C^-[[z^{-1}]] &= A[[z^{-1}]] \otimes_{A[z^{-1}]} C^- \ , \\ C^-((z^{-1})) &= A((z^{-1})) \otimes_{A[z^{-1}]} C^- \ = \ A((z^{-1})) \otimes_{A[z,z^{-1}]} C \ . \end{aligned}$$

By hypothesis $H_*(C^-((z^{-1}))) = 0$, so i is an $A[z^{-1}]$-module chain equivalence. Let

$$j \ : \ C^+ \cap C^- \longrightarrow C^+ \oplus C^-[[z^{-1}]]$$

be the A-module chain map defined by inclusions in each component. The algebraic mapping cones of i and j are chain equivalent A-module chain complexes, since $C/C^- \cong C^+/(C^+ \cap C^-)$. But i is a chain equivalence, so that j is also a chain equivalence. Since $C^+ \cap C^-$ is a finite f.g. free A-module chain complex this shows that both C^+ and $C^-[[z^{-1}]]$ are A-finitely dominated. Similarly, $A((z)) \otimes_{A[z,z^{-1}]} C \simeq 0$ implies that C^- is A-finitely dominated.

□

Corollary *A connected finite CW complex X with universal cover $\widetilde{X}$ and fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$ is a band if and only if*

$$H_*(\mathbb{Z}[\pi]((z)) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\widetilde{X})) \ = \ H_*(\mathbb{Z}[\pi]((z^{-1})) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\widetilde{X})) \ = \ 0 \ .$$

□

Example For any f.g. projective A-module $P = \mathrm{im}(p = p^2 : A^k \longrightarrow A^k)$ there is defined a chain complex band

$$C \ : \ \dots \longrightarrow 0 \longrightarrow C_1 \ = \ A[z,z^{-1}]^k \xrightarrow{d=1-p+(1-z)p} C_0 \ = \ A[z,z^{-1}]^k$$

with $H_0(C) = P$ and $H_0(A((z)) \otimes_{A[z,z^{-1}]} C) = H_0(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) = 0$, with

$$(1 \otimes d)^{-1} \ = \ 1 + p \sum_{j=1}^{\infty} z^j \ : \ A((z)) \otimes_{A[z,z^{-1}]} C_0 \longrightarrow A((z)) \otimes_{A[z,z^{-1}]} C_1 \ ,$$

$$(1 \otimes d)^{-1} \ = \ 1 - p \sum_{j=-\infty}^{0} z^j \ : \ A((z^{-1})) \otimes_{A[z,z^{-1}]} C_0 \longrightarrow A((z^{-1})) \otimes_{A[z,z^{-1}]} C_1 \ .$$

This is an algebraic analogue of the Mather construction of a CW band X in the homotopy type of $Y \times S^1$ for a finitely dominated CW complex Y.

□

§6. **Ends of complexes**

The proof of the Theorem in §5 will now be related to the chain complex properties of tame ends, as promised in the Introduction.

Given a space W let

$$W^\infty \;=\; W \cup \{\infty\}$$

be the one-point compactification.

Definition (Quinn [8]) (i) The *homotopy link of* ∞ *in* W^∞ is the space $e(W)$ of proper paths

$$\omega \;:\; (0,1] \longrightarrow W$$

or equivalently maps

$$\omega \;:\; ([0,1],\{0\}) \longrightarrow (W^\infty,\{\infty\})$$

such that $\omega^{-1}(\infty) = \{0\}$.
(ii) A non-compact space W is *tame at* ∞ if there exists a closed cocompact subspace $V \subseteq W$ such that the inclusion $V \times \{1\} \longrightarrow W$ extends to a proper map $q : V \times (0,1] \longrightarrow W$, or equivalently a map

$$\bar{q} \;:\; (V \times (0,1])^\infty \;=\; V^\infty \wedge [0,1] \longrightarrow W^\infty$$

such that $(\bar{q})^{-1}(\infty) = \infty$.

□

If W is tame at ∞ the homotopy link is such that there is defined a homotopy pushout

$$\begin{array}{ccc} e(W) & \longrightarrow & \{\infty\} \\ {\scriptstyle p_W}\big\downarrow & & \big\downarrow \\ W & \xrightarrow{\;\;i\;\;} & W^\infty \end{array}$$

with $i : W \longrightarrow W^\infty$ the inclusion and

$$p_W \;:\; e(W) \longrightarrow W \;;\; \omega \longrightarrow \omega(1)\;.$$

For such W the homology groups of $e(W)$ fit into an exact sequence

$$\dots \longrightarrow H_n(e(W)) \longrightarrow H_n(W) \longrightarrow H_n^{lf}(W) \longrightarrow H_{n-1}(e(W)) \longrightarrow \dots ,$$

with $H_*^{lf}(W) = H_*(W^\infty, \{\infty\})$ the locally finite homology groups of W.

The homotopy link is a homotopy theoretic model for the topology of an end of a non-compact space which is tame at ∞. An infinite cyclic cover $W = \overline{X}$ of a connected finite CW complex X has two ends. To each path $\omega \in e(W)$ assign a sign $\pm$ according as to which end contains $\omega((0,1])$, and lift $p_W : e(W) \longrightarrow W$ to a map

$$\overline{p}_W \ : \ e(W) \longrightarrow W \times \{\pm\} \ = \ W \sqcup W \ ; \ \omega \longrightarrow (\omega(1), \pm) \ .$$

If W is finitely dominated then W is tame at ∞, and $\overline{p}_W$ is a homotopy equivalence.

The combinatorial and chain level properties of the homotopy link construction are investigated in Hughes and Ranicki [3], including the definition of the *end complex* $e(C)$ of a based free A-module chain complex C. If

$$C_r \ = \ \sum_{I_r} A \ \ (r \in \mathbb{Z})$$

then the A-module chain complex $e(C)$ is defined to be the algebraic mapping cone of the inclusion $i : C \longrightarrow C^{lf}$

$$e(C) \ = \ \mathcal{C}(i : C \longrightarrow C^{lf})_{*+1}$$

with C^{lf} the locally finite chain complex given by

$$C_r^{lf} \ = \ \prod_{I_r} A \ \ (r \in \mathbb{Z}) \ .$$

It is shown in [3] that if W is tame at ∞, and $\widetilde{W}$ is the universal cover of W, then the pullback cover $\widetilde{e(W)}$ of $e(W)$ is such that

$$H_*(\widetilde{e(W)}) \ = \ H_*(e(C(\widetilde{W})))$$

with $\widetilde{p}_W : H_*(\widetilde{e(W)}) \longrightarrow H_*(\widetilde{W})$ induced by the projection $e(C(\widetilde{W})) \longrightarrow C(\widetilde{W})$.

Let X be a finite CW complex with an infinite cyclic cover $W = \overline{X}$ classified by

$$c \ = \ \text{projection} \ : \ \pi_1(X) \ = \ \pi \times \mathbb{Z} \longrightarrow \mathbb{Z} \ ,$$

with a finite fundamental domain $(X_Y;Y,\zeta Y)$ such that

$$W \;=\; W^+\cup W^- \;=\; \bigcup_{j=-\infty}^{\infty}\zeta^j(X_Y;Y,\zeta Y)\ ,$$

$$W^+ \;=\; \bigcup_{j=0}^{\infty}\zeta^j X_Y\quad,\quad W^- \;=\; \bigcup_{j=-\infty}^{-1}\zeta^j X_Y\quad,\quad W^+\cap W^- \;=\; Y\ ,$$

$$\pi_1(W) \;=\; \pi_1(W^+) \;=\; \pi_1(W^-) \;=\; \pi_1(Y) \;=\; \pi_1(X_Y) \;=\; \pi\ .$$

Then W is finitely dominated if and only if W^+ and W^- are finitely dominated, in which case the composites

$$e(W^+)\xrightarrow{p_{W^+}} W^+\longrightarrow W\ ,$$

$$e(W^-)\xrightarrow{p_{W^-}} W^-\longrightarrow W\ ,$$

$$e(W^+)\sqcup e(W^-)\longrightarrow e(W)$$

are homotopy equivalences. Let $\widetilde{Y},\widetilde{X}_Y,\widetilde{W}$ be the universal covers of Y,X_Y,W, so that

$$\widetilde{W} \;=\; \widetilde{W}^+\cup_{\widetilde{Y}}\widetilde{W}^- \;=\; \bigcup_{j=-\infty}^{\infty}\zeta^j(\widetilde{X}_Y;\widetilde{Y},\zeta\widetilde{Y})\ ,$$

$$\widetilde{W}^+ \;=\; \bigcup_{j=0}^{\infty}\zeta^j\widetilde{X}_Y\quad,\quad \widetilde{W}^- \;=\; \bigcup_{j=-\infty}^{-1}\zeta^j\widetilde{X}_Y\quad,\quad \widetilde{W}^+\cap\widetilde{W}^- \;=\; \widetilde{Y}\ .$$

Let $A=\mathbb{Z}[\pi]$, so that $A[z,z^{-1}]=\mathbb{Z}[\pi\times\mathbb{Z}]$. The cellular chain complexes

$$C(\widetilde{W}^+) \;=\; C^+\quad,\quad C(\widetilde{W}^-) \;=\; C^-$$

are such that C^+ (resp. C^-) is a based f.g. free $A[z]$- (resp. $A[z^{-1}]$-) module chain complex. The cellular chain complex of $\widetilde{W}$ is given by

$$C(\widetilde{W}) \;=\; C \;=\; A[z,z^{-1}]\otimes_{A[z]}C^+ \;=\; A[z,z^{-1}]\otimes_{A[z^{-1}]}C^-\ ,$$

and the locally π-finite cellular chain complexes of $\widetilde{W}^+$, $\widetilde{W}^-$ are given by

$$C(\widetilde{W}^+)^{lf} \;=\; A[[z]]\otimes_{A[z]}C^+\quad,\quad C(\widetilde{W}^-)^{lf} \;=\; A[[z^{-1}]]\otimes_{A[z^{-1}]}C^-\ .$$

The end complexes of C^+, C^- are given by

$$e(C^+) \;=\; \mathcal{C}(i:C^+\longrightarrow A[[z]]\otimes_{A[z]}C^+)_{*+1}\ ,$$

$$e(C^-) \;=\; \mathcal{C}(i:C^-\longrightarrow A[[z^{-1}]]\otimes_{A[z^{-1}]}C^-)_{*+1}\ .$$

The condition $H_*(A((z^{-1}))\otimes_{A[z]}C^+)=0$ for C^+ to be A-finitely dominated is just that the composite

$$e(C^-)\xrightarrow{\text{proj.}} C^-\xrightarrow{\text{incl.}} C$$

be a homology equivalence. The two conditions of the Theorem for a finite f.g. free $A[z,z^{-1}]$-module chain complex C to be A-finitely dominated

$$H_*(A((z))\otimes_{A[z,z^{-1}]} C) \;=\; H_*(A((z^{-1}))\otimes_{A[z,z^{-1}]} C) \;=\; 0$$

are just that both the chain maps $e(C^+)\longrightarrow C$, $e(C^-)\longrightarrow C$ be homology equivalences, for any Mayer-Vietoris exact sequence

$$0 \longrightarrow C^+\cap C^- \longrightarrow C^+\oplus C^- \longrightarrow C \longrightarrow 0$$

given by algebraic transversality, with $C^+\cap C^-$ a finite f.g. free A-module chain complex.

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