On the Novikov conjecture

Andrew Ranicki (Edinburgh)

Introduction

The Hirzebruch theorem expresses the signature $\sigma(N) \in \mathbb{Z}$ of a $4k$-dimensional manifold N^{4k} in terms of the $\mathcal{L}$-genus $\mathcal{L}(N) \in H^{4*}(N;\mathbb{Q})$. The signature theorem plays a central role in the classification of simply-connected manifolds. The 'higher signatures' of a manifold M with fundamental group $\pi_1(M) = \pi$ are the signatures of submanifolds $N^{4k} \subset M$ which are determined by the cohomology $H^*(B\pi;\mathbb{Q})$. The Novikov conjecture on the homotopy invariance of the higher signatures is of great importance in understanding the connection between the algebraic and geometric topology of high-dimensional non-simply-connected manifolds. Progress in the field is measured by the class of groups π for which the conjecture has been verified. A wide variety of methods has been used to attack the conjecture, such as surgery theory, elliptic operators, C^*-algebras, differential geometry, hyperbolic geometry, bounded/controlled topology, and algebra.

The diffeomorphism class of a closed differentiable m-dimensional manifold M^m is distinguished in its homotopy type up to a finite number of possibilities by the rational Pontrjagin classes $p_*(M) \in H^{4*}(M;\mathbb{Q})$. Thom proved that the rational Pontrjagin classes $p_*(M)$ are combinatorial invariants by showing that they determine and are determined by the signatures of closed $4k$-dimensional submanifolds $N^{4k} \subset M \times \mathbb{R}^j$ (j large) with trivial normal bundle. A homotopy equivalence of manifolds only preserves the global algebraic topology, and so need not preserve the local algebraic topology given by the Pontrjagin classes. The Browder-Novikov-Sullivan-Wall surgery theory shows that modulo torsion invariants for $m \geq 5$ a homotopy equivalence of closed differentiable m-dimensional manifolds is homotopic to a diffeomorphism if and only if it preserves the signatures of submanifolds and the non-simply-connected surgery obstruction is in the image of the assembly map; this map is onto in the simply-connected case. (Here, torsion means both Whitehead groups and finite groups). Novikov proved the topological invariance of the rational Pontrjagin classes by showing that a homeomorphism preserves signatures of submanifolds, using the fundamental group and non-compact manifold topology.

The object of this expository paper is to outline the relationship between the topological invariance of the rational Pontrjagin classes, the Novikov conjecture, the algebraic theory of surgery of Ranicki [16],[17],[18], the bounded/controlled topology of non-compact manifolds, and the recent proof of the conjecture by Carlsson and

This is an expanded version of the talks I gave at the Oberwolfach meetings 'Algebraic K-theory', 28 June, 1993 and 'Novikov conjectures, index theory and rigidity', 6 September, 1993.

Pedersen [7] for a geometrically defined class of infinite torsion-free groups π with $B\pi$ a finite complex and $E\pi$ a non-compact space with a compactification of a certain type.

The assembly maps in quadratic L-theory

$$A \ :\ H_*(X;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])\ ;\ \Gamma \longrightarrow q_!p^!\Gamma \quad (\pi = \pi_1(X))$$

are defined in Ranicki [18] for any topological space X, abstracting a geometric construction of Quinn. The surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ are the cobordism groups of quadratic Poincaré complexes C over $\mathbb{Z}[\pi]$ (= $\mathbb{Z}[\pi]$-module chain complexes with Poincaré duality). The generalized homology groups $H_*(X;\mathbb{L}.(\mathbb{Z}))$ with coefficients in the simply-connected surgery spectrum $\mathbb{L}.(\mathbb{Z})$ are the cobordism groups of sheaves Γ over X of quadratic Poincaré complexes over $\mathbb{Z}$. The assembly map A sends such a sheaf Γ to the quadratic Poincaré complex $A(\Gamma) = q_!p^!\Gamma$ over $\mathbb{Z}[\pi]$, with $p : \widetilde{X} \longrightarrow X$ the universal covering projection and $q : \widetilde{X} \longrightarrow \{\text{pt.}\}$ the unique map.

Novikov conjecture for a group π
The assembly maps for the classifying space $B\pi$

$$A \ :\ H_*(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$$

are rational split injections.

□

The rational Novikov conjecture is trivially true for finite groups π; it has been verified for groups which have strong geometric properties.

Integral Novikov conjecture for a group π
The assembly maps $A : H_*(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$ are split injections.

□

The integral Novikov conjecture is known to be false for finite groups π; it has been verified for torsion-free groups which have strong geometric properties.

The verification of the integral Novikov conjecture π requires the construction of a 'disassembly' map

$$B \ :\ L_m(\mathbb{Z}[\pi]) \longrightarrow H_m(B\pi;\mathbb{L}.(\mathbb{Z}))\ ;\ C \longrightarrow B(C)$$

such that $BA = 1$. Such a map B has to send a quadratic Poincaré complex C over $\mathbb{Z}[\pi]$ to a sheaf $B(C)$ over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$, with $BA(\Gamma)$ cobordant to Γ for any sheaf Γ over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$. It is possible to construct such B for any group π which has sufficiently geometry that manifolds with fundamental group π have rigidity, meaning that homotopy equivalences can be deformed to homeomorphisms. Novikov [13] constructed B algebraically in the case of a free abelian group $\pi = \mathbb{Z}^n$, when $B\pi = T^n$ and A is an isomorphism. See Farrell and Jones [8] for a geometric construction of B in the case when $B\pi$ is realized by a compact aspherical Riemannian manifold all of whose sectional curvatures are nonpositive (when A is also an isomorphism), and the connection with the original Mostow rigidity theorem for hyperbolic manifolds.

The locally finite assembly maps in quadratic L-theory

$$A^{lf} \; : \; H^{lf}_*(X; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{C}_X(\mathbb{Z}))$$

are defined in Ranicki [18] for any metric space X, using the X-graded $\mathbb{Z}$-module category $\mathbb{C}_X(\mathbb{Z})$ of Pedersen and Weibel [14]. The locally finite generalized homology groups $H^{lf}_*(X; \mathbb{L}.(\mathbb{Z}))$ are the cobordism groups of locally finite sheaves Γ over X of quadratic Poincaré complexes over $\mathbb{Z}$. It was shown in Ranicki [17] that A^{lf} is an isomorphism for $X = O(K) \subseteq \mathbb{R}^{N+1}$ the open cone of a compact polyhedron $K \subseteq S^N$, which corresponds to the topological invariance of the rational Pontrjagin classes (see 8.7 below). It is easier to prove that the locally finite assembly maps A^{lf} are isomorphisms than the ordinary assembly maps A. This is an algebraic reflection of the observed fact that rigidity theorems deforming homotopy equivalences to homeomorphisms are easier to prove for non-compact manifolds than for compact manifolds.

Carlsson and Pedersen [7] prove the integral Novikov conjecture for groups π with $B\pi$ a finite complex realized by a compact metric space such that the universal cover $E = E\pi$ admits a contractible π-equivariant compactification $\overline{E}$ with the metric such that compact sets in E become small near the boundary $\partial E = \overline{E} \backslash E$. Bounded/controlled algebra is used to prove that A^{lf} is an isomorphism for $X = E$, and equivariant topology is used to construct an algebraic disassembly map B by means of $(A^{lf})^{-1}$. The conditions on the compactification allow E-bounded algebra/topology to be deformed to ∂E-controlled algebra/topology, i.e. to pass from homotopy equivalences to homeomorphisms. See Ferry and Weinberger [10] for a more geometric approach, which outlines a proof of the integral Novikov conjecture (for essentially the same class of groups as in [7]). The computation $Wh_{-*}(\{1\}) = 0$ of Bass, Heller and Swan [3] is an essential ingredient of both [7] and [10], since the lower K-groups of $\mathbb{Z}$ are potential obstructions to the disassembly of quadratic Poincaré complexes over $\mathbb{Z}$ in bounded algebra, or equivalently to compactifying simply-connected open manifolds in bounded topology.

In dealing with vector bundles, manifolds, homotopy equivalences etc. only the oriented and orientation-preserving cases are considered. Manifolds are understood to be compact and differentiable, unless specified otherwise. Also, only topological spaces which are finite-dimensional locally finite polyhedra or topological manifolds are considered.

§1. Pontrjagin classes

The **Pontrjagin classes** of an m-plane bundle $\eta : X \longrightarrow BO(m)$ over a connected space X are characteristic classes

$$p_*(\eta) \in H^{4*}(X) \ .$$

See Milnor and Stasheff [11] for the textbook account of the Pontrjagin classes. The rational Pontrjagin character defines an isomorphism

$$\mathrm{ph} \ : \ KO(X) \otimes \mathbb{Q} \ = \ [X, \mathbb{Z} \times BO] \otimes \mathbb{Q} \longrightarrow H^{4*}(X;\mathbb{Q}) \ .$$

The **Pontrjagin classes** of an m-dimensional differentiable manifold M are the Pontrjagin classes of the tangent m-plane bundle $\tau_M : M \longrightarrow BO(m)$

$$p_*(M) \ = \ p_*(\tau_M) \in H^{4*}(M) \ .$$

By construction, the Pontrjagin classes are invariants of the differentiable structure of M: if $h : M' \longrightarrow M$ is a diffeomorphism then

$$\tau_{M'} \ = \ h^*\tau_M \ : \ M' \longrightarrow BO(m) \ ,$$
$$p_*(M') \ = \ h^*p_*(M) \in H^{4*}(M') \ .$$

§2. Signature

Definition 2.1 The **intersection form** of a closed $4k$-dimensional manifold N^{4k} is the nondegenerate symmetric form

$$\phi \ : \ H^{2k}(N;\mathbb{Q}) \times H^{2k}(N;\mathbb{Q}) \longrightarrow \mathbb{Q} \ ; \ (x,y) \longrightarrow \langle x \cup y, [N] \rangle$$

on the finite-dimensional $\mathbb{Q}$-vector space $H^{2k}(N;\mathbb{Q})$. The **signature** of N^{4k} is

$$\sigma(N) \ = \ \mathrm{signature}(H^{2k}(N;\mathbb{Q}), \phi) \in \mathbb{Z} \ .$$

□

Remarks 2.2 (i) An m-dimensional geometric Poincaré complex X is a finite CW complex with a fundamental class $[X] \in H_m(X)$ inducing isomorphisms

$$[X] \cap - \ : \ H^*(X) \longrightarrow H_{m-*}(X) \ .$$

Closed topological manifolds are the prime examples of geometric Poincaré complexes. The intersection form $(H^{2k}(X;\mathbb{Q}), \phi)$ and the signature $\sigma(X) \in \mathbb{Z}$ are defined for any $4k$-dimensional geometric Poincaré complex X, and are homotopy invariants of X.
(ii) The intersection form and signature are also defined for any $4k$-dimensional geometric Poincaré pair $(X, \partial X)$, such as a manifold with boundary $(M, \partial M)$.

□

Signature Theorem 2.3 (Hirzebruch) *The signature of a closed differentiable man-*

ifold N^{4k} is the evaluation of the $\mathcal{L}$-genus $\mathcal{L}(N) \in H^{4*}(N;\mathbb{Q})$ on $[N] \in H_{4k}(N;\mathbb{Q})$

$$\sigma(N) \;=\; \langle \mathcal{L}(N), [N] \rangle \in \mathbb{Z} \ .$$

The $\mathcal{L}$-genus determines and is determined by the Pontrjagin classes

$$p_*(N) \;=\; p_*(\tau_N) \in H^{4*}(N) \ .$$

□

The $\mathcal{L}$-genus $\mathcal{L}_k$ is a polynomial in the Pontrjagin classes $p_1, p_2, \ldots, p_k$. The first two $\mathcal{L}$-polynomials are given by

$$\mathcal{L}_1 \;=\; \frac{1}{3}p_1 \quad , \quad \mathcal{L}_2 \;=\; \frac{1}{45}(7p_2 - (p_1)^2) \ .$$

Transversality Theorem 2.4 *A continuous map $h : M'^m \longrightarrow M^m$ of differentiable m-dimensional manifolds is homotopic to a differentiable map. Given an n-dimensional submanifold $N^n \subset M^m$ it is possible choose the homotopy in such a way that the differentiable map (also denoted by h) is transverse regular at N, with the restriction*

$$f \;=\; h| \;:\; N'^n \;=\; h^{-1}(N) \longrightarrow N^n$$

a degree 1 map of n-dimensional manifolds which is covered by a map of the normal $(m-n)$-plane bundles $\nu_{N' \subset M'} \longrightarrow \nu_{N \subset M}$.

□

Definition 2.5 A submanifold $N^n \subset M^m \times \mathbb{R}^j$ is **special** if it is closed, $n = 4k$ and the normal bundle is trivial

$$\nu_{N \subset M} \;=\; \epsilon^i \;:\; N \longrightarrow BO(i) \;\; (i = m + j - 4k) \ .$$

□

Proposition 2.6 (Thom) *The rational Pontrjagin classes and the $\mathcal{L}$-genus of a manifold M are determined by the signatures of the special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$.*
Proof The $\mathbb{Q}$-vector space $H_{4k}(M;\mathbb{Q})$ is spanned by the homology classes $[N]$ of special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$, and

$$\mathcal{L}(M) \in H^{4k}(M;\mathbb{Q}) \;=\; \mathrm{Hom}_{\mathbb{Q}}(H_{4k}(M;\mathbb{Q}),\mathbb{Q})$$

is determined by

$$\mathcal{L}(M) \;:\; H_{4k}(M;\mathbb{Q}) \longrightarrow \mathbb{Q} \ ; \ [N] \longrightarrow \langle \mathcal{L}(M), [N] \rangle \;=\; \langle \mathcal{L}(N), [N] \rangle \;=\; \sigma(N) \ .$$

□

Theorem 2.7 (Thom) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are combinatorial invariants.*

Proof Let M^m be a differentiable m-dimensional manifold. The Pontrjagin classes and the $\mathcal{L}$-genus of a special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$ are the images in $H^{4*}(N;\mathbb{Q})$ of the Pontrjagin classes and the $\mathcal{L}$-genus of M. The signature $\sigma(N)$ depends only on the homology class $[N] \subset H_{4k}(M;\mathbb{Q})$ represented by N. The oriented cobordism spectrum MSO is given rationally by a wedge of Eilenberg-MacLane spectra

$$MSO \otimes \mathbb{Q} \;=\; \bigvee_k K(\mathbb{Q}^{\pi_k}, 4k)$$

with π_k the number of partitions of k. Transversality also works in the PL category, with the oriented PL cobordism spectrum $MSPL$ such that

$$MSPL \otimes \mathbb{Q} \;=\; MSO \otimes \mathbb{Q}\ .$$

Thus only special PL submanifolds $N^{4k} \subset M \times \mathbb{R}^j$ need be considered in 2.6, and if $h : M' \longrightarrow M$ is a PL homeomorphism then $p_*(M') = h^* p_*(M)$, $\mathcal{L}(M') = h^*\mathcal{L}(M)$.

□

Remark 2.8 Thom used PL transversality and the Hirzebruch signature theorem to define rational Pontrjagin classes $p_*(M)$ and the $\mathcal{L}$-genus $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$ for a PL manifold M. It is not possible to prove the topological invariance of the rational Pontrjagin classes by a mimicry of Thom's PL transversality argument: on the contrary, topological invariance is required for topological transversality and

$$MSTOP \otimes \mathbb{Q} \;=\; MSO \otimes \mathbb{Q}\ .$$

□

Proposition 2.9 (Dold, Milnor) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are not homotopy invariants.*

Proof The stable classifying space G/O for fibre homotopy trivialized vector bundles is such that there is defined a fibration

$$G/O \longrightarrow BO \longrightarrow BG$$

with an exact sequence

$$\dots \longrightarrow \pi_{n+1}(BG) \longrightarrow \pi_n(G/O) \longrightarrow \pi_n(BO) \longrightarrow \pi_n(BG) \longrightarrow \dots\ .$$

The homotopy groups of the stable classifying space BG for spherical fibrations are the stable homotopy groups of spheres

$$\pi_*(BG) \;=\; \pi^S_{*-1}\ ,$$

so that by Serre's finiteness theorem

$$\pi_*(BG) \otimes \mathbb{Q} \;=\; \pi^S_{*-1} \otimes \mathbb{Q} \;=\; 0 \quad (* > 1)\ .$$

By Bott periodicity $\pi_{4k}(BO) = \mathbb{Z}$, detected by the kth Pontrjagin class p_k. For any $k \geq 1$ there exists a fibre homotopy trivial $(j+1)$-plane bundle $\eta : S^{4k} \longrightarrow BO(j+1)$ (j large) over S^{4k} with

$$p_k(\eta) \neq 0 \in H^{4k}(S^{4k}) \;=\; \mathbb{Z}\ .$$

The sphere bundle $S(\eta)$ is a closed $(4k+j)$-dimensional manifold which is homotopy equivalent to $S(\epsilon^{j+1}) = S^{4k} \times S^j$ with

$$p_k(S(\eta)) \ = \ -p_k(\eta) \neq 0 \in H^{4k}(S^{4k} \times S^j) \ = \ \mathbb{Z} \ .$$

□

§3. Splitting

Let M^m be an m-dimensional manifold, and let $N^n \subset M^m$ be an n-dimensional submanifold. A homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds can be made transverse regular at $N \subset M$, with the restriction

$$f \ = \ h| \ : \ N' \ = \ h^{-1}(N) \longrightarrow N$$

a degree 1 map of n-dimensional manifolds. Let $i : N \longrightarrow M$, $i' : N' \longrightarrow M'$ be the inclusions. For any embedding $M' \subset S^{m+k}$ (k large) define a map of $(m-n+k)$-plane bundles covering f

$$b \ : \ \nu_{N' \subset S^{m+k}} \ = \ \nu_{N' \subset M'} \oplus i'^*(\nu_{M' \subset S^{m+k}}) \longrightarrow \eta \ = \ \nu_{N \subset M} \oplus (h^{-1}i)^*(\nu_{M' \subset S^{m+k}}) \ ,$$

so that $(f,b) : N' \longrightarrow N$ is a normal map. In general, (f,b) is not a homotopy equivalence.

Definition 3.1 A homotopy equivalence $h : M' \longrightarrow M$ of manifolds **splits** along a submanifold $N \subset M$ if transversality can be applied in such a way that the restriction $h| = (f,b) : N' \longrightarrow N$ is a homotopy equivalence.

□

If $h : M' \longrightarrow M$ is a homotopy equivalence of m-dimensional manifolds which is homotopic to a diffeomorphism then h splits along every n-dimensional submanifold $N \subset M$. A homotopy equivalence which does not split along a submanifold cannot be homotopic to a diffeomorphism.

The exotic spheres provide a concrete construction of a homotopy equivalence $h : M'^m \longrightarrow M^m$ which does not split along a special submanifold $N^{4k} \subset M^m$, with $p_*(M') \neq h^* p_*(M)$, $\mathcal{L}(M') \neq h^* \mathcal{L}(M)$, as follows:

Example 3.2 Let (W^8, Σ^7) be the framed 3-connected 8-dimensional differentiable manifold with signature $\sigma(W) = 8$ obtained by the Milnor E_8-plumbing of 8 copies of $\tau_{S^4} : S^4 \longrightarrow BO(4)$, with boundary $\partial W = \Sigma^7$ the homotopy 7-sphere generating the exotic sphere group $\Theta^7 = \mathbb{Z}_{28}$. The 28-fold connected sum $\#_{28}\Sigma^7$ is diffeomorphic to the standard 7-sphere S^7. Let $\eta : S^8 \longrightarrow BO(j+1)$ (j large) be a fibre homotopy trivial $(j+1)$-plane bundle over S^8 such that

$$\eta \in \ker(J : \pi_8(BO) \longrightarrow \pi_S^7) \ = \ 240\mathbb{Z} \subset \pi_8(BO) \ = \ \mathbb{Z}$$

is the generator with

$$p_2(\eta) \;=\; -1440 \in H^8(S^8) \;=\; \mathbb{Z}\ .$$

The sphere bundle is a closed $(8+j)$-dimensional differentiable manifold $M' = S(\eta)$ with a homotopy equivalence

$$h\ :\ M' \;=\; S(\eta) \longrightarrow M \;=\; S(\epsilon^{j+1}) \;=\; S^8 \times S^j$$

which does not split along the special submanifold

$$N^8 \;=\; S^8 \times \{\text{pt.}\} \subset M^{8+j} \;=\; S^8 \times S^j\ .$$

The inverse image of N is the special submanifold

$$N'^{\,8} \;=\; h^{-1}(N) \;=\; \#_{28} W \cup D^8 \subset M'^{\,8+j}$$

with

$$\begin{aligned}\mathcal{L}_2(M') \;&=\; \sigma(N') \;=\; \frac{7}{45}\langle -p_2(\eta), [S^8]\rangle \;=\; 28\cdot\sigma(W) \;=\; 224\\ &\neq\; h^*\mathcal{L}_2(M) \;=\; \sigma(N) \;=\; 0\ \in H^8(M';\mathbb{Q}) \;=\; \mathbb{Q}\ ,\\ p_2(M') \;&=\; 1440 \neq h^* p_2(M) \;=\; 0 \in H^8(M') \;=\; \mathbb{Z}\ .\end{aligned}$$

(Thus h is not homotopic to a diffeomorphism. In fact, by the topological invariance of the rational Pontrjagin classes h is not homotopic to a homeomorphism.)

□

If $h : M' \longrightarrow M$ is a homotopy equivalence of m-dimensional manifolds and $N \subset M$ is an n-dimensional submanifold such that the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(N)])$ of the restriction

$$(f,b) \;=\; h|\ :\ N' \;=\; h^{-1}(N) \longrightarrow N$$

is non-zero then h does not split along N. For $m-n \geq 3$ and $n \geq 5$ h splits if and only if $\sigma_*(f,b) = 0$.

Example 3.3 If $m - 4k \geq 3$ and $k \geq 2$ a homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds splits along a $4k$-dimensional submanifold $N^{4k} \subset M$ if and only if the simply-connected surgery obstruction

$$\sigma_*(f,b) \;=\; \frac{1}{8}(\sigma(N') - \sigma(N)) \in L_{4k}(\mathbb{Z}) = \mathbb{Z}$$

is 0, which for special $N \subset M$ is equivalent to

$$\langle (h^{-1})^*\mathcal{L}_k(M') - \mathcal{L}_k(M), [N]\rangle \;=\; 0 \in \mathbb{Q}\ .$$

□

See Chapter 23 of Ranicki [18] for an account of the Browder-Wall surgery obstruction theory for splitting homotopy equivalences along submanifolds.

§4. Topological invariance

A homeomorphism of differentiable manifolds cannot in general be approximated by a diffeomorphism, by virtue of the exotic spheres. Thus it is not at all obvious that the $\mathcal{L}$-genus $\mathcal{L}(M)$ and the rational Pontrjagin classes $p_*(M)$ are topological invariants of a differentiable manifold M.

Surgery theory for simply-connected compact manifolds is adequate for the construction and classification of exotic spheres. The applications to topological invariance properties require a surgery theory for non-compact manifolds, such as the universal covers of compact manifolds with infinite fundamental group. The applications require the vanishing of the lower Whitehead groups $Wh_{-*}(\{1\})$ to split homotopy equivalences of open manifolds.

The lower K-groups $K_{-*}(A)$ of Bass [2, XII] are defined to be such that

$$K_1(A[\mathbb{Z}^i]) \;=\; \sum_{j=0}^{i} \binom{i}{j} K_{1-j}(A) \oplus \text{Nil-groups}$$

for any ring A. For a group ring $A = \mathbb{Z}[\pi]$

$$Wh(\pi \times \mathbb{Z}^i) \;=\; \sum_{j=0}^{i} \binom{i}{j} Wh_{1-j}(\pi) \oplus \text{Nil-groups}$$

with the lower Whitehead groups are defined by

$$Wh_{1-j}(\pi) \;=\; \begin{cases} Wh(\pi) & \text{if } j = 0 \\ \widetilde{K}_0(\mathbb{Z}[\pi]) & \text{if } j = 1 \\ K_{1-j}(\mathbb{Z}[\pi]) & \text{if } j \geq 2 \;. \end{cases}$$

Bass, Heller and Swan [3] proved that $Wh(\mathbb{Z}^i) = 0$ for all $i \geq 1$, so that

$$Wh_{1-*}(\{1\}) \;=\; 0 \;.$$

(In fact, Novikov [12] gave an independent proof that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^i]) = 0$, and hence that $Wh_{1-*}(\{1\}) = 0$.)

The unobstructed simply-connected case $\pi_1(N) = \{1\}$ of the following result is the essential step in the proof of the topological invariance of the rational Pontrjagin classes.

Proposition 4.1 (Browder [5], Novikov [12], Siebenmann [20]) *Let N be a compact n-dimensional manifold, and let W be an open $(n+i)$-dimensional manifold with an $\mathbb{R}^i$-bounded homotopy equivalence $f : W \longrightarrow N \times \mathbb{R}^i$ $(i \geq 1)$. The lower Whitehead torsion $\tau_{1-i}(f) \in Wh_{1-i}(\pi_1(N))$ is such that $\tau_{1-i}(f) = 0$ if (and for $n \geq 5$ only if) f splits along $N \times \{0\} \subset N \times \mathbb{R}^i$.*

□

The original results were not formulated in the language of bounded topology - see Ranicki [17] for an exposition of the applications of lower K- and L-theory to splitting theorems for homotopy equivalences of non-compact manifolds in bounded topology, including the definition of the lower Whitehead torsion. There is a brief account in §8 below. Proposition 4.1 applies to a homeomorphism $f : W \longrightarrow N \times \mathbb{R}^i$ because it is possible to approximate f by a differentiable $\mathbb{R}^i$-bounded homotopy equivalence.

Theorem 4.2 (Novikov [12]) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are topological invariants.*
Proof By 2.6 it suffices to prove that the signatures of special submanifolds are homeomorphism invariant, i.e. that if $h : M'^m \longrightarrow M^m$ is a homeomorphism of differentiable (or PL) manifolds then $\sigma(N) = \sigma(N') \in \mathbb{Z}$ for any special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$, with

$$N' \ = \ h'^{-1}(N) \subset M' \times \mathbb{R}^j$$

the transverse inverse image of any differentiable (or PL) approximation $h' : M' \times \mathbb{R}^j \longrightarrow M \times \mathbb{R}^j$ to $h \times 1_{\mathbb{R}^j}$. Every special submanifold is (ambient) cobordant to a simply-connected one, so it may be assumed that N is simply-connected, $\pi_1(N) = \{1\}$. The homeomorphism

$$f \ = \ (h \times 1_{\mathbb{R}^j})| \ : \ W \ = \ (h \times 1_{\mathbb{R}^j})^{-1}(N \times \mathbb{R}^i) \longrightarrow N \times \mathbb{R}^i \quad (i = m + j - 4k)$$

is an $\mathbb{R}^i$-bounded homotopy equivalence. The splitting obstruction given by 4.1 is $\tau_{1-i} \in Wh_{1-i}(\pi_1(N)) = 0$. Thus h' can be chosen such that $h'| : N' \longrightarrow N$ is a homotopy equivalence, and $\sigma(N) = \sigma(N') \in \mathbb{Z}$. (See 8.8 below for an L-theoretic version of this argument).

□

Remarks 4.3 (i) Novikov's proof of the topological invariance of the rational Pontrjagin classes led to the disproof of the manifold Hauptvermutung by Casson, Sullivan and Siebenmann. A homeomorphism of PL manifolds cannot in general be approximated by a PL homeomorphism. Thus it is also not at all obvious that the $\mathcal{L}$-genus and the rational Pontrjagin classes are topological invariants of a PL manifold. Although the original proof of topological invariance was in the differentiable category, it applies equally well in the PL category.
(ii) Novikov's proof of the topological invariance of the rational Pontrjagin classes also led to the subsequent development by Kirby and Siebenmann of the classification theory of high-dimensional topological manifolds. It is now possible to define the $\mathcal{L}$-genus and the rational Pontrjagin classes for a topological manifold, and the Hirzebruch signature theorem $\sigma(N) = \langle \mathcal{L}(N), [N] \rangle$ also holds for topological manifolds N^{4k}.
(iii) The computation $Wh_{1-*}(\{1\}) = 0$ used in the proof of the topological invariance

of the rational Pontrjagin classes and of Whitehead torsion (Chapman) plays the role of the computation $Wh(\{1\}) = 0$ used in the proof of the combinatorial invariance of Whitehead torsion - see Ranicki and Yamasaki [18] for an exposition.

□

§5. Homotopy invariance

Definition 5.1 The **higher $\mathcal{L}$-genus** of an m-dimensional manifold M with fundamental group $\pi_1(M) = \pi$ is

$$\mathcal{L}_\pi(M) \;=\; f_*(\mathcal{L}(M) \cap [M]) \in H_{m-4*}(B\pi;\mathbb{Q}) \;,$$

with $f : M \longrightarrow B\pi$ classifying the universal cover $\widetilde{M}$, and $\mathcal{L}(M) \cap [M] \in H_{m-4*}(M;\mathbb{Q})$ the Poincaré dual of the $\mathcal{L}$-genus $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$.

□

Conjecture 5.2 (Novikov [13, §11]) *The higher $\mathcal{L}$-genus is a homotopy invariant: if $h : M'^m \longrightarrow M^m$ is a homotopy equivalence of m-dimensional manifolds then*

$$\mathcal{L}_\pi(M) \;=\; \mathcal{L}_\pi(M') \in H_{m-4*}(B\pi;\mathbb{Q}) \;.$$

□

Definition 5.3 A submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$ is **π-special** if it is special and the Poincaré dual $[N]^* \in H^{m-4k}(M;\mathbb{Q})$ of $[N] \in H_{4k}(M;\mathbb{Q})$ is such that

$$[N]^* \in \mathrm{im}(f^* : H^{m-4k}(B\pi;\mathbb{Q}) \longrightarrow H^{m-4k}(M;\mathbb{Q})) \;.$$

The **higher signatures** of M are the signatures $\sigma(N) \in \mathbb{Z}$ of the π-special manifolds $N \subset M \times \mathbb{R}^j$.

□

Remarks 5.4 (i) The higher $\mathcal{L}$-genus of an m-dimensional manifold M with $\pi_1(M) = \pi$ is detected by the higher signatures. As before, let $f : M \longrightarrow B\pi$ classify the universal cover $\widetilde{M}$ of M. The $\mathbb{Q}$-vector space $H^{m-4k}(B\pi;\mathbb{Q})$ is spanned by the elements of type $x = e^*(1)$ for a proper map

$$e \;:\; B\pi \times \mathbb{R}^j \longrightarrow \mathbb{R}^i \;\; (i = m + j - 4k)$$

with large j. (It is convenient to assume here that $B\pi$ is compact). For any such x, e the composite

$$e(f \times 1) \;:\; M \times \mathbb{R}^j \xrightarrow{f \times 1} B\pi \times \mathbb{R}^j \xrightarrow{e} \mathbb{R}^i$$

can be made transverse regular at $0 \in \mathbb{R}^i$, with

$$N^{4k} \;=\; (e(f \times 1))^{-1}(0) \subset M \times \mathbb{R}^j$$

a π-special submanifold. The higher $\mathcal{L}$-genus of M is such that

$$\begin{aligned}\mathcal{L}_\pi(M)\ &: H^{m-4k}(B\pi;\mathbb{Q}) \longrightarrow \mathbb{Q}\ ;\\ &x \longrightarrow \langle x, \mathcal{L}_\pi(M)\rangle \ =\ \langle \mathcal{L}(M)\cup f^*(x), [M]\rangle \ =\ \langle \mathcal{L}(N), [N]\rangle \ =\ \sigma(N)\ .\end{aligned}$$

(ii) The Novikov conjecture is equivalent to the homotopy invariance of the higher signatures: if $h : M'^m \longrightarrow M^m$ is a homotopy equivalence then

$$\sigma(N) \ =\ \sigma(N') \in \mathbb{Z}$$

for any π-special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$, with the inverse image

$$N' \ =\ (h\times 1)^{-1}(N) \subset M' \times \mathbb{R}^j$$

also a π-special submanifold.

□

See Chapter 24 of Ranicki [18] for a more detailed account of the higher signatures.

§6. Cobordism invariance

Very early on in the history of the Novikov conjecture (essentially already in [13]) it was recognized that the conjecture is equivalent to the algebraic Poincaré cobordism invariance of the higher $\mathcal{L}$-genus, and also to the splitting of the rational assembly map $A_\pi : H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q}$.

See Ranicki [16] for the **symmetric** (resp. **quadratic**) **L-groups** $L^m(R)$ (resp. $L_m(R)$) of a ring with involution R, which are the cobordism groups of m-dimensional symmetric (resp. quadratic) Poincaré complexes (C,ϕ) consisting of an m-dimensional f.g. free R-module chain complex C with a symmetric (resp. quadratic) Poincaré duality $\phi : C^{m-*} \simeq C$. The symmetrization maps $1+T : L_m(R) \longrightarrow L^m(R)$ are isomorphisms modulo 8-torsion. The quadratic L-groups $L_m(R)$ are the Wall surgery obstruction groups, and depend only on $m(\bmod 4)$. The symmetric L-groups $L^m(R)$ were introduced by Mishchenko, and are not 4-periodic in general. The L-groups of $\mathbb{Z}$ are given by

$$L^m(\mathbb{Z}) \ =\ \begin{cases}\mathbb{Z} & \text{if } m\equiv 0(\bmod 4)\\ \mathbb{Z}_2 & \text{if } m \equiv 1(\bmod 4)\\ 0 & \text{otherwise}\end{cases}\quad ,\quad L_m(\mathbb{Z}) \ =\ \begin{cases}\mathbb{Z} & \text{if } m\equiv 0(\bmod 4)\\ \mathbb{Z}_2 & \text{if } m \equiv 2(\bmod 4)\\ 0 & \text{otherwise}\ .\end{cases}$$

Definition 6.1 The **symmetric signature** of an m-dimensional geometric Poincaré complex X with universal cover $\widetilde{X}$ is the symmetric Poincaré cobordism class

$$\sigma^*(X) \ =\ (C(\widetilde{X}),\phi) \in L^m(\mathbb{Z}[\pi_1(X)])$$

with ϕ the m-dimensional symmetric structure of the Poincaré duality chain equivalence $[X]\cap - : C(\widetilde{X})^{m-*} \longrightarrow C(\widetilde{X})$.

□

The standard algebraic mapping cylinder argument shows:

Proposition 6.2 *The symmetric signature is both a cobordism and a homotopy invariant of a geometric Poincaré complex.*

□

The symmetric signature is a non-simply-connected generalization of the signature; for $m = 4k$ the natural map $L^m(\mathbb{Z}[\pi_1(X)]) \longrightarrow L^m(\mathbb{Z}) = \mathbb{Z}$ sends $\sigma^*(X)$ to the signature $\sigma(X)$.

Definition 6.3 The **quadratic signature** of a normal map of m-dimensional manifolds with boundary $(f,b) : (M',\partial M') \longrightarrow (M,\partial M)$ and with $\partial f : \partial M' \longrightarrow \partial M$ a homotopy equivalence is the cobordism class of the quadratic Poincaré complex kernel

$$\sigma_*(f,b) \;=\; (C(f^!),\psi) \in L_m(\mathbb{Z}[\pi_1(M)]) \;,$$

with ψ the quadratic structure on the algebraic mapping cone $C(f^!)$ of the Umkehr $\mathbb{Z}[\pi_1(M)]$-module chain map

$$f^! \;:\; C(\widetilde{M}) \;\simeq\; C(\widetilde{M},\partial\widetilde{M})^{m-*} \xrightarrow{\widetilde{f}^*} C(\widetilde{M}',\partial\widetilde{M}')^{m-*} \;\simeq\; C(\widetilde{M}') \;.$$

□

Proposition 6.4 *The quadratic signature of a normal map (f,b) is the Wall surgery obstruction, such that $\sigma_*(f,b) = 0$ if (and for $m \geq 5$ only if) (f,b) is normal bordant to a homotopy equivalence.*

□

The symmetrization of the quadratic signature is the symmetric signature

$$(1+T)\sigma_*(f,b) \;=\; \sigma^*(M' \cup_{\partial f} -M) \in L^m(\mathbb{Z}[\pi_1(M)])$$

where $-M$ refers to M with the opposite orientation $[-M] = -[M]$.

The rational surgery obstruction of a normal map $(f,b) : M' \longrightarrow M$ of closed m-dimensional manifolds with fundamental group π

$$\sigma_*(f,b) \otimes \mathbb{Q} \in L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

depends only on the difference of the higher $\mathcal{L}$-genera

$$\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M) \in H_{m-4*}(B\pi;\mathbb{Q}) \;.$$

For any finitely presented group π the $\mathbb{Q}$-vector space $H_{m-4*}(B\pi;\mathbb{Q})$ is spanned by the differences $\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M)$ for normal maps $(f,b) : M' \longrightarrow M$ of closed m-dimensional manifolds with fundamental group π.

Definition 6.5 The **rational assembly map** in quadratic L-theory is

$$\begin{aligned} A_\pi \ &: H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \ ; \\ &\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M) \longrightarrow \sigma_*(f,b) \otimes \mathbb{Q} \ = \ \frac{1}{8}(\sigma^*(M') - \sigma^*(M)) \ , \end{aligned}$$

with

$$A_\pi \mathcal{L}_\pi(M) \ = \ \frac{1}{8}\sigma^*(M) \in L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \ = \ L^m(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \ .$$

□

The kernel of A_π is spanned by the differences $\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M)$ for homotopy equivalent m-dimensional manifolds M, M' with fundamental group π.

Proposition 6.6 (Novikov [13]) *The Novikov conjecture holds for π if and only if the rational assembly map*

$$A_\pi \ : \ H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

is a split injection for each $m \pmod 4$.

□

§7. The integral Novikov conjecture

The integral versions of the topological invariance of the rational Pontrjagin classes and of the Novikov conjecture on the homotopy invariance of the higher signatures involve the algebraic L-spectra defined in Ranicki [18].

The **symmetric L-spectrum** $\mathbb{L}^\cdot(R)$ of a ring with involution R is defined in [18] using n-ads of symmetric forms over $\mathbb{Z}$, with homotopy groups

$$\pi_*(\mathbb{L}^\cdot(R)) \ = \ L^*(R) \ .$$

The **generalized homology spectrum** $\mathbb{H}_.(X;\mathbb{L}^\cdot(R))$ of a topological space X is defined in [18] using sheaves over X of symmetric Poincaré complexes over R, with homotopy groups

$$\pi_*(\mathbb{H}_.(X;\mathbb{L}^\cdot(R))) \ = \ H_*(X;\mathbb{L}^\cdot(R))$$

the cobordism groups of such sheaves. The assembly map

$$A \ : \ \mathbb{H}_.(X;\mathbb{L}^\cdot(R)) \longrightarrow \mathbb{L}^\cdot(R[\pi_1(X)])$$

is defined by pulling back symmetric Poincaré sheaf over X to the universal cover $\widetilde{X}$, and then assembling the stalks to obtain a symmetric Poincaré complex over $R[\pi_1(X)]$. Similarly for the **quadratic L-spectrum** $\mathbb{L}_.(R)$.

The 0th space of the quadratic L-spectrum $\mathbb{L}.(\mathbb{Z})$ is such that

$$\mathbb{L}_0(\mathbb{Z}) \simeq L_0(\mathbb{Z}) \times G/TOP .$$

As usual, G/TOP is the classifying space for fibre homotopy trivialized topological bundles, with a fibration sequence

$$G/TOP \longrightarrow BTOP \longrightarrow BG .$$

Let $\mathbb{L}. = \mathbb{L}.\langle 1\rangle(\mathbb{Z})$ be the 1-connective cover of $\mathbb{L}.(\mathbb{Z})$, with 0th space such that

$$\mathbb{L}_0 \simeq G/TOP .$$

For any space X define the **structure spectrum**

$$\mathbb{S}.(X) = \text{homotopy cofibre}(A : \mathbb{H}.(X;\mathbb{L}.) \longrightarrow \mathbb{L}.(\mathbb{Z}[\pi_1(X)])) ,$$

to fit into a cofibration sequence of spectra

$$\mathbb{H}.(X;\mathbb{L}.) \xrightarrow{A} \mathbb{L}.(\mathbb{Z}[\pi_1(X)])) \longrightarrow \mathbb{S}.(X) ,$$

with A the spectrum level assembly map. The **structure groups**

$$\mathbb{S}_*(X) = \pi_*(\mathbb{S}.(X))$$

are the cobordism groups of sheaves over X of quadratic Poincaré complexes over $\mathbb{Z}$ such that the assembly quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$ is contractible. The structure groups are the relative groups in the **algebraic surgery exact sequence**

$$\dots \longrightarrow \mathbb{S}_{m+1}(X) \longrightarrow H_m(X;\mathbb{L}.) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathbb{S}_m(X) \longrightarrow \dots .$$

If X is an m-dimensional CW complex then $H_*(X;\mathbb{L}.) = H_*(X;\mathbb{L}.(\mathbb{Z}))$ for $* > m$ and $\mathbb{S}_*(X) = \mathbb{S}_{*+4}(X)$ for $* > m+1$.

Proposition 7.1 (Ranicki [18]) (i) *An m-dimensional geometric Poincaré complex X has a* **total surgery obstruction**

$$s(X) \in \mathbb{S}_m(X)$$

such that $s(X) = 0$ if (and for $m \geq 5$ only if) X is homotopy equivalent to a closed m-dimensional topological manfold.
(ii) *A closed m-dimensional topological manifold M has a* **symmetric L-theory orientation**

$$[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))$$

which is represented by the symmetric Poincaré orientation sheaf, with assembly the symmetric signature

$$A([M]_{\mathbb{L}}) = \sigma^*(M) \in L^m(\mathbb{Z}[\pi_1(M)]) .$$

(iii) *A normal map $(f,b) : M' \longrightarrow M$ of closed m-dimensional topological manifolds has a* **normal invariant**

$$[f,b]_{\mathbb{L}} \in H_m(M;\mathbb{L}.) = [M, G/TOP]$$

which is represented by the sheaf over M of the quadratic Poincaré complex kernels over $\mathbb{Z}$ of the normal maps

$$(f,b) \;=\; h| \;:\; N' \;=\; h^{-1}(N) \longrightarrow N \;\; (N^n \subset M^m)\ ,$$

with assembly the surgery obstruction

$$A([f,b]_{\mathbb{L}}) \;=\; \sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(M)])\ .$$

The symmetrization of the normal invariant is the difference of the symmetric L-theory orientations

$$(1+T)[f,b]_{\mathbb{L}} \;=\; f_*[M']_{\mathbb{L}} - [M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))\ .$$

(iv) *A homotopy equivalence $h : M' \longrightarrow M$ of closed m-dimensional topological manifolds has a* **structure invariant** *$s(h) \in \mathbb{S}_{m+1}(M)$, which is represented by the $\mathbb{Z}[\pi_1(M)]$-contractible quadratic Poincaré kernel sheaf of* (iii) *and is such that $s(h) = 0$ if (and for $m \geq 5$ only if) h is h-cobordant to a homeomorphism. Moreover, for $m \geq 5$ every element $x \in \mathbb{S}_{m+1}(M)$ is the structure invariant $x = s(h)$ of a homotopy equivalence $h : M' \longrightarrow M$.*

□

Remarks 7.2 (i) The symmetric and quadratic L-spectra of $\mathbb{Z}$ are given rationally by

$$\mathbb{L}^{\cdot}(\mathbb{Z}) \otimes \mathbb{Q} \;=\; \mathbb{L}_{\cdot}(\mathbb{Z}) \otimes \mathbb{Q} \;=\; \bigvee_k K(\mathbb{Q},4k)\ ,$$

so that for any space X

$$H_m(X;\mathbb{L}^{\cdot}(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_m(X;\mathbb{L}_{\cdot}(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(X;\mathbb{Q})\ .$$

(ii) The symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))$ of a closed m-dimensional topological manifold M is an integral refinement of the $\mathcal{L}$-genus

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; [M] \cap \mathcal{L}(M) \in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(M;\mathbb{Q})$$

detected by the signatures $\sigma(N)$ of special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$. As before, let $\pi_1(M) = \pi$ and let $f : M \longrightarrow B\pi$ be the classifying map of the universal cover $\widetilde{M}$. The image $f_*[M]_{\mathbb{L}} \in H_m(B\pi;\mathbb{L}^{\cdot}(\mathbb{Z}))$ is an integral refinement of the higher $\mathcal{L}$-genus

$$f_*[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; \mathcal{L}_{\pi}(M) \in H_m(B\pi;\mathbb{L}^{\cdot}(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(B\pi;\mathbb{Q})$$

detected by the signatures $\sigma(N)$ of π-special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$.

(iii) The normal invariant $[f,b]_{\mathbb{L}} \in H_m(M;\mathbb{L}_{\cdot})$ of an m-dimensional normal map $(f,b) : M' \longrightarrow M$ is given rationally by the difference of the Poincaré duals of the $\mathcal{L}$-genera

$$\begin{aligned}[f,b]_{\mathbb{L}} \otimes \mathbb{Q} \;&=\; f_*(\mathcal{L}(M') \cap [M']) - (\mathcal{L}(M) \cap [M]) \\ &\in H_m(M;\mathbb{L}_{\cdot}(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(M;\mathbb{Q})\ .\end{aligned}$$

(iv) The quadratic L-theory invariant of a normal map $(f,b) : M' \longrightarrow M$ of closed m-dimensional manifolds is given rationally by the difference of the Poincaré duals of the $\mathcal{L}$-genera

$$\begin{aligned}[f,b]_{\mathbb{L}}\otimes\mathbb{Q} &= f_*(\mathcal{L}(M')\cap[M'])-(\mathcal{L}(M)\cap[M]) \\ &\in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))\otimes\mathbb{Q} \;=\; H_{m-4*}(M;\mathbb{Q})\ .\end{aligned}$$

(v) The construction and the verification of the combinatorial invariance of the symmetric L-theory orientation $[M]_{\mathbb{L}}\in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))$ is quite straightforward for a PL manifold M. The construction and topological invariance of $[M]_{\mathbb{L}}$ for a topological manifold M is much more complicated - see §8 below.
(vi) For $m\geq 5$ an m-dimensional geometric Poincaré complex X is homotopy equivalent to a closed m-dimensional manifold M if and only if there exists a symmetric L-theory orientation $[X]_{\mathbb{L}}\in H_m(X;\mathbb{L}^{\cdot}(\mathbb{Z}))$ such that $A([X]_{\mathbb{L}})=\sigma^*(X)\in L^m(\mathbb{Z}[\pi_1(X)])$, modulo 2-primary torsion invariants.

□

Integral Novikov conjecture 7.3 *The assembly map in quadratic L-theory*

$$A\ :\ H_*(B\pi;\mathbb{L}.(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[\pi])$$

is a split injection.

□

The algebraic surgery exact sequence for the classifying space $B\pi$ of a group π

$$\dots\longrightarrow\mathbb{S}_{m+1}(B\pi)\longrightarrow H_m(B\pi;\mathbb{L}.)\xrightarrow{A}L_m(\mathbb{Z}[\pi])\longrightarrow\mathbb{S}_m(B\pi)\longrightarrow\dots$$

is such that

$$\mathrm{im}(\mathbb{S}_{m+1}(B\pi)\longrightarrow H_m(B\pi;\mathbb{L}.)) \;=\; \ker(A:H_m(B\pi;\mathbb{L}.)\longrightarrow L_m(\mathbb{Z}[\pi]))\subseteq L_m(\mathbb{Z}[\pi])$$

consists of the images of the normal invariants

$$f_*[s(h)] \;=\; f_*[h,b]_{\mathbb{L}}\in H_m(B\pi;\mathbb{L}.)$$

of all homotopy equivalences $h:M'\longrightarrow M$ of m-dimensional topological manifolds with $\pi_1(M)=\pi$, and with $f:M\longrightarrow B\pi$ classifying the universal cover.

Remarks 7.4 (i) The integral Novikov conjecture for π implies the original Novikov conjecture for π, since the integral assembly map A induces the rational assembly map

$$A\otimes 1\ :\ H_m(B\pi;\mathbb{L}.(\mathbb{Z}))\otimes\mathbb{Q} \;=\; H_{m-4*}(B\pi;\mathbb{Q})\longrightarrow L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q}\ .$$

(ii) The integral Novikov conjecture is generally false if π has torsion, e.g. if $\pi=\mathbb{Z}_2$.
(iii) The integral Novikov conjecture has been verified for many torsion-free groups π, starting with the free abelian case $\pi=\mathbb{Z}^i$ (when A is an isomorphism).

□

The realization theorem of Wall [21] identifies $L_m(\mathbb{Z}[\pi])$ for a finitely presented group π and $m \geq 5$ with a bordism group of normal maps $(f, b) : (M, \partial M) \longrightarrow (N, \partial N)$ of compact m-dimensional manifolds with boundary which restrict to a homotopy equivalence $\partial f = f| : \partial M \longrightarrow \partial N$ on the boundary, with $\pi_1(N) = \pi$. The generalized homology groups $H_*(B\pi; \mathbb{L}.)$ have a similar description, with the added condition that $\partial f : \partial M \longrightarrow \partial N$ be a homeomorphism (including $\partial M = \partial N = \emptyset$ as a special case). If π is sufficiently geometric for manifolds with fundamental group π to be rigid it is possible to split A by geometry. Suppose π is such that there is a systematic procedure for associating to any homotopy equivalence $f' : M' \longrightarrow N'$ of closed m'-dimensional manifolds with $\pi_1(N) = \pi$ an $(m'+1)$-dimensional normal bordism

$$(g', c') \ : \ (W; M'_0, M'_1) \longrightarrow N' \times ([0,1]; \{0\}, \{1\})$$

from $g'_0 = f' : M'_0 = M' \longrightarrow N'$ to a homeomorphism $g'_1 : M' \longrightarrow N'$, at least for $m' \geq 5$. Then there is defined a direct sum system

$$H_m(B\pi; \mathbb{L}.) \underset{B}{\overset{A}{\rightleftarrows}} L_m(\mathbb{Z}[\pi]) \underset{D}{\overset{C}{\rightleftarrows}} \mathbb{S}_m(B\pi)$$

verifying the integral Novikov conjecture for π, with

$$\begin{aligned}
B \ &: \ L_m(\mathbb{Z}[\pi]) \longrightarrow H_m(B\pi; \mathbb{L}.) \ ; \ (f, b) \longrightarrow (f, b) \cup (g', c') \ \ (f' = \partial f) \ , \\
C \ &: \ L_m(\mathbb{Z}[\pi]) \longrightarrow \mathbb{S}_m(B\pi) \ ; \\
&\qquad \sigma_*((f, b) : (M, \partial M) \longrightarrow (N, \partial N)) \longrightarrow s(\partial f : \partial M \longrightarrow \partial N) \ , \\
D \ &: \ \mathbb{S}_m(B\pi) \longrightarrow L_m(\mathbb{Z}[\pi]) \ ; \ s(f') \longrightarrow \sigma_*(g', c') \ \ (m' = m - 1) \ .
\end{aligned}$$

The chain complex treatment in Ranicki [16], [18] of the surgery obstruction of Wall [21] associates a quadratic Poincaré complex $\sigma_*(f, b)$ over $\mathbb{Z}[\pi]$ (resp. a sheaf over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$) to any normal map $(f, b) : (M, \partial M) \longrightarrow (N, \partial N)$ with ∂f a homotopy equivalence (resp. a homeomorphism) and $\pi_1(N) = \pi$. In principle, this allows the translation into algebra of any geometric construction of a disassembly map B.

§8. With one bound

The applications of bounded and controlled algebra to splitting theorems in topology and the Novikov conjectures depend on the development of an algebraic theory of transversality. The splitting obstructions are the same in algebra and topology, but it is necessary to use algebra to prove that the obstructions vanish in particular cases.

Given a metric space X and a ring R let $\mathbb{C}_X(R)$ be the **X-bounded free R-module** additive category of Pedersen and Weibel [14]. An object in $\mathbb{C}_X(R)$ is a direct sum of f.g. free R-modules graded by X

$$A = \sum_{x \in X} A(x)$$

such that $A(K) = \sum_{x \in K} A(x)$ is a f.g. free R-module for every subspace $K \subseteq X$ of finite diameter. A morphism in $\mathbb{C}_X(R)$

$$f = \{f(y,x)\} : A = \sum_{x \in X} A(x) \longrightarrow B = \sum_{y \in X} B(y)$$

is an R-module morphism such that there exists a number $b > 0$ with $f(y,x) = 0 : A(x) \longrightarrow B(y)$ for all $x, y \in X$ with $d(x,y) > b$.

A **proper eventually Lipschitz map** $f : X \longrightarrow Y$ of metric spaces is a function (not necessarily continuous) such that the inverse image of a bounded set is a bounded set, and there exist numbers $r, k > 0$ depending only on f such that for all $s > r$ and all $x, y \in X$ with $d(x,y) < s$ it is the case that $d(f(x), f(y)) < ks$. Such a map induces a functor

$$f_! : \mathbb{C}_X(R) \longrightarrow \mathbb{C}_Y(R) ; \ M = \sum_{x \in X} M(x) \longrightarrow f_! M = \sum_{y \in Y} \left(\sum_{x \in f^{-1}(y)} M(x) \right) .$$

Suppose given a metric space X with a decomposition $X = X^+ \cup X^-$ define for any $b \geq 0$ the subspaces

$$\begin{aligned} X_b^+ &= \{x \in X \,|\, d(x,y) \leq b \text{ for some } y \in X^+\} , \\ X_b^- &= \{x \in X \,|\, d(x,z) \leq b \text{ for some } z \in X^-\} , \\ Y_b &= \{x \in X \,|\, d(x,y) \leq b \text{ and } d(x,z) \leq b \text{ for some } y \in X^+, z \in X^-\} . \end{aligned}$$

The inclusions $X^+ \longrightarrow X_b^+$, $X^- \longrightarrow X_b^-$ are homotopy equivalences in the proper eventually Lipschitz category, so that

$$K_*(\mathbb{C}_{X_b^+}(R)) = K_*(\mathbb{C}_{X^+}(R)) \ , \ K_*(\mathbb{C}_{X_b^-}(R)) = K_*(\mathbb{C}_{X^-}(R)) .$$

Every finite chain complex C in $\mathbb{C}_X(R)$ is such that there exist subcomplexes $C^+, C^- \subseteq C$ with $C^\pm$ defined in $\mathbb{C}_{X_b^\pm}(R)$ and $C^+ \cap C^-$ defined in $\mathbb{C}_{Y_b}(R)$ for some $b \geq 0$. Thus C admits a 'Mayer-Vietoris presentation'

$$0 \longrightarrow C^+ \cap C^- \longrightarrow C^+ \oplus C^- \longrightarrow C \longrightarrow 0 .$$

If C is contractible there is an algebraic K-theory obstruction to the existence of such a presentation with $C^+, C^-, C^+ \cap C^-$ contractible, which depends only on the Whitehead torsion $\tau(C) \in K_1(\mathbb{C}_X(R))$. Specifically, if $\mathbb{P}_X(R)$ denotes the idempotent completion of $\mathbb{C}_X(R)$, the obstruction is the element

$$\partial\tau(C) \in \varinjlim_b K_0(\mathbb{P}_{Y_b}(R)) ,$$

with ∂ the connecting map in a Mayer-Vietoris exact sequence of algebraic K-groups

$$\begin{aligned}\ldots \longrightarrow K_1(\mathbb{C}_{X^+}(R)) \oplus K_1(\mathbb{C}_{X^-}(R)) \longrightarrow K_1(\mathbb{C}_X(R))\\ \xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{Y_b}(R)) \longrightarrow K_0(\mathbb{P}_{X^+}(R)) \oplus K_0(\mathbb{P}_{X^-}(R)) \longrightarrow \ldots .\end{aligned}$$

See Ranicki [17] for further details. (The corresponding splitting obstruction in controlled K-theory is developed in Ranicki and Yamasaki [18]).

A CW complex M is X**-bounded** if it is equipped with a proper map $M \longrightarrow X$ such that the diameters of the images of the cells of M are uniformly bounded in X, so that the cellular chain complex $C(M)$ is defined in $\mathbb{C}_X(\mathbb{Z})$. Similarly for cellular maps, with induced chain maps in $\mathbb{C}_X(\mathbb{Z})$.

If $f : M \longrightarrow N$ is an X-bounded homotopy equivalence of X-bounded CW complexes which are locally simply-connected the X-bounded Whitehead torsion is given by

$$\tau(f) \ = \ \tau(C) \in K_1(\mathbb{C}_X(\mathbb{Z}))$$

with $C = \mathcal{C}(f : C(M) \longrightarrow C(N))$ the algebraic mapping cone of the induced chain equivalence $f : C(M) \longrightarrow C(N)$ in $\mathbb{C}_X(\mathbb{Z})$. If $X = X^+ \cup X^-$ the algebraic splitting obstruction

$$\partial\tau(f) \in \varinjlim_b K_0(\mathbb{P}_{Y_b}(\mathbb{Z}))$$

is such that $\partial\tau(f) = 0$ if and only if f is X-bounded homotopic to an X-bounded homotopy equivalence (also denoted by f) such that the restrictions $f| : f^{-1}(Y) \longrightarrow Y$ are Y-bounded homotopy equivalences, with $Y = X^+, X^-, Y_b$ (for some $b \geq 0$). There is a corresponding algebraic L-theory splitting obstruction for homotopy equivalences of manifolds, using algebraic Poincaré complexes.

An involution on the ground ring R induces a duality involution on the X-bounded R-module category

$$* \ : \ \mathbb{C}_X(R) \longrightarrow \mathbb{C}_X(R) \ ; \ M \ = \ \sum_{x \in X} M(x) \longrightarrow M^* \ = \ \sum_{x \in X} M(x)^* \ ,$$

with $M(x)^* = \mathrm{Hom}_R(M(x), R)$.

Definition 8.1 (Ranicki [17]) The X**-bounded symmetric** L**-groups** $L^*(\mathbb{C}_X(R))$ are the cobordism groups of symmetric Poincaré complexes in $\mathbb{C}_X(R)$. Similarly for the X**-bounded quadratic** L**-groups** $L_*(\mathbb{C}_X(R))$.

□

The symmetrization maps $1 + T : L_*(\mathbb{C}_X(R)) \longrightarrow L^*(\mathbb{C}_X(R))$ are isomorphisms modulo 8-torsion. For compact X $\mathbb{C}_X(R)$ is equivalent to the category of f.g. free

R-modules and

$$L^*(\mathbb{C}_X(R)) \;=\; L^*(R) \;\;,\;\; L_*(\mathbb{C}_X(R)) \;=\; L_*(R)\;.$$

The functor

$$\{\text{metric spaces and proper eventually Lipschitz maps}\}$$
$$\longrightarrow \{\mathbb{Z}\text{-graded abelian groups}\}\;;\; X \longrightarrow L_*(\mathbb{C}_X(R))$$

was shown in Ranicki [17] to be within a bounded distance (in the non-technical sense) of being a generalized homology theory, by an extension to algebraic L-theory of the corresponding properties of the algebraic K-theory obtained by Pedersen and Weibel [14]. The functor is homotopy invariant, and has the following bounded excision property:

Proposition 8.2 (Ranicki [17, 14.2]) *For any metric space X and any decomposition $X = X^+ \cup X^-$ there is defined a Mayer-Vietoris exact sequence in bounded L-theory*

$$\dots \longrightarrow L_n(\mathbb{C}_{X^+}(R)) \oplus L_n(\mathbb{C}_{X^-}(R)) \longrightarrow L_n(\mathbb{C}_X(R))$$
$$\xrightarrow{\partial} \varinjlim_b L^{J_b}_{n-1}(\mathbb{P}_{Y_b}(R)) \longrightarrow L_{n-1}(\mathbb{C}_{X^+}(R)) \oplus L_{n-1}(\mathbb{C}_{X^-}(R)) \longrightarrow \dots\;,$$

with

$$\begin{aligned} Y_b &= \{x \in X \,|\, d(x,y) \le b \text{ and } d(x,z) \le b \text{ for some } y \in X^+, z \in X^-\}\;, \\ J_b &= \ker(\widetilde{K}_0(\mathbb{P}_{Y_b}(R)) \longrightarrow \widetilde{K}_0(\mathbb{P}_X(R))) \\ &\subseteq \widetilde{K}_0(\mathbb{P}_{Y_b}(R)) \;=\; \operatorname{coker}(K_0(\mathbb{C}_{Y_b}(R)) \longrightarrow K_0(\mathbb{P}_{Y_b}(R)))\;. \end{aligned}$$

The J_b-intermediate quadratic L-groups $L^{J_b}_*(\mathbb{P}_{Y_b}(R))$ are such that there is defined a Rothenberg-type exact sequence

$$\dots \longrightarrow L_n(\mathbb{C}_{Y_b}(R)) \longrightarrow L^{J_b}_n(\mathbb{P}_{Y_b}(R)) \longrightarrow \widehat{H}^n(\mathbb{Z}_2; J_b) \longrightarrow L_{n-1}(\mathbb{C}_{Y_b}(R)) \longrightarrow \dots$$

with $\widehat{H}^*(\mathbb{Z}_2; J_b)$ the Tate $\mathbb{Z}_2$-cohomology groups of the duality involution $* : J_b \longrightarrow J_b$.

□

Definition 8.3 The X**-bounded symmetric signature** of an m-dimensional X-bounded geometric Poincaré complex M is the cobordism class

$$\sigma^*(M) \;=\; (C(M), \phi) \in L^m(\mathbb{C}_X(\mathbb{Z}))\;,$$

with ϕ the symmetric structure of the Poincaré duality chain equivalence $[M] \cap - : C(M)^{m-*} \longrightarrow C(M)$.

□

The standard algebraic mapping cylinder argument shows:

Proposition 8.4 *The X-bounded symmetric signature is an X-bounded homotopy invariant of an X-bounded geometric Poincaré complex.*

□

Definition 8.5 The **X-bounded quadratic signature** of a normal map of X-bounded m-dimensional manifolds with boundary $(f,b):(M',\partial M')\longrightarrow(M,\partial M)$ and with $\partial f:\partial M'\longrightarrow\partial M$ an X-bounded homotopy equivalence is the quadratic Poincaré cobordism class

$$\sigma_*(f,b) \;=\; (C(f^!),\psi)\in L_m(\mathbb{C}_X(\mathbb{Z}))\;,$$

with ψ the quadratic structure on the algebraic mapping cone $C(f^!)$ of the Umkehr chain map in $\mathbb{C}_X(\mathbb{Z})$

$$f^! \;:\; C(M) \;\simeq\; C(M,\partial M)^{m-*} \xrightarrow{f^*} C(M',\partial M')^{m-*} \;\simeq\; C(M')\;.$$

□

Proposition 8.6 *The X-bounded quadratic signature is the bounded surgery obstruction of Ferry and Pedersen [9], such that $\sigma_*(f,b)=0$ if (and for $m\geq 5$, $\pi_1(M)=\{1\}$ only if) (f,b) is normal bordant to an X-bounded homotopy equivalence.*

□

The symmetrization of the X-bounded quadratic signature is the X-bounded symmetric signature

$$(1+T)\sigma_*(f,b) \;=\; \sigma^*(M'\cup_{\partial f} -M)\in L^m(\mathbb{C}_X(\mathbb{Z}))\;.$$

See Ranicki [18, Appendix C5] for the construction of **the locally finite assembly maps**

$$A^{lf} \;:\; \mathbb{H}^{lf}_{\cdot}(X;\mathbb{L}_{\cdot}(\mathbb{Z})) \longrightarrow \mathbb{L}_{\cdot}(\mathbb{C}_X(\mathbb{Z}))$$

for any metric space X, with the locally finite homology spectrum defined using locally finite sheaves over X of quadratic Poincaré complexes over $\mathbb{Z}$. The **X-bounded structure groups**

$$\mathbb{S}^b_*(X) \;=\; \pi_*(A^{lf}:\mathbb{H}^{lf}_{\cdot}(X;\mathbb{L}_{\cdot}(\mathbb{Z}))\longrightarrow\mathbb{L}_{\cdot}(\mathbb{C}_X(\mathbb{Z})))$$

are the relative groups in the **X-bounded algebraic surgery exact sequence**

$$\dots \longrightarrow \mathbb{S}^b_{m+1}(X) \longrightarrow H^{lf}_m(X;\mathbb{L}_{\cdot}(\mathbb{Z})) \xrightarrow{A^{lf}} L_m(\mathbb{C}_X(\mathbb{Z})) \longrightarrow \mathbb{S}^b_m(X) \longrightarrow \dots\;.$$

The **structure invariant** $s^b(h)\in\mathbb{S}^b_{m+1}(X)$ of an X-bounded homotopy equivalence $h:M'\longrightarrow M$ of X-bounded m-dimensional manifolds is such that $s^b(h)=0$ if (and for $m\geq 5$, $\pi_1(M)=\{1\}$ only if) h is X-bounded homotopic to a homeomorphism.

For any subspace $K\subseteq S^N$ define the **open cone** metric space

$$O(K) \;=\; \{tx\,|\,x\in K, t\geq 0\}\subseteq\mathbb{R}^{N+1}\;,$$

such that for compact K

$$H^{lf}_{*+1}(O(K);\mathbb{L}.(\mathbb{Z})) \ = \ \widetilde{H}_*(K;\mathbb{L}.(\mathbb{Z})) \ .$$

In particular, $O(S^N)=\mathbb{R}^{N+1}$ and

$$H^{lf}_{*+1}(O(S^N);\mathbb{L}.(\mathbb{Z})) \ = \ \widetilde{H}_*(S^N;\mathbb{L}.(\mathbb{Z})) \ = \ L_{*-N}(\mathbb{Z}) \ .$$

Proposition 8.7 (Ranicki [17],[18]) (i) *The locally finite assembly maps*

$$A^{lf} \ : \ H^{lf}_*(O(K);\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{C}_{O(K)}(\mathbb{Z}))$$

are isomorphisms for any compact polyhedron $K\subseteq S^N$, with $\mathbb{S}^b_*(O(K))=0$. Similarly for symmetric L-theory.
(ii) *The symmetric L-theory orientation $[M]_{\mathbb{L}}\in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z}))$ of a closed m-dimensional manifold M is a topological invariant.*
Proof (i) For any ring with involution R every quadratic complex (C,ψ) in $\mathbb{C}_{O(K)}(R)$ is cobordant to the assembly $A(\Gamma)$ of a locally finite sheaf Γ over $O(K)$ of quadratic complexes over R. If (C,ψ) is a quadratic Poincaré complex it may not be possible to choose Γ such that each of the stalks is a quadratic Poincaré complex over R - the reduced lower K-groups $\widetilde{K}_{-*}(R)$ are the potential obstructions to such a quadratic Poincaré disassembly. This is an $O(K)$-bounded algebraic L-theory version of the lower Whitehead torsion obstruction (4.1) to splitting $O(K)$-bounded homotopy equivalences of open manifolds. For $R=\mathbb{Z}$ the obstruction groups are $\widetilde{K}_{-*}(\mathbb{Z})=Wh_{-*}(\{1\})=0$ by Bass, Heller and Swan [3]. See [18, Appendix C14] for further details.
(ii) Let $M_+=M\cup\{\text{pt.}\}$. Regard $M\times\mathbb{R}$ as an $(m+1)$-dimensional $O(M_+)$-bounded geometric Poincaré complex via the projection $M\times\mathbb{R}\longrightarrow O(M_+)$, with $O(M_+)$ defined using any embedding $M_+\subset S^N$ (N large). The symmetric L-theory orientation of M is the $O(M_+)$-bounded symmetric signature of $M\times\mathbb{R}$

$$\begin{aligned}\sigma^*(M\times\mathbb{R}) \ &= \ [M]_{\mathbb{L}} \\ &\in L^{m+1}(\mathbb{C}_{O(M_+)}(\mathbb{Z})) \ = \ H^{lf}_{m+1}(O(M_+);\mathbb{L}^{\cdot}(\mathbb{Z})) \ = \ H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z})) \ .\end{aligned}$$

A homeomorphism $h:M'\longrightarrow M$ determines an $O(M_+)$-bounded homotopy equivalence $h\times 1:M'\times\mathbb{R}\longrightarrow M\times\mathbb{R}$, so that

$$\begin{aligned}[M]_{\mathbb{L}} \ &= \ \sigma^*(M\times\mathbb{R}) \ = \ (h\times 1)_*\sigma^*(M'\times\mathbb{R}) \ = \ h_*[M']_{\mathbb{L}} \\ &\in H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z})) \ = \ L^{m+1}(\mathbb{C}_{O(M_+)}(\mathbb{Z})) \ .\end{aligned}$$

See [18, Appendix C16] for further details.

□

Remark 8.8 If $N^{4k}\subset M^m\times\mathbb{R}^j$ is a special submanifold there exists a proper map

$$e \ : \ M\times\mathbb{R}^j \longrightarrow \mathbb{R}^i \ \ (i \ = \ m+j-4k)$$

such that $N = e^{-1}(0)$, and there is defined a commutative diagram

$$\begin{array}{ccc} H_m(M;\mathbb{L}^\cdot(\mathbb{Z})) = H^{lf}_{m+j}(M\times\mathbb{R}^j;\mathbb{L}^\cdot(\mathbb{Z})) & & \\ \Big\downarrow & \overset{e_*}{\searrow} & \\ & & H^{lf}_{m+j}(\mathbb{R}^i;\mathbb{L}^\cdot(\mathbb{Z})) = L^{4k}(\mathbb{Z}) \\ & \underset{A}{\nearrow} & \\ H^{lf}_{m+j}(M\times\mathbb{R}^j,(M\times\mathbb{R}^j)\backslash N;\mathbb{L}^\cdot(\mathbb{Z})) = H_{4k}(N;\mathbb{L}^\cdot(\mathbb{Z})) & & \end{array}$$

with A the simply-connected symmetric L-theory assembly map. The symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^\cdot(\mathbb{Z}))$ has image the signature of N

$$e_*([M]_{\mathbb{L}}) = A([N]_{\mathbb{L}}) = \sigma(N) \in L^{4k}(\mathbb{Z}) = \mathbb{Z} .$$

The topological invariance of the symmetric L-theory orientation $[M]_{\mathbb{L}}$ thus implies the topological invariance of the signatures $\sigma(N)$ of special submanifolds, and hence the topological invariance of the $\mathcal{L}$-genus and the rational Pontrjagin classes $\mathcal{L}(M), p_*(M) \in H^{4*}(M;\mathbb{Q})$ (as in 4.2).

□

The **homotopy fixed set** of a pointed space X with π-action is

$$X^{h\pi} = \mathrm{map}_\pi(E\pi_+, X) ,$$

with $E\pi_+ = E\pi \cup \{\mathrm{pt.}\}$.

Let K be a connected compact polyhedron, regarded as a metric space. The action of the fundamental group $\pi = \pi_1(K)$ on the universal cover $\widetilde{K}$ induces an action of π on the spectrum $\mathbb{L}_\cdot(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))$, with the fixed point spectrum such that

$$\mathbb{L}_\cdot(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^\pi \simeq \mathbb{L}_\cdot(\mathbb{C}_K(\mathbb{Z}[\pi])) \simeq \mathbb{L}_\cdot(\mathbb{Z}[\pi]) .$$

The action of π on the cofibration sequence of spectra

$$\mathbb{H}^{lf}_\cdot(\widetilde{K};\mathbb{L}_\cdot(\mathbb{Z})) \xrightarrow{A^{lf}} \mathbb{L}_\cdot(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})) \longrightarrow \mathbb{S}^b(\widetilde{K})$$

determines a cofibration sequence of the homotopy fixed point spectra

$$\mathbb{H}^{lf}_\cdot(\widetilde{K},\mathbb{L}_\cdot(\mathbb{Z}))^{h\pi} \xrightarrow{A^{lf}} \mathbb{L}_\cdot(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^{h\pi} \longrightarrow \mathbb{S}^b(\widetilde{K})^{h\pi}$$

with a homotopy equivalence

$$\mathbb{H}^{lf}_\cdot(\widetilde{K},\mathbb{L}_\cdot(\mathbb{Z}))^{h\pi} \simeq \mathbb{H}_\cdot(K,\mathbb{L}_\cdot(\mathbb{Z})) .$$

The infinite transfer maps of Ranicki [18, p. 328]

$$\mathrm{trf} : L_*(\mathbb{Z}[\pi]) = L_*(\mathbb{C}_K(\mathbb{Z}[\pi])) = L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^\pi) \longrightarrow L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))$$

extend to define a natural transformation of algebraic surgery exact sequences

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathbb{S}_{m+1}(K) & \longrightarrow & H_m(K;\mathbb{L}.(\mathbb{Z})) & \xrightarrow{A} & L_m(\mathbb{Z}[\pi]) & \longrightarrow & \mathbb{S}_m(K) & \longrightarrow & \dots \\
 & & \Big\downarrow \mathrm{trf} & & \Big\downarrow \mathrm{trf}\ \cong & & \Big\downarrow \mathrm{trf} & & \Big\downarrow \mathrm{trf} & & \\
\dots & \longrightarrow & \mathbb{S}^{b,h\pi}_{m+1}(\widetilde{K}) & \longrightarrow & H^{lf,h\pi}_m(\widetilde{K};\mathbb{L}.(\mathbb{Z})) & \xrightarrow{A^{lf}} & L_m(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^{h\pi}) & \longrightarrow & \mathbb{S}^{b,h\pi}_m(\widetilde{K}) & \longrightarrow & \dots
\end{array}$$

with

$$\mathbb{S}^{b,h\pi}_*(\widetilde{K}) \;=\; \pi_*(\mathbb{S}^b(\widetilde{K})^{h\pi}) \;\;,\;\; L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^{h\pi}) \;=\; \pi_*(\mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^{h\pi}) \;.$$

The composite

$$\mathbb{S}_{m+1}(K) \xrightarrow{\mathrm{trf}} \mathbb{S}^{b,h\pi}_{m+1}(\widetilde{K}) \longrightarrow \mathbb{S}^b_{m+1}(\widetilde{K})$$

sends the structure invariant $s(h) \in \mathbb{S}_{m+1}(K)$ of a homotopy equivalence $h : M' \longrightarrow M$ of compact m-dimensional manifolds with a π_1-isomorphism reference map $M \longrightarrow K$ to the $\widetilde{K}$-bounded structure invariant $s^b(\widetilde{h}) \in \mathbb{S}^b_{m+1}(\widetilde{K})$ of the induced $\widetilde{K}$-bounded homotopy equivalence $\widetilde{h} : \widetilde{M'} \longrightarrow \widetilde{M}$ of the universal covers.

Proposition 8.9 *If π is a group such that the universal cover $E\pi$ of the classifying space $B\pi$ is realized by a contractible metric space E with a free π-action and such that the locally finite assembly maps*

$$A^{lf}_\pi \;:\; H^{lf}_*(E;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{C}_E(\mathbb{Z}))$$

are isomorphisms then the integral Novikov conjecture holds for π.

Proof The $E\pi$-bounded structure spectrum $\mathbb{S}^b(E\pi)$ is contractible, and hence so is the homotopy fixed point spectrum $\mathbb{S}^b(E\pi)^{h\pi}$. The locally finite assembly map

$$A^{lf}_\pi \;:\; \mathbb{H}.^{lf}(E\pi;\mathbb{L}.(\mathbb{Z}))^{h\pi} \longrightarrow \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi}$$

is a homotopy equivalence, so that there are defined homotopy equivalences

$$\mathbb{H}.(B\pi;\mathbb{L}.(\mathbb{Z})) \;\simeq\; \mathbb{H}.^{lf}(E\pi;\mathbb{L}.(\mathbb{Z}))^{h\pi} \;\simeq\; \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi} \;.$$

The infinite transfer maps

$$\mathrm{trf} \;:\; \mathbb{L}.(\mathbb{Z}[\pi]) \;\simeq\; \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{\pi} \longrightarrow \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi} \;\simeq\; \mathbb{H}.(B\pi;\mathbb{L}.(\mathbb{Z}))$$

induce splitting maps $\mathrm{trf} : L_*(\mathbb{Z}[\pi]) \longrightarrow H_*(B\pi;\mathbb{L}.(\mathbb{Z}))$ for the assembly maps $A : H_*(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$.

□

Example 8.10 Let $\pi = \mathbb{Z}^n$, so that

$$B\pi \;=\; T^n \;\;,\;\; E \;=\; E\pi \;=\; \mathbb{R}^n \;.$$

Compactify E by adding the $(n-1)$-sphere at infinity

$$\overline{E} \ = \ \mathbb{R}^n \cup S^{n-1} \ = \ D^n \ ,$$

extending the free $\mathbb{Z}^n$-action on $\mathbb{R}^n$ by the identity on $\partial E = S^{n-1}$. In this case the locally finite assembly isomorphisms

$$\begin{array}{l} A^{lf} \ : \ H_*(D^n, S^{n-1}; \mathbb{L}.(\mathbb{Z})) \ = \ H^{lf}_*(\mathbb{R}^n; \mathbb{L}.(\mathbb{Z})) \ = \ \widetilde{H}_{*-1}(S^{n-1}; \mathbb{L}.(\mathbb{Z})) \ = \ L_{*-n}(\mathbb{Z}) \\ \qquad\qquad\qquad \longrightarrow L_*(\mathbb{B}_{D^n, S^{n-1}}(\mathbb{Z})) \ = \ L_*(\mathbb{C}_{\mathbb{R}^n}(\mathbb{Z})) \end{array}$$

and the assembly isomorphisms

$$A \ : \ H_*(T^n; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\mathbb{Z}^n])$$

were already obtained in Ranicki [17], using the identification of the $\mathbb{R}^n$-bounded L-groups of a ring with involution A with the lower L-groups $L^{\langle -* \rangle}_*(A)$

$$L_*(\mathbb{C}_{\mathbb{R}^n}(A)) \ = \ L^{\langle 1-n \rangle}_*(A)$$

and the splitting theorem

$$L_*(A[\mathbb{Z}^n]) \ = \ \sum_{k=0}^{n} \binom{n}{k} L^{\langle 1-k \rangle}_{*-k}(A) \ ,$$

with $L^{\langle -* \rangle}_*(\mathbb{Z}) = L_*(\mathbb{Z})$ by virtue of $Wh_{-*}(\{1\}) = 0$.

□

Theorem 8.11 (Carlsson and Pedersen [7]) *Let π is a group such that the universal cover $E\pi$ of the classifying space $B\pi$ is realized by a contractible metric space E with a free π-action, and with a compactification $\overline{E}$ such that:*

(a) *the free π-action on E extends to a π-action on $\overline{E}$ (which need not be free),*
(b) *$\overline{E}$ is contractible,*
(c) *compact subsets of E become small near the boundary $\partial E = \overline{E} \backslash E$, i.e. for every point $y \in \partial E$, every compact subset $K \subseteq E$ and for every neighbourhood U of y in ∂E, there exists a neighbourhood V of y in X so that if $g \in \pi$ and $g(K) \cap V \neq \emptyset$ then $g(K) \subset U$.*

Then the integral Novikov conjecture holds for π.

□

The proof of the Novikov conjectures by Carlsson and Pedersen [7] uses infinite transfer maps (as in 8.9), but with the continuously controlled category $\mathbb{B}_{X,Y}(\mathbb{Z})$ of Anderson, Connolly, Ferry and Pedersen [1] replacing the bounded category $\mathbb{C}_E(\mathbb{Z})$ of Pedersen and Weibel [14]. For a compact metrizable space X and a closed dense subspace $Y \subseteq X$ $\mathbb{B}_{X,Y}(\mathbb{Z})$ is the category with the same objects as $\mathbb{C}_E(\mathbb{Z})$, where $E = X \backslash Y$. A morphism in $\mathbb{B}_{X,Y}(\mathbb{Z})$

$$f \ = \ \{f(x', x)\} \ : \ A \ = \ \sum_{x \in E} A(x) \longrightarrow B \ = \ \sum_{x' \in E} B(x')$$

is a $\mathbb{Z}$-module morphism such that for every $y \in Y$ and every neighbourhood $U \subseteq X$ of y there is a neighbourhood $V \subseteq U$ such that $f(A(V)) \subseteq B(U)$.

The algebraic transversality of Ranicki [17], [18] is extended in [7, 5.4] to prove that the assembly maps $A : H_*^{lf}(E; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{B}_{X,Y}(\mathbb{Z}))$ are isomorphisms, with $E = E\pi$ and $(X, Y) = (\overline{E}, \partial E)$ - this is the essential step in the proof of 8.11. As in 4.1 and 8.7 (i) there are potential lower Whitehead torsion obstructions to splitting, which are avoided by the computation $Wh_{-*}(\{1\}) = 0$ of Bass, Heller and Swan [3]. The assembly map $A : H_m^{lf,\pi}(E; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_m(\mathbb{B}_{\overline{E},\partial E}(\mathbb{Z})^{h\pi})$ in the commutative square

$$\begin{array}{ccc} H_m(B\pi; \mathbb{L}.(\mathbb{Z})) & \xrightarrow{\quad A \quad} & L_m(\mathbb{Z}[\pi]) \\ \mathrm{trf} \Big\downarrow \cong & & \Big\downarrow \mathrm{trf} \\ H_m^{lf,h\pi}(E; \mathbb{L}.(\mathbb{Z})) & \xrightarrow{\quad A \quad} & L_m(\mathbb{B}_{\overline{E},\partial E}(\mathbb{Z})^{h\pi}) \end{array}$$

is an isomorphism, giving the splitting of the assembly map $A : H_m(B\pi; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_m(\mathbb{Z}[\pi])$.

Example 8.12 Negatively curved groups in the sense of Gromov are the main examples of groups π satisfying the conditions of Theorem 8.11. See Bestvina and Mess [4].

□

Example 8.13 For any integer $g \geq 1$ let

$$\pi_g = \{a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g]\}$$

be the fundamental group of the closed oriented surface M_g of genus g, so that

$$B\pi_g = M_g \ , \ E = E\pi_g = \mathbb{R}^2 \ .$$

For $g = 1$ $M_g = T^2$, as already considered in 8.10. For $g \geq 2$ M_g has a hyperbolic structure, and the free action of π_g on $E = \mathbb{R}^2 = \mathrm{int}(D^2)$ extends to a (non-free) action on $\overline{E} = D^2$, with the identity on $\partial E = S^1$. The hypotheses of 8.11 (and 8.12) are satisfied, so that the assembly maps $A : H_*(B\pi_g; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi_g])$ are split injections. In fact, these assembly maps are isomorphisms, which may be verified as follows. By the Freiheitssatz the subgroup $\rho_g \subset \pi_g$ generated by $a_1, b_1, \dots, a_g$ is free, so that π_g is a twisted extension of ρ_g by $\mathbb{Z} = \{b_g\}$ and $\mathbb{Z}[\pi_g]$ is a twisted Laurent polynomial extension of $\mathbb{Z}[\rho_g]$. The assembly maps $A : H_*(B\rho_g; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\rho_g])$ are isomorphisms by Cappell [6], and the exact sequence of Ranicki [15] for the L-theory of twisted Laurent polynomial extensions combined with $Wh(\pi_g) = 0$ extends this to prove that the assembly maps $A : H_*(B\pi_g; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi_g])$ are isomorphisms.

□

References

[1] D. R. Anderson, F. X. Connolly, S. Ferry and E. K. Pedersen **Algebraic K-theory with continuous control at infinity** J. Pure and App. Alg. (to appear)

[2] H. Bass **Algebraic K-theory** Benjamin (1969)

[3] −−, A. Heller and R. Swan **The Whitehead group of a polynomial extension** Publ. Math. I. H. E. S. 22, 61–80 (1964)

[4] M. Bestvina and G. Mess **The boundary of negatively curved groups** Journal of A. M. S. 4, 469–481 (1991)

[5] W. Browder **Structures on** $M \times \mathbb{R}$ Proc. Camb. Phil. Soc. 61, 337–345 (1965)

[6] S. Cappell **Manifolds with fundamental group a generalized free product.** Bull. A. M. S. 80, 1193–1198 (1974)

[7] G. Carlsson and E. K. Pedersen **Controlled algebra and the Novikov conjectures for K- and L-theory** Topology (to appear)

[8] F. T. Farrell and L. E. Jones **Rigidity in Geometry and Topology** Proc. 1990 I. C. M., Kyoto, 653–663 (1991)

[9] S. Ferry and E. K. Pedersen **Epsilon surgery I.** Math. Gott. 17 (1990)

[10] −− and S. Weinberger **A coarse approach to the Novikov conjecture** in this volume

[11] J. Milnor and J. Stasheff **Characteristic classes** Ann. Math. Stud. 76, Princeton University Press (1974)

[12] S. P. Novikov **On manifolds with free abelian fundamental group and applications (Pontrjagin classes, smoothings, high–dimensional knots)** Izv. Akad. Nauk SSSR, ser. mat. 30, 208–246 (1966)
English translation: A. M. S. Transl. (2) 67, 1–42 (1967)

[13] −− **The algebraic construction and properties of hermitian analogues of K-theory for rings with involution, from the point of view of the hamiltonian formalism. Some applications to differential topology and the theory of characteristic classes.** Izv. Akad. Nauk SSSR, ser. mat. 34, 253–288, 478–500 (1970)

[14] E. K. Pedersen and C. Weibel **K-theory homology of spaces** Proc. 1986 Arcata Algebraic Topology Conf., Springer Lecture Notes 1370, 346-361 (1989)

[15] A. A. Ranicki **Algebraic L-theory III. Twisted Laurent extensions** Proc. 1972 Battelle Seattle K-theory Conf. , Vol. III, Springer Lecture Notes 343, 412–463 (1973)

[16] −− **The algebraic theory of surgery** Proc. Lond. Math. Soc. 40 (3), 87–287 (1980)

[17] −− **Lower K- and L-theory** London Math. Soc. Lecture Notes 178, Cambridge University Press (1992)

[18] −− **Algebraic L-theory and topological manifolds** Cambridge Tracts in Mathematics 102, Cambridge University Press (1992)

[19] −− and M. Yamasaki **Controlled K-theory** Topology and Its Applications (to

appear)
[20] L. Siebenmann **Infinite simple homotopy types** Indag. Math. 32, 479–495 (1970)
[21] C. T. C. Wall **Surgery on compact manifolds** Academic Press (1971)