Annals of Mathematics, **0** (1993), 1–22

$\mathbf{K_4(Z)}$ is the trivial group

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Introduction

We prove the following theorem about Quillen's higher algebraic K-groups of the rational integers.

Main Theorem. $K_4(\mathbb{Z}) = 0$.

$K_i(\mathbb{Z})$ is the ith homotopy group of the algebraic K-theory spectrum $K(\mathbb{Z})$. The determination of these groups is a fundamental problem, but has proved to be difficult in higher degrees. When $i = 0$, 1 or 2 the groups were computed as soon as they were defined (*cf.* Milnor's book [10]), and are $K_0(\mathbb{Z}) \cong \mathbb{Z}$, $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$ and $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$. $K_3(\mathbb{Z})$ was found to be $\mathbb{Z}/48$ by Lee and Szczarba in [7]. We recover their result in this paper. $K_4(\mathbb{Z})$ was previously known to be a finite two– and three–torsion group, with three–torsion of order at most three, by the results of Quillen [13], Lee and Szczarba [8], and Soulé [22].

Until now, these were the only complete calculations of algebraic K-groups of the integers, although all the elements predicted to exist in $K_*(\mathbb{Z})$ by the Lichtenbaum–Quillen conjectures made in [9] and [14] are known to be detected by Borel regulators (by Borel [2]) or by étale K-theory (by Dwyer and Friedlander [5]; see also Soulé [21]). See Dwyer and Friedlander [6] and Mitchell [12] for an extension of the Lichtenbaum–Quillen conjectures to the prime two, and for the expected structure of the groups $K_i(\mathbb{Z})$ for all i. Our theorem confirms the predictions for $i = 4$.

We now outline the argument. It relies on results from two preceding papers by the author, [16] and [17].

First, in [16] we constructed a spectrum level rank filtration of prespectra

$$* \simeq F_0K(R) \longrightarrow F_1K(R) \longrightarrow \dots \longrightarrow F_kK(R) \longrightarrow \dots \longrightarrow K(R)$$

exhausting the free algebraic K-theory spectrum of R. This construction works for rings R such that there only exists injections $R^i \to R^j$ with free cokernel when $i \leq j$. For such rings free R-modules have a well–defined and well–behaved rank. By the free algebraic K-theory of R we mean the K-theory made from the category of finitely generated free R-modules, rather than projective

R-modules, but this K-theory is isomorphic to Quillen's in positive degrees, and for $R = \mathbb{Z}$ the two theories agree. Conceptually, the kth stage $F_kK(R)$ of the rank filtration is then the prespectrum built as a subspectrum of $K(R)$ using only the subcategory of finitely generated free R-modules of rank at most k.

We have the following description of the subquotients of the rank filtration. For every k there is a prespectrum with $GL_k(R)$-action $D(R^k)$, which we call the stable building. Its nth space $D(R^k)_n$ is called the n-dimensional building, by analogy with Tits buildings. There is then a homotopy equivalence of prespectra

$$F_kK(R)/F_{k-1}K(R) \simeq EGL_k(R)_+ \wedge_{GL_k(R)} D(R^k).$$

Hence the map $F_{k-1}K(R) \to F_kK(R)$ is at least as highly connected as the prespectrum $D(R^k)$, since the homotopy orbit construction on the right preserves connectivity.

Second, in Theorem 1.1 of [17] we computed the spectrum homology of the third stage of this filtration in the case $R = \mathbb{Z}$. The result is

$$H_*^{spec}(F_3K(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus T_4, \dots),$$

where T_4 is some group of order dividing four. The essential inputs for this calculation were the group homology of $GL_3(\mathbb{Z})$ as determined by Soulé in [20], the group homology of various parabolic subgroups of $GL_3(\mathbb{Z})$, and their precise interplay. To extend the computation above to $F_4K(\mathbb{Z})$ would require further such information about the group homology of $GL_4(\mathbb{Z})$. The calculations in [17] can be somewhat simplified by the methods of the present paper, *e.g.* by utilizing Proposition 3.2 below.

Third, we need to know something about the connectivity of the inclusion $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$. We conjectured in Conjecture 12.3 of [16] that $D(R^k)$ has the non–equivariant homotopy type of a wedge of $(2k-2)$-spheres for suitable R, including the case $R = \mathbb{Z}$. We refer to this assertion as the connectivity conjecture. Assuming this conjecture, the map $F_kK(\mathbb{Z}) \to F_{k+1}K(\mathbb{Z})$ would be $(2k-1)$-connected for all k, and in particular $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$ would be five–connected. As a consequence, the spectrum homology computation above would imply that $K_4(\mathbb{Z}) = 0$ and $K_5(\mathbb{Z}) \cong \mathbb{Z} \oplus T_8$, where T_8 is a group of order dividing eight. This argument was given in [17]. However, we do not currently have a proof of this conjecture about the connectivity of the stable buildings.

Instead, in this paper we prove enough of the connectivity conjecture to conclude that the map $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$ is at least four–connected. Thus the spectrum homology of $K(\mathbb{Z})$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \dots)$ as a graded group. Our main theorem then follows by using the Atiyah–Hirzebruch spectral sequence for stable homotopy theory, as applied to the spectrum $K(\mathbb{Z})$. We determine the first few differentials and extensions in this spectral sequence by a comparison with Bökstedt's spectrum $JK(\mathbb{Z})$, which can be thought of as a model for étale K-theory. This argument uses his two–complete splitting of the looped underlying space $\Omega JK(\mathbb{Z})$ off from $\Omega K(\mathbb{Z})$, as given in [4], and is explained in Section 6 of this paper.

The paper is organized as follows.

In Section 1 we review the spectrum level rank filtration from [16] and define the n-dimensional buildings. We describe how to associate a partially ordered set (poset) to each simplex of an n-dimensional building, and use this to obtain a filtration on these buildings indexed by isomorphism classes of posets, called the poset filtration.

Section 2 proves a version of the Solomon-Tits theorem for principal ideal domains. In Section 3 we define a coarser filtration on the stable buildings than the poset filtration, called the component filtration. We give a way to analyze the component filtration in Section 4, by comparing it with a combinatorially defined filtration on the so-called partition lattice.

In Section 5 we apply these ideas to prove connectivity results for stable buildings when R is a principal ideal domain. Proposition 5.1 proves the easy half of the connectivity conjecture in this case, namely that $D(R^k)$ is at least $(k-1)$-connected when $k \geq 2$. Proposition 5.2 states that $D(R^4)$ is at least four–connected. It follows that every map $F_kK(\mathbb{Z}) \to F_{k+1}K(\mathbb{Z})$ is at least four–connected for $k \geq 3$, and thus $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$ is four–connected.

Finally, in Section 6 we recall the computations from [17], and make the comparison with Bökstedt's $JK(\mathbb{Z})$, to evaluate the Atiyah–Hirzebruch spectral sequence abutting to $K_*(\mathbb{Z})$ through total degree four. This argument only uses the four–connectivity of the map $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$, and may if desired be read before the more combinatorial Sections 2 through 5.

1. The spectrum level rank filtration

We recall the results we will need from [16], referring the reader to that paper for a more detailed presentation.

The spectrum level rank filtration is defined by means of a particular model for the algebraic K-theory spectrum, with nth space given by iterating Waldhausen's $S_\bullet$-construction n times. Let $\mathcal{F}(R)$ be the category of finitely generated free R-modules. It is a category with cofibrations in Waldhausen's sense, using injections with free cokernels as the cofibrations. See Waldhausen's foundational paper [24] for these notions, and for the precise definitions we are thinking of when writing "roughly" in the paragraph below. We call a homomorphism $f : U \to V$ of finitely generated free R-modules a cofibration if it is injective and has free cokernel, and use the feathered arrow $U \rightarrowtail V$ to denote this situation.

Given a category with cofibrations $\mathcal{C}$, the $S_\bullet$-construction gives a simplicial category with cofibrations denoted $S_\bullet\mathcal{C}$. Let $[q] = \{0 \to 1 \to \cdots \to q\}$. In simplicial degree q, $S_q\mathcal{C}$ is roughly the category of diagrams

$$X \colon [q] \longrightarrow \mathcal{C},$$

with $X(0) = 0$ and the morphisms $X(i) \rightarrowtail X(i+1)$ in the diagram given by cofibrations. Iterating the $S_\bullet$-construction n times, and taking the diagonal, we obtain a simplicial category $S^n_\bullet\mathcal{C}$ whose objects are roughly n-dimensional cubical diagrams

$$X \colon [q]^n \longrightarrow \mathcal{C}.$$

Here $[q]^n$ is the cartesian product of n copies of $[q]$. The nth space $K(R)_n$ of the free algebraic K-theory spectrum is set to be the geometric realization of the nerve of the isomorphism subcategory of the n-fold iterated $S_\bullet$-construction on $\mathcal{F}(R)$.

$$K(R)_n = |NiS^n_\bullet\mathcal{F}(R)|$$

As usual, the prefix i denotes the isomorphism subcategory, N denotes nerve, and the vertical bars denote geometric realization.

Note. We assume throughout the paper that the unital, associative ring R is such that there are no cofibrations $R^i \rightarrowtail R^j$ unless $i \le j$. This excludes certain large rings, such as endomorphism rings of infinite dimensional vector spaces, but includes all commutative rings.

For every rank k let $F_kK(R)_n \subset K(R)_n$ be the subspace realizing the nerve of the simplicial subcategory of $iS^n_\bullet\mathcal{F}(R)$ consisting of diagrams of modules in $\mathcal{F}(R)$ of rank at most k. Momentarily fixing k, the collection of spaces $\{F_kK(R)_n\}_n$ forms a prespectrum $F_kK(R)$, which is the kth stage of the spectrum level rank filtration. The sequence of prespectra

$$* \simeq F_0K(R) \longrightarrow F_1K(R) \longrightarrow \dots \longrightarrow F_kK(R) \longrightarrow \dots \longrightarrow K(R)$$

is then induced by the obvious inclusions.

The n-dimensional building of rank k, denoted $D(R^k)_n$, can be defined as the simplicial object set of a full subcategory of $S^n_\bullet\mathcal{F}(R)$. The full subcategory has as objects the diagrams $X \colon [q]^n \longrightarrow \mathcal{F}(R)$ where the top module $X(q, \dots, q)$ equals R^k and the morphisms in the diagram are inclusions (as opposed to mere injections). Whenever it is convenient, we will think of $D(R^k)_n$

and the related simplicial sets defined below as spaces, by passing to geometric realization in the usual way. We now give an explicit reformulation of the definition of these generalized buildings.

Suppose given an increasing sequence $1 \le i_1 < \cdots < i_m \le n$ and an object $(a_1, \ldots, a_n) \in [q]^n$ with $0 \le a_s < q$ for all s. These data determine a functor $u\colon [1]^m \to [q]^n$, given by $u(\epsilon_1, \ldots, \epsilon_m) = (u_1, \ldots, u_n)$ with $\epsilon_s \in \{0,1\}$ for all s, where $u_s = a_s + \epsilon_t$ if $s = i_t$ for some $1 \le t \le m$, and $u_s = a_s$ otherwise. We call a functor $u\colon [1]^m \to [q]^n$ arising in this way a little subcube of $[q]^n$.

Let $\mathrm{Sub}(R^k)$ be the subcategory of $\mathcal{F}(R)$ of free submodules of R^k and genuine inclusions.

Definition 1.1. The n-dimensional building of rank k, written $D(R^k)_n$, is the simplicial set with q-simplices the set of all diagrams

$$X\colon [q]^n \longrightarrow \mathrm{Sub}(R^k)$$

such that

(i) $X(a_1, \ldots, a_n) = 0$ if some $a_s = 0$,
(ii) $X(q, \ldots, q) = R^k$, and
(iii) for every little subcube $u\colon [1]^m \to [q]^n$, the induced homomorphism $\delta(X, u)$ from the iterated pushout of $X \circ u$ over $[1]^m - (1, \ldots, 1)$ to $X(u(1, \ldots, 1))$ is a cofibration.

In addition there is a base point $*$ in every simplicial degree.

(Condition (iii) is inherited from the manner in which $S_\bullet\mathcal{C}$ is made into a simplicial category with cofibrations.)

The simplicial face maps d_i for $1 \le i \le q$ are given by deleting the modules $X(a_1, \ldots, a_n)$ where some $a_s = i$. Any diagram then appearing with $X(q, \ldots, q) \ne R^k$ is identified with the base point. The zeroth face map d_0 maps every nondegenerate simplex to the base point, and is determined on degenerate simplices by the simplicial identities. The simplicial degeneracy maps s_j are given by repeating the modules $X(a_1, \ldots, a_n)$ where some $a_s = j$.

The spaces $D(R^k)_n$ for varying n assemble into a prespectrum, which we denote $D(R^k)$ and refer to as the stable building. $GL_k(R)$ acts on each $D(R^k)_n$ through its action on submodules of R^k. The following is Proposition 3.8 of [16].

PROPOSITION 1.2. *$F_kK(R)/F_{k-1}K(R) \simeq EGL_k(R)_+ \wedge_{GL_k(R)} D(R^k)$.*

Fix a q-simplex $X \ne *$ in $D(R^k)_n$. By induction X is determined by the locations $u(1, \ldots, 1)$ of little subcubes $u\colon [1]^n \to [q]^n$ such that the homomorphism $\delta(X, u)$ is not an isomorphism (see Definition 1.1(iii)), together with the modules $X(u(1, \ldots, 1))$ appearing there. The multiplicity of such a location $\vec{p} = u(1, \ldots, 1)$ is defined to be the rank of the cokernel of $\delta(X, u)$, and the sum of these multiplicities equals k by Corollary 5.5 of [16].

Hence we can associate to X an unordered k-tuple of locations $\{\vec{p}_1, \dots, \vec{p}_k\}$ in the indexing category $[q]^n$, called the special locations of the simplex. (These were called pick sites in [16].) Each special location occurs as often in this list as its multiplicity. The special locations inherit a partial ordering from the product partial ordering on $[q]^n$, which in turn determines a partial ordering on the set $\{1, \dots, k\}$ by defining $i \prec j$ for $i,\, j \in \{1, \dots, k\}$ if and only if $\vec{p}_i$ precedes $\vec{p}_j$ in $[q]^n$. This defines the poset associated to X, denoted $\omega(X)$. It is only defined modulo indeterminacy caused by renumbering the locations $\{\vec{p}_1, \dots, \vec{p}_k\}$, *i.e.* up to isomorphism.

Note that what we for brevity are calling a poset is usually called a quasi–ordering. We do not insist that if two elements i and j of a poset satisfy $i \prec j$ and $j \prec i$ then $i = j$. We say that one partial ordering on a set is stronger than another partial ordering on the same set if the former can be obtained by adjoining relations to the latter.

The simplicial face maps can only introduce additional relations between the special locations, or equivalently strengthen the associated poset. Fixing a poset ω we can therefore define a subspace $F_{[\omega]}D(R^k)_n$ of $D(R^k)_n$ consisting of simplices X whose associated poset $\omega(X)$ is at least as strong as ω, up to isomorphism. The resulting filtration $\{F_{[\omega]}D(R^k)_n\}_{[\omega]}$ of $D(R^k)_n$ indexed by the isomorphism classes of partial orderings on $\{1, \dots, k\}$ is the poset filtration on the n-dimensional building. Let $F^{[\omega]}D(R^k)_n$ be the $[\omega]$th filtration subquotient of this filtration, *i.e.* the quotient of $F_{[\omega]}D(R^k)_n$ obtained by collapsing all simplices with associated poset strictly stronger than ω (up to isomorphism) to the base point.

To analyze the poset filtration we make a comparison with the corresponding construction on the K-theory of finite sets, where we completely determined the homotopy types of the filtration subquotients in [16].

Let $\mathrm{Sub}(\{1, \dots, k\})$ be the category of subsets of $\{1, \dots, k\}$ and inclusions.

Definition 1.3. The n-dimensional apartment of rank k, denoted A^k_n, is the simplicial set with q-simplices the set of all diagrams

$$X \colon [q]^n \longrightarrow \mathrm{Sub}(\{1, \dots, k\})$$

such that

(i) $X(a_1, \dots, a_n) = \emptyset$ if some $a_s = 0$,

(ii) $X(q, \dots, q) = \{1, \dots, k\}$, and

(iii) for every little subcube $u \colon [1]^m \to [q]^n$, the induced map $\alpha(X, u)$ from the iterated pushout of $X \circ u$ over $[1]^m - (1, \dots, 1)$ to $X(u(1, \dots, 1))$ is an injection.

In addition there is a base point $*$ in every simplicial degree. The simplicial structure maps are defined by deletion and repetition, as for the n-dimensional building.

Now we have Proposition 4.5 of [16].

PROPOSITION 1.4. *A^k_n is homeomorphic to an nk-sphere.*

We can think of A^k_n as a subspace of $D(R^k)_n$ by identifying a subset U of $\{1,\dots,k\}$ with the free submodule of R^k where the coordinates not in U are zero. More generally, let a line $L \subseteq R^k$ be a free rank one submodule with free quotient. Then given a decomposition of R^k as the direct sum of k lines $L_1,\dots,L_k$, we can embed A^k_n into $D(R^k)_n$ by taking a subset $U \subseteq \{1,\dots,k\}$ to the direct sum of the L_s with $s \in U$. The union of the images of these embeddings $\ell\colon A^k_n \to D(R^k)_n$ covers $D(R^k)_n$.

The apartments also admit a poset filtration. If X is a simplex in A^k_n, the ith special location $\vec{p}_i \in [q]^n$ is the minimal location where i appears in the set $X(a_1,\dots,a_n)$. So we have a canonical numbering of the special locations, and the associated poset $\omega(X)$ is well defined, not only up to isomorphism. Let $F_\omega A^k_n \subseteq A^k_n$ be the subspace of simplices whose associated poset is at least as strong as ω. Let $F^\omega A^k_n$ be the quotient of $F_\omega A^k_n$ obtained by collapsing all simplices with associated poset strictly stronger than ω to the base point. The embeddings $A^k_n \to D(R^k)_n$ above clearly take $F_\omega A^k_n$ into $F_{[\omega]} D(R^k)_n$.

The apartments A^k_n for varying n assemble into a prespectrum A^k, called the stable apartment, which embeds into the stable building as just indicated. There are similar prespectra assembled from the subspaces and subquotients of the poset filtrations, denoted by omitting the subscript n from the notations.

We repeat that what we are calling a poset is often called a quasi–ordering. See Section 7 of [16] for more detail on our terminology on posets. Given two elements i and j of a poset with $i \prec j$ and $j \prec i$ we say that i and j are equivalent, but do not insist that $i = j$. We call a partial ordering indiscrete if all its elements are equivalent, *i.e.* if every element precedes every other element. A partial ordering is discrete if every element precedes no other element. We call a partial ordering linear if every element precedes or is preceded by every other. The components of a partial ordering are (the vertex sets of) the path components of the geometric realization of its nerve when viewed as a category. Let $\mathrm{comp}(\omega)$ be the number of components of ω, and let $\mathrm{size}(\omega) = \mathrm{comp}(\omega) +$ (the number of equivalence classes in ω) $-\,2$.

The following description of the stable homotopy types in the poset filtration on A^k_n combines Propositions 9.1 and 11.12 with Corollary 13.13 from [16].

PROPOSITION 1.5. *Let ω be a partial ordering on $\{1,\dots,k\}$.*

(i) If ω is indiscrete, then the prespectrum $F_\omega A^k$ is homotopy equivalent to the sphere spectrum $\Sigma^\infty S^0$. Otherwise this prespectrum is contractible.

(ii) If every component of ω *is linear, then the prespectrum* $F^\omega A^k$ *is homotopy equivalent to the suspension spectrum on a wedge of* $(\mathrm{comp}(\omega)-1)!$ *spheres of dimension* $\mathrm{size}(\omega)$*. Otherwise this prespectrum is contractible.*

In particular, when ω is the discrete partial ordering on $\{1,\dots,k\}$ with k components, $H^{spec}_*(F^\omega A^k)$ is concentrated in degree $(2k-2)$, where it is a free $\mathbb{Z}$-module of rank $(k-1)!$ which we denote W_k. The permutation group Σ_k acts on W_k through the permutations on $\mathrm{Sub}(\{1,\dots,k\})$. W_1 is the trivial representation, W_2 is the sign representation, and generally W_k maps under complexification to the Σ_k-representation induced up from a faithful one–dimensional representation over the usual inclusion $\mathbb{Z}/k \to \Sigma_k$. See also Corollary 4.7.

The poset filtrations on buildings and apartments are closely related. Proposition 8.6 of [16] gives the following splitting result, where we omit to specify which wedge summands occur, as this would require introducing some otherwise extraneous notation.

Proposition 1.6. $F^{[\omega]}D(R^k)_n \simeq \vee F^\omega A^k_n$. *Hence* $F^{[\omega]}D(R^k)$ *is contractible if* ω *is not componentwise linear.*

This completes our review of results from [16].

2. Tits buildings for PIDs

We will need some results on Tits buildings for principal ideal domains. We adapt Quillen's proof in [13] of the Solomon–Tits theorem from [19] to this more general case.

Recall that when R is a field, the Tits building $B(R^k)$ is the nerve of the set of proper nontrivial submodules of R^k, partially ordered with respect to inclusion. We now extend the definition to general R.

Suppose $U \subseteq V$ is an inclusion of finitely generated free R-modules with free quotient, *i.e.* $U \rightarrowtail V$. Then let $b(U,V)$ be the set of free R-modules W with $U \subset W \subset V$, such that the inclusions are cofibrations, partially ordered by setting $W_1 \prec W_2$ if $W_1 \subseteq W_2$ and the inclusion is a cofibration. Let $B(U,V)$ be the geometric realization of the nerve of $b(U,V)$. We write $B(V) = B(0,V)$.

The following lemma then proves that some constructions which can be made for vector spaces also make sense in our context of modules over a principal ideal domain.

Lemma 2.1. *Let* R *be a principal ideal domain. Suppose* $R^k = U \oplus V$ *is a direct sum decomposition, with* U *and* V *free submodules of* R^k*. Then* $f\colon B(U) \to B(V,R^k)$ *given by* $W \mapsto V \oplus W$*, and* $g\colon B(V,R^k) \to B(U)$ *given by* $W \mapsto U \cap W$*, are well defined maps which are mutual inverses.*

Proof. f is clearly well defined. To see that g is well defined it suffices to know that given cofibrations $0 \rightarrowtail U \rightarrowtail R^k$ and $0 \rightarrowtail W \rightarrowtail R^k$, there are cofibrations $0 \rightarrowtail U \cap W \rightarrowtail R^k$. $U \cap W$ is free since $U \cap W \subseteq U$, U is free, and R is a principal ideal domain. Hence $0 \rightarrowtail U \cap W$. The inclusion $U \rightarrowtail R^k$ induces an injection $U/(U \cap W) \to R^k/W$, and R^k/W is free, so $U/(U \cap W)$ is free. Hence $U \cap W \rightarrowtail U$ is a cofibration, and so is the composite $U \cap W \rightarrowtail U \rightarrowtail R^k$. This proves that g is well defined.

Next we prove that $gf = 1$. Given $0 \rightarrowtail W \rightarrowtail U$ and $R^k = U \oplus V$ as above, we need to prove that $W = U \cap (V \oplus W)$. The inclusion $W \subseteq U \cap (V \oplus W)$ is obvious. For the reverse inclusion, let $x = v + w$ with $x \in U$, $v \in V$, and $w \in W \subset U$. Then $v = x - w \in U \cap V = 0$, so $x = w \in W$.

Finally we prove $fg = 1$. The inclusion $V \oplus (U \cap W) \subseteq W$ is clear. For the reverse inclusion, consider the maps induced by inclusions $V \to W/(U \cap W) \to R^k/U$. The second map, and the composite, are both isomorphisms; hence so is the first. Thus $V \oplus (U \cap W) = W$. □

For the next results we fix a maximal flag in R^k as follows. Choose the usual ordered basis for the free module R^k, and for $s < k$ denote by R^s the free submodule spanned by the first s basis vectors.

PROPOSITION 2.2. *Let R be a principal ideal domain. Then there is a natural homotopy equivalence*

$$B(R^k) \xrightarrow{\simeq} \bigvee_L \Sigma B(R^{k-1})$$

where the wedge sum runs over all lines $L \subset R^k$ such that $R^{k-1} \oplus L = R^k$ is a direct sum decomposition of R^k.

Proof. Let $\bar{b}(0, R^k)$ be the set theoretic complement in $b(0, R^k)$ of the set of lines L indexing the wedge sum above, with the inherited subset partial ordering. Let $\bar{B}(R^k) \subset B(R^k)$ be the geometric realization of the nerve of $\bar{b}(0, R^k)$.

Then $\bar{B}(R^k)$ deformation retracts to the cone on $B(R^{k-1})$ with vertex point R^{k-1}, by the map induced by intersection with R^{k-1}. On the poset level this is $W \mapsto R^{k-1} \cap W$. This is immediate from Lemma 2.1, by means of the natural transformation $R^{k-1} \cap W \rightarrowtail W$ of functors mapping $\bar{b}(0, R^k)$ onto $b(R^{k-1})$ union a terminal point represented by R^{k-1} itself. Hence $\bar{B}(R^k)$ is contractible.

The quotient of $B(R^k)$ by $\bar{B}(R^k)$ splits as a wedge over the lines L of suspended copies of the spaces $B(L, R^k)$, each of which is homeomorphic to $B(R^{k-1})$ by the lemma above. □

For chosen sum decomposition of R^k into a sum of lines $\oplus_{s=1}^k L_s$, and $U \subseteq \{1, \dots, k\}$, let $L_U = \oplus_{s \in U} L_s \subseteq R^k$. There is then a subspace of $B(R^k)$ called an apartment (in Tits' sense), which is homeomorphic to S^{k-2}, and realizes the nerve of the set of modules of the form L_U with $\emptyset \neq U \subset \{1, \dots, k\}$. Let Apt^k be the nerve of the poset of proper nontrivial subsets of $\{1, \dots, k\}$, partially ordered by inclusion. Then Apt^k is homeomorphic to S^{k-2}, and the association $U \mapsto L_U$ takes Apt^k homeomorphically onto the Tits apartment above. In more detail, the chain

$$\Delta = \{1\} \subset \{1,2\} \subset \cdots \subset \{1, \dots, k-1\}$$

is a (maximal) $(k-2)$-simplex in Apt^k, and its orbits under the obvious Σ_k-action cover Apt^k.

The inclusion of the wedge of some of the Tits apartments into $B(R^k)$ is a homotopy equivalence, as stated below, which now follows by induction from Proposition 2.2.

THEOREM 2.3 (Tits–Solomon). *Let R be a principal ideal domain. Then $B(R^k) \simeq \vee S^{k-2}$, with one wedge summand (apartment) for each sum decomposition $\oplus_{s=1}^k L_s = R^k$ into lines with $L_s \subseteq R^s \subseteq R^k$ for all s.* □

Definition 2.4. We will say that R has spherical Tits buildings if $B(R^k)$ has the non–equivariant homotopy type of a wedge of $(k-2)$-spheres, for all $k \geq 1$.

By the Theorem above, principal ideal domains have spherical Tits buildings. We proved in [18] that nilpotent rings of the form $\mathbb{Z}/p^n$ have spherical Tits buildings.

3. The component filtration

A central observation in this paper is that there is a useful filtration on the n-dimensional building by the number of components of the posets associated to its simplices, for which the subquotients have a description given in terms of Tits buildings and known homotopy types arising from the poset filtration on stable apartments. We obtain our connectivity results about stable buildings by means of this filtration.

Let ${}_cD(R^k)_n$ be the union of the $F_{[\omega]}D(R^k)_n$ over the posets ω with at most c components, *i.e.* whose realization viewed as a category has at most c path components. Let ${}^cD(R^k)_n = {}_cD(R^k)_n / {}_{c-1}D(R^k)_n$ be the filtration subquotient consisting of simplices whose associated poset has precisely c components.

Similarly let ${}_cA^k_n$ be the subspace of A^k_n of simplices whose associated poset has at most c components, and let ${}^cA^k_n = {}_cA^k_n / {}_{c-1}A^k_n$ be the filtration subquotient. Stably we obtain a filtration $\{{}_cD(R^k)\}_c$ on the prespectrum $D(R^k)$, with subquotient prespectra ${}^cD(R^k)$, and similarly for A^k.

When V is a finitely generated free R-module, let $\Sigma B(V)$ denote the suspended Tits building. We will use as a model for $\Sigma B(V)$ the nerve of the poset of nontrivial, not necessarily proper submodules of V, divided out by the Tits building $B(V)$ realized as the nerve of the poset of proper nontrivial submodules of V, all subject to the usual cofibration conditions. The containing space is contractible due to the presence of the maximal element V, so the quotient has the desired homotopy type.

PROPOSITION 3.1. *There is a natural chain of maps from ${}^cD(R^k)_n$ to*

$$\bigvee \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge {}^cA^c_n$$

inducing a weak homotopy equivalence of prespectra with $GL_k(R)$-action. The wedge sum runs over all unordered sets $\{V_1, \dots, V_c\}$ of free nontrivial submodules of R^k with $\oplus_{s=1}^c V_s = R^k$, and $GL_k(R)$ permutes the wedge summands via the action on the V_s. Hence

$${}^cD(R^k) \simeq \bigvee \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge {}^cA^c.$$

If R has spherical Tits buildings, then ${}^cD(R^k) \simeq \vee \Sigma^\infty S^{k+c-2}$.

Proof. Consider the subspace $Y \subseteq {}^cD(R^k)_n$ of simplices with poset consisting of precisely c linear components. The componentwise linear posets are the strongest among those with precisely c components. Hence Y is a subspace, and by Proposition 1.6 the inclusion induces a homotopy equivalence of prespectra, respecting the group action.

The special locations of a simplex $X \neq *$ in Y lie in c linear chains, each unrelated to (not succeeding or preceding) the others in the n-dimensional indexing cube $[q]^n$. Such a simplex is determined by two things: first, the c flags of submodules of R^k appearing at these special locations, and second, where in $[q]^n$ these special locations are.

The former amounts to a c-tuple of simplices in $\Sigma B(V_1)$ through $\Sigma B(V_c)$ respectively, where $V_1, \dots, V_c$ are the maximal submodules in the c flags. This is a simplex in $\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c)$.

The latter deformation retracts to its simplicial subset of c-tuples of unrelated constant linear chains of special locations, or equivalently the simplicial set ${}^cA^c_n$ of c unrelated special locations in $[q]^n$. This proves the splitting result.

Finally suppose that each $\Sigma B(V_s)$ is a wedge of $(k_s - 1)$-spheres, where k_s is the rank of V_s. By Proposition 1.5(ii) we have ${}^cA^c \simeq \vee \Sigma^\infty S^{2c-2}$. Hence

${}^cD(R^k)$ is non–equivariantly homotopy equivalent to the suspension spectrum on a wedge of spheres of dimension $\sum_{s=1}^{c}(k_s - 1) + (2c - 2) = k + c - 2$. □

When R has spherical Tits buildings, the Steinberg module $\mathrm{St}(R^k)$ is defined as the $GL_k(R)$-representation $\widetilde{H}_{k-2}(B(R^k))$. Σ_k acts as the sign representation $\mathbb{Z}_{\mathrm{sgn}}$ on $\widetilde{H}_{k-2}(Apt^k)$. Also recall the Σ_k-representation $W_k = H^{spec}_{2k-2}({}^kA^k)$ from Section 1. It is free abelian of rank $(k-1)!$.

The component filtration $\{{}_cD(R^k)\}_c$ determines a spectral sequence in spectrum homology, which degenerates to a horizontal row by Proposition 3.1. Hence the E^1-term (E^1_*, d^1_*) is a naturally defined chain complex whose homology computes $H^{spec}_*(D(R^k))$. This proves the following two results.

Proposition 3.2. *Suppose R has spherical Tits buildings. The complex (Z_*, d_*) of $\mathbb{Z}GL_k(R)$-modules associated to the component filtration on $D(R^k)$ satisfies*

$$Z_{k+c-2} \cong \bigoplus \mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_c) \otimes W_c$$

where the sum runs over all unordered decompositions of R^k as the direct sum of c nontrivial free submodules $V_1, \dots, V_c$. The homology groups $H_*(Z_*, d_*)$ are naturally isomorphic to $H^{spec}_*(D(R^k))$. □

Corollary 3.3. *Suppose R has spherical Tits buildings. Then $D(R^k)$ is $(k-2)$-connected for all $k \geq 1$.*

There are similar results for the component filtration on the stable apartment prespectra A^k, which we now explain.

Given a sum decomposition $R^k = \oplus_{s=1}^{k} L_s$, we have an associated embedding $\ell\colon A^k_n \to D(R^k)_n$, which maps onto the subspace of $D(R^k)_n$ where all the submodules of R^k occurring are of the form $L_U = \oplus_{s\in U} L_s$. The associated component filtration on A^k has subquotients

$${}^cA^k \simeq \bigvee \Sigma Apt^{k_1} \wedge \cdots \wedge \Sigma Apt^{k_c} \wedge {}^cA^c.$$

Here the wedge sum runs over partitions of R^k into unordered sums of nontrivial submodules V_s all of the form L_U, and the factor $\Sigma Apt^{k_s} \cong S^{k_s-1}$ embeds into $\Sigma B(V_s)$ as described at the end of Section 2. As before, k_s is the rank of V_s.

Clearly this can be rewritten as the wedge sum over all unordered partitions of $\{1, \dots, k\}$ into c nonempty subsets $U_1, \dots, U_c$. So taking the homology of these subquotients, we obtain the following result.

Proposition 3.4. *The complex (Y_*, d_*) of $\mathbb{Z}\Sigma_k$-modules associated to the component filtration on A^k satisfies*

$$Y_{k+c-2} \cong \bigoplus \mathbb{Z}_{\mathrm{sgn}} \otimes W_c$$

where the sum runs over all unordered partitions of $\{1,\dots,k\}$ *into* c *nonempty subsets. It maps naturally to* (Z_*, d_*) *when given a choice of sum decomposition* $R^k = \oplus_{s=1}^k L_s$*. The complex* (Y_*, d_*) *is exact when* $k \geq 2$*.*

Proof. The only remaining observation is that the prespectrum A^k is contractible for $k \geq 2$ by Proposition 1.4, so the homology of the complex (Y_*, d_*) vanishes in these cases. □

4. The partition lattice

There is a combinatorial description of the complex in Proposition 3.4 in terms of the poset of partitions on $\{1,\dots,k\}$, which we now describe. We refer to *e.g.* the paper [23] by Stanley for further detail on partitions.

Let Π_k be the set of partitions β of $\{1,\dots,k\}$ into disjoint nonempty subsets covering $\{1,\dots,k\}$. We call these subsets the components of the partition β. We partially order Π_k by reverse refinement, *i.e.* $\beta_1 \prec \beta_2$ if β_2 refines β_1. Π_k with this partial ordering is called the partition lattice on k elements.

Σ_k acts on Π_k through its action on $\{1,\dots,k\}$. For a partition $\beta \in \Pi_k$, let $(\Sigma_k)_\beta$ be its stabilizer for this group action. Let $|\beta|$ be the number of components of β, so $(\Sigma_k)_\beta$ naturally acts on $\{1,\dots,|\beta|\}$ when given an enumeration of the set of components of β.

Let $\iota_k \in \Pi_k$ be the initial, indiscrete partition with one component, and let $\delta_k \in \Pi_k$ be the terminal, discrete partition with k components. Let $\Pi'_k = \Pi_k - \{\iota_k\}$ and $\Pi''_k = \Pi_k - \{\iota_k, \delta_k\}$ be given the subset partial orderings. Write $N\Pi_k$, $N\Pi'_k$ and $N\Pi''_k$ for the nerves of the respective posets. Then $N\Pi_k$ and $N\Pi'_k$ are contractible, and $N\Pi''_k$ has reduced homology concentrated in degree $(k-3)$ since the partition lattice is Cohen–Macaulay; compare Theorem 1.2 and Section 3 of [23]. We write

$$W_k^\pi = \widetilde{H}_{k-3}(N\Pi''_k)$$

for the homology of the partition lattice, viewed as a $\mathbb{Z}\Sigma_k$-module. The notation is motivated by Corollary 4.7 below. W_k^π has rank $(k-1)!$ and restricts to the regular representation on $\Sigma_{k-1} \subset \Sigma_k$.

For $\beta \in \Pi_k$ let $F_\beta N\Pi'_k \subseteq N\Pi'_k$ be the nerve of the subposet of Π'_k consisting of all $\beta_1 \in \Pi'_k$ with $\beta_1 \prec \beta$. Then $\{F_\beta N\Pi'_k\}_\beta$ is a filtration of $N\Pi'_k$ indexed by all partitions $\beta \in \Pi$, and every $F_\beta N\Pi'_k$ for $\beta \neq \iota_k$ is contractible, due to the presence of the terminal element β. $F_{\iota_k} N\Pi'_k$ is empty, and it will be convenient to think of it as a (-1)-sphere.

Suppose $1 \leq c \leq k$. Let ${}_cN\Pi'_k \subseteq N\Pi'_k$ be the nerve of the subposet of Π'_k consisting of partitions with at most c components. If β has c numbered components, we can identify the partitions $\beta_1 \prec \beta$ with partitions of $\{1, \dots, c\}$ by only keeping track of components. Thus the filtration subquotient ${}_cN\Pi'_k / {}_{c-1}N\Pi'_k$ splits as a wedge of copies of $N\Pi'_c / N\Pi''_c \simeq \Sigma N\Pi''_c$ indexed by the partitions of k into c components. Hence its reduced homology is concentrated in degree $(c-2)$.

For H a subgroup of G and V an H-module, we write $\mathrm{ind}_H^G(V) = \mathbb{Z}G \otimes_H V$ for the induced G-module. Let $p(k)$ be a set of orbit representatives for the Σ_k-action on Π_k, and let $p_c(k) \subseteq p(k)$ be the subset of partitions with c components.

Proposition 4.1. *The complex* (X_*, d_*) *of* $\mathbb{Z}\Sigma_k$*-modules associated to the component filtration* $\{{}_cN\Pi'_k\}_c$ *on the nerve of the partition lattice satisfies*

$$X_{c-2} \cong \bigoplus_{\beta \in p_c(k)} \mathrm{ind}_{(\Sigma_k)_\beta}^{\Sigma_k}(W_c^\pi).$$

This complex is exact for $k \geq 2$. □

By identifying $F_\beta N\Pi'_k$ with $N\Pi'_c$ in the case where β has c components we can inductively describe the boundary map $d_{c-2} \colon X_{c-2} \to X_{c-3}$ on the summand corresponding to β in the complex above. When $c < k$ it is induced up from the top boundary map d_{c-2} in the corresponding complex for the partition lattice on c elements, via the projection $(\Sigma_k)_\beta \to \Sigma_c$. In the case $c = k$, the top boundary map d_{k-2} is determined by exactness of the complex (X_*, d_*). We make the case $k = 4$ explicit for later use.

Example 4.2. Suppose $k = 4$. The proposition above describes the exact complex (X_*, d_*) as

$$0 \longrightarrow W_4^\pi \xrightarrow{d_2} \mathrm{ind}_{\Sigma_2 \times \Sigma_2}^{\Sigma_4}(W_3^\pi) \xrightarrow{d_1} \mathrm{ind}_{\Sigma_3 \times 1}^{\Sigma_4}(W_2^\pi) \oplus \mathrm{ind}_{\Sigma_2 \wr \Sigma_2}^{\Sigma_4}(W_2^\pi) \xrightarrow{d_0} W_1^\pi \longrightarrow 0.$$

We claim that we can choose additive generators w_1 and w_2 for $W_3^\pi \cong \mathbb{Z}^2$, and $\mathbb{Z}\Sigma_4$-module generators $x_1 \otimes w_1$ and $x_1 \otimes w_2$ in X_1, $x_0^1 \in \mathrm{ind}_{\Sigma_3 \times 1}^{\Sigma_4}(W_2^\pi)$ and $x_0^2 \in \mathrm{ind}_{\Sigma_2 \wr \Sigma_2}^{\Sigma_4}(W_2^\pi)$ in X_0, and finally $x_{-1} \in X_{-1}$, such that the boundary maps take the following form

$$\begin{aligned} d_1(x_1 \otimes w_1) &= x_0^1 - x_0^2 \qquad & d_0(x_0^1) &= x_{-1} \\ d_1(x_1 \otimes w_2) &= x_0^1 - (34) \cdot x_0^1 \qquad & d_0(x_0^2) &= x_{-1}. \end{aligned}$$

Here $(34) \cdot x_0^1$ means the image of x_0^1 under the action of the transposition $(34) \in \Sigma_4$ on X_0.

To see this, let β be the partition with components $\{1, 2\}$, $\{3\}$ and $\{4\}$, and make a sketch of $F_\beta N\Pi'_4$. Choose x_0^1 and x_0^2 to generate the summands of

X_0 corresponding to the partitions $\{1,2,3\},\{4\}$ and $\{1,2\},\{3,4\}$ respectively. These generators can be chosen to have common image x_{-1} under d_0. Then the existence of appropriate classes w_1, w_2 and x_1 is immediate from the exactness of the complex (X_*, d_*) associated to $F_\beta N\Pi'_4$.

We now make analogous definitions for subspectra of A^k.

Definition 4.3. We say that a partial ordering ω respects a partition β if the components of ω refine β. For every partition $\beta \in \Pi_k$ let $F_\beta A^k \subseteq A^k$ be the union of the prespectra $F_\omega A^k$ with ω respecting β. Then $\{F_\beta A^k\}_\beta$ is a filtration of A^k indexed by the partitions $\beta \in \Pi_k$.

LEMMA 4.4. *Let $\beta \in \Pi_k$ be a partition of $\{1,\dots,k\}$.*
(i) If β is not indiscrete, then the prespectrum $F_\beta A^k$ is contractible.
(ii) If β is indiscrete, then $F_\beta A^k$ is homotopy equivalent to the suspension spectrum on ΣApt^k, as a prespectrum with Σ_k-action.

Proof. Suppose β is not indiscrete, so β has two or more components. Then $F_\beta A^k$ is covered by the $F_\omega A^k$ where ω runs through all partial orderings with the same components as β. The multiple intersections in this covering are all of the form $F_{\omega_0} A^k$ where ω_0 has at least two components, hence contractible by Proposition 1.5(i). So by a Mayer–Vietoris argument, the union $F_\beta A^k$ is also contractible.

Next suppose $\beta = \iota_k$ is indiscrete. $F_\beta A^k$ is then the union of all $F_\omega A^k$ with ω connected. By Proposition 1.5(i) again, we may replace $F_\beta A^k$ by the union of the $F_\omega A^k$ with ω linear (hence connected). Again multiple intersections in this covering have the form $F_{\omega_0} A^k$, which will be contractible unless ω_0 is indiscrete. In this case it is homotopy equivalent to the sphere spectrum with trivial group action.

A linear poset ω determines an increasing chain of nontrivial subsets $\{j \mid j \prec i\} \subseteq \{1,\dots,k\}$ as i varies, ending with the full subset. Conversely such a chain determines the linear poset. The indiscrete poset corresponds to the constant chain. These chains are the simplicies of the quotient space formed from the nerve of the poset of nontrivial subsets of $\{1,\dots,k\}$ by collapsing the nerve of the subposet of proper, nontrivial subsets of $\{1,\dots,k\}$. Hence we can think of these chains as the simplices in the suspension of the Tits apartment Apt^k.

We now build $F_\beta A^k$ by starting with the sphere spectrum and inductively adjoining contractible spectra for every such chain of subsets of $\{1,\dots,k\}$. Working with Σ_k-orbits of such chains, this recovery can be made Σ_k-equivariant, and we recover $F_\beta A^k$ as being homotopy equivalent to the suspension spectrum on ΣApt^k, as a prespectrum with Σ_k-action. $\square$

The following lemma asserts that the filtrations on the nerve of the partition lattice and on the stable apartment are essentially the same.

By a homotopy equivalence f of filtered objects, we mean one having a homotopy inverse g and homotopies $f \circ g \simeq 1$ and $g \circ f \simeq 1$, all respecting the given filtrations.

Lemma 4.5. *There is a homotopy equivalence of filtered prespectra with Σ_k-action*

$$\{F_\beta A^k\}_\beta \simeq \{\Sigma^\infty(F_\beta N\Pi'_k \wedge \Sigma^2 Apt^k)\}_\beta.$$

Proof. We proceed by an increasing induction over Σ_k-orbits of partitions $\beta \in \Pi_k$. The induction begins for $\beta = \iota_k$ indiscrete by choosing a homotopy equivalence $F_\beta A^k \simeq \Sigma^\infty \Sigma Apt^k$ of spectra with Σ_k-action, using part (ii) of Lemma 4.4. The induction step follows by part (i) of the same lemma, as there is no obstruction to extending a map from a subspectrum of, say, $F_\beta A^k$, to the whole of this contractible spectrum. □

Proposition 4.6. *The chain complex (Y_*, d_*) associated to the component filtration on the stable apartment A^k is isomorphic as a complex of $\mathbb{Z}\Sigma_k$-modules to the chain complex (X_*, d_*) associated to the partition lattice, tensored with the sign representation $\Sigma^k \mathbb{Z}_{\text{sgn}}$ concentrated in degree k.*

$$(Y_*, d_*) \cong (X_*, d_*) \otimes \Sigma^k \mathbb{Z}_{\text{sgn}}$$

Proof. These chain complexes are derived from the filtered complexes in the preceding lemma. Together with the observation that $\widetilde{H}_k(\Sigma^2 Apt^k) \cong \mathbb{Z}_{\text{sgn}}$ as a $\mathbb{Z}\Sigma_k$-representation, this proves the proposition. □

Corollary 4.7. *$W_k \cong W_k^\pi \otimes \mathbb{Z}_{\text{sgn}}$ as $\mathbb{Z}\Sigma_k$-modules.*

5. Connectivity results

We can now prove the following connectivity results for stable buildings.

Proposition 5.1. *Let R be a principal ideal domain, and suppose $k \geq 2$. Then $D(R^k)$ is at least $(k-1)$-connected.*

Proof. We have to prove that the last differential $d_k \colon Z_k \to Z_{k-1}$ in the complex computing $H_*^{spec}(D(R^k))$ is onto. Consider the various injective chain maps $\ell_* : Y_* \to Z_*$ associated to those direct sum decompositions $R^k = \oplus_{s=1}^k L_s$

for which $L_s \subseteq R^s \subseteq R^k$ for every s.

$$\begin{tikzcd}
\dots \arrow[r] & Y_k \arrow[r, "d_k"] \arrow[d, "\ell_*"] & Y_{k-1} \arrow[r] \arrow[d, "\ell_*"] & 0 \\
\dots \arrow[r] & Z_k \arrow[r, "d_k"] & Z_{k-1} \arrow[r] & 0
\end{tikzcd}$$

By the Solomon–Tits theorem in the explicit form of Theorem 2.3, $Z_{k-1} = \mathrm{St}(R^k)$ is generated by the images of $Y_{k-1} = \mathbb{Z}_{\mathrm{sgn}}$ under these embeddings. By the exactness statement in Proposition 3.4, the differential $d_k \colon Y_k \to Y_{k-1}$ is onto. Hence $d_k \colon Z_k \to Z_{k-1}$ is onto because the ℓ_* are chain maps. □

PROPOSITION 5.2. *Let R be a principal ideal domain. Then $D(R^4)$ is at least four–connected.*

Proof. We establish $H_4^{spec}(D(R^4)) = 0$ by proving exactness at Z_4 in the complex below. This is the complex (Z_*, d_*) associated to the component filtration on $D(R^4)$, whose homology computes $H_*^{spec}(D(R^4))$.

$$0 \longrightarrow Z_6 \xrightarrow{d_6} Z_5 \xrightarrow{d_5} Z_4 \xrightarrow{d_4} Z_3 \longrightarrow 0$$

The relevant modules are given by Proposition 3.2.

$$\begin{aligned}
Z_3 &= \mathrm{St}(R^4) \\
Z_4 &= \bigoplus_{\{H,L_4\}} (\mathrm{St}(H) \otimes \mathrm{St}(L_4) \otimes W_2) \oplus \bigoplus_{\{P,Q\}} (\mathrm{St}(P) \otimes \mathrm{St}(Q) \otimes W_2) \\
Z_5 &= \bigoplus_{\{P,L_3,L_4\}} \mathrm{St}(P) \otimes \mathrm{St}(L_3) \otimes \mathrm{St}(L_4) \otimes W_3
\end{aligned}$$

These sums run over all unordered sum decompositions of R^4, with L_1, L_2, L_3 and L_4 lines, and $P = L_1 \oplus L_2$, $Q = L_3 \oplus L_4$, and $H = L_1 \oplus L_2 \oplus L_3$.

The boundary maps in the complex are induced from those of the exact complex (Y_*, d_*) of Proposition 3.4. These are in turn determined by the complex (X_*, d_*) arising from the component filtration on the partition lattice, and this complex was made explicit in the case $k = 4$ in Example 4.2.

In particular we can take $\mathbb{Z}\Sigma_4$-module generators $y_5 \otimes w_1$ and $y_5 \otimes w_2$ for Y_5, y_4^1 and y_4^2 for Y_4, and y_3 for Y_3, corresponding to the generators for (X_*, d_*) by Proposition 4.6. Here w_1 and w_2 are the basis elements for W_3 chosen in Example 4.2. Then the relevant boundary maps in (Y_*, d_*) satisfy

$$\begin{aligned}
d_5(y_5 \otimes w_1) &= y_4^1 - y_4^2 & \qquad d_4(y_4^1) &= y_3 \\
d_5(y_5 \otimes w_2) &= y_4^1 - (34) \cdot y_4^1 & \qquad d_4(y_4^2) &= y_3.
\end{aligned}$$

Letting the lines L_1, L_2, L_3, L_4 run through the various sum decompositions of R^4, the images of the chosen generators for Y_* will map under the associated embeddings ℓ_* to a generating set for Z_*, in every degree $*$. This follows from Theorem 2.3. We let

$$\begin{aligned}(z_5 \otimes w_i)(L_1, \dots, L_4) &= \ell_*(y_5 \otimes w_i) \\ z_4^i(L_1, \dots, L_4) &= \ell_*(y_4^i) \\ z_3(L_1, \dots, L_4) &= \ell_*(y_3)\end{aligned}$$

for $i = 1$, 2. The ℓ_* are chain maps by Proposition 3.4, so

$$d_5((z_5 \otimes w_1)(L_1, \dots, L_4)) = z_4^1(L_1, \dots, L_4) - z_4^2(L_1, \dots, L_4)$$

and hence $z_4^1 \equiv z_4^2$ modulo $\mathrm{im}(d_5)$ in Z_4.

By Theorem 2.3, d_4 has a section $s\colon Z_3 \to Z_4$ determined by

$$s(z_3(L_1, \dots, L_4)) = z_4^1(L_1, \dots, L_4),$$

when $L_1 \oplus L_2 \oplus L_3 = H = R^3$. The image of s consists of the summands $\mathrm{St}(H) \otimes \mathrm{St}(L_4) \otimes W_2$ of Z_4 for which $H = R^3$. To prove exactness at Z_4 it suffices to prove that $\mathrm{im}(s)$ generates Z_4 modulo $\mathrm{im}(d_5)$.

Consider a class $z_4^1(L_1, L_2, L_3, L_4) \in \mathrm{St}(H) \otimes \mathrm{St}(L_4) \otimes W_2$ for $H = L_1 \oplus L_2 \oplus L_3$. If $H = R^3$, the class is already in $\mathrm{im}(s)$. Otherwise $H \cap R^3$ is free of rank two and includes by a cofibration into H, by Lemma 2.1. So by Proposition 2.2 applied to $B(H)$, the class $z_4^1(L_1, L_2, L_3, L_4)$ can be written as a sum of terms of the form $z_4^1(M_1, M_2, M_3, L_4)$ with M_1, $M_2 \subset H \cap R^3$ and $M_3 \subset H$.

Now let $U = M_1 \oplus M_2$, and $V = M_3 \oplus L_4$. Modulo $\mathrm{im}(d_5)$

$$z_4^1(M_1, M_2, M_3, L_4) \equiv z_4^2(M_1, M_2, M_3, L_4) \in \mathrm{St}(U) \otimes \mathrm{St}(V) \otimes W_2.$$

We have $U \subset R^3$, so $V \not\subset R^3$ and $V \cap R^3$ is free of rank one and includes by a cofibration into V. Here we are using Lemma 2.1 again. Hence, by Proposition 2.2 applied to $B(V)$, the class $z_4^2(M_1, M_2, M_3, L_4)$ is a sum of terms $z_4^2(M_1, M_2, N_3, N_4)$ with $N_3 \subset V \cap R^3$. Each such term is in turn congruent modulo $\mathrm{im}(d_5)$ to $z_4^1(M_1, M_2, N_3, N_4)$ with $M_1 \oplus M_2 \oplus N_3 = R^3$, which lies in $\mathrm{im}(s)$ as desired. □

Remark 5.3. The proof of Proposition 5.2 can be extended to show that $D(R^k)$ is k-connected for all $k \geq 3$, thus proving Conjecture 12.3 of [16] for $k = 1$, 2 and 3, when R is a principal ideal domain. When $k = 4$, the conjecture asserts that $D(R^4)$ is expected to be five–connected.

6. K-theory of the integers

In this section we prove the theorem $K_4(\mathbb{Z}) = 0$.

Proof of Main Theorem. Associated to the spectrum level rank filtration $\{F_sK(R)\}_s$ there is a spectrum homology spectral sequence

$$E^1_{s,t} = H^{spec}_{s+t}(F_{s+1}K(R)/F_sK(R)) \Longrightarrow H^{spec}_{s+t}(K(R)).$$

By Proposition 1.2 the subquotient $F_{s+1}K(R)/F_sK(R)$ is at least as highly connected as $D(R^{s+1})$. When R is a principal ideal domain, $D(R^{s+1})$ is at least $(s-1)$-connected by Corollary 3.3. Hence this spectral sequence is located in the first quadrant.

In the proof of Theorem 1.1 of [17], we computed the spectrum homology of the first three subquotients of the rank filtration on $K(\mathbb{Z})$, and the differentials terminating below total degree five in the spectral sequence. The result is displayed as the first three columns in the spectral sequence below.

5	$\mathbb{Z}/2$	$\longleftarrow (\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	?	?	?
4	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	?	?	?
3	$\mathbb{Z}/2$	$\longleftarrow (\mathbb{Z}/2)^2$	$\longleftarrow \mathbb{Z}$	?	?	?
2	0	$\mathbb{Z}/6$	$\longleftarrow \mathbb{Z}/3$	?	?	?
1	$\mathbb{Z}/2$	$\longleftarrow \mathbb{Z}/2$	0	0	?	?
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

The differentials shown are all nonzero. The d^1-differentials originating in bidegrees $(2,4)$ and $(2,5)$ are unknown. The group $\mathbb{Z}/2$ in bidegree $(2,5)$ was not given in the abovementioned proof, but follows immediately from the diagram preceding Lemma 6.10 of [17], and that lemma. Conjecture 12.3 of [16] would imply a vanishing line of slope one in this spectral sequence.

By Proposition 5.1 the stable building $D(\mathbb{Z}^k)$ is at least $(k-1)$-connected for $k \geq 2$, and by Proposition 5.2 the stable building $D(\mathbb{Z}^4)$ is at least four–connected. This accounts for the vanishing of the remaining groups shown as trivial in the spectral sequence.

Hence $H^{spec}_*(K(\mathbb{Z}))$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}\oplus ?, \dots)$ in view of Borel's rational result from [2]. The Atiyah–Hirzebruch spectral sequence for stable homotopy theory

$$E^2_{s,t} = H^{spec}_s(K(\mathbb{Z}); \pi^S_t) \Longrightarrow K_{s+t}(\mathbb{Z})$$

abuts to the homotopy groups of the K-theory spectrum, which are the higher algebraic K-groups of $\mathbb{Z}$. In low degrees it looks as follows.

6	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
2	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
1	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus ?$
	0	1	2	3	4	5

The terms in the zeroth column through degree three survive to E^∞ because the image of J injects into $K_*(\mathbb{Z})$ below degree eight by Quillen's letter [15]. See also Mitchell's paper [11]. We claim that the only nonzero differential terminating below degree five is

$$d^2_{5,0}\colon H^{spec}_5(K(\mathbb{Z})) \longrightarrow H^{spec}_3(K(\mathbb{Z});\pi^S_1).$$

Thus there are no classes surviving to E^∞ in total degree four, and we can conclude that $K_4(\mathbb{Z}) = 0$.

To establish the first nonzero differential we compare $K(\mathbb{Z})$ with Bökstedt's spectrum $JK(\mathbb{Z})$. The non–triviality of the extension in $K_3(\mathbb{Z})$ also follows from this comparison. At the prime two, $JK(\mathbb{Z})$ is defined in [4] as the homotopy fiber of the composite

$$ko \xrightarrow{\psi^3-1} bspin \xrightarrow{c} bsu$$

where ψ^3 is the Adams operation and c is complexification. One easily finds

$$JK_*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/16, 0, \mathbb{Z}, \dots)$$

modulo odd torsion, and

$$H^{spec}_*(JK(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}, \dots).$$

There is a canonical map $\Phi\colon K(\mathbb{Z})^\wedge_2 \to JK(\mathbb{Z})^\wedge_2$ of two–completed spectra in the sense of Bousfield and Kan [3]. Bökstedt constructed a section of two–completed looped underlying spaces

$$f\colon \Omega JK(\mathbb{Z})^\wedge_2 \longrightarrow \Omega K(\mathbb{Z})^\wedge_2$$

with $\Omega\Phi \circ f \simeq 1$. Hence $JK_3(\mathbb{Z}) \cong \mathbb{Z}/16$ splits off $K_3(\mathbb{Z})$ and we recover the result $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ of Lee and Szczarba [7]. Thus Φ is at least four–connected.

Let $\eta\colon S^3 \to S^2$ denote the Hopf map, whose stable class generates π^S_1, and let $\lambda\colon S^3 \to K(\mathbb{Z})$ represent a generator of $K_3(\mathbb{Z})$. Then $\lambda' = \Phi \circ \lambda$ generates $JK_3(\mathbb{Z})$. The class $\eta \cdot \lambda$ in $E^2_{3,1}$ is represented by the composite

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{\lambda'} \Omega JK(\mathbb{Z})^\wedge_2 \xrightarrow{f} \Omega K(\mathbb{Z})^\wedge_2.$$

The composite of the first two maps lies in $JK_4(\mathbb{Z}) = 0$. Hence the class $\eta \cdot \lambda \in E^2_{3,1}$ cannot survive to E^∞, and the only possible differential that can affect it is a d^2 from $E^2_{5,0}$.

This concludes the proof of our theorem. □

We add some miscellaneous remarks.

Lemma 6.1. *The spectrum map $\Phi : K(\mathbb{Z})^\wedge_2 \to JK(\mathbb{Z})^\wedge_2$ induces an isomorphism on homotopy through degree four, and a split surjection in degree five, hence is at least five–connected.* □

Lemma 6.2. *There are surjective differentials $d^2_{5,0} \colon E^2_{5,0} \to E^2_{3,1}$ and $d^2_{5,1} \colon E^2_{5,1} \to E^2_{3,2}$ in the Atiyah–Hirzebruch spectral sequence above. In particular we can find a class $\beta \in H^{spec}_5(K(\mathbb{Z}))$ generating this group modulo its torsion subgroup, such that $d^2_{5,0}(\beta) = \eta \cdot \lambda$ and $d^2_{5,1}(\eta \cdot \beta) = \eta^2 \cdot \lambda$.*

Proof. By Arlettaz' Theorem 1.5 in [1], the Hurewicz homomorphism $K_5(\mathbb{Z}) \to H_5(GL(\mathbb{Z}))$ induces multiplication by two on the torsion free part, *i.e.* on these groups modulo torsion. The composite $K_5(\mathbb{Z}) \twoheadrightarrow E^\infty_{5,0} \subseteq H^{spec}_5(K(\mathbb{Z}))$ arising from the Atiyah–Hirzebruch spectral sequence is the composite of this Hurewicz map followed by stabilization $H_5(GL(\mathbb{Z})) \to H^{spec}_5(K(\mathbb{Z}))$ from group homology to spectrum homology. Hence this composite is not a surjection on the torsion free part, and the surjective differential $d^2_{5,0}$ cannot factor through the torsion in $E^2_{5,0}$. Thus there exists a class $\beta \in H^{spec}_5(K(\mathbb{Z}))$ generating the torsion–free quotient and with $d^2(\beta) \neq 0$. This class then has the stated properties. □

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(Received ??)
(Revised ??)