PRODUCTS AND DUALITY IN CATEGORIES WITH COFIBRATIONS AND WEAK EQUIVALENCES

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Abstract. The natural transformation Ξ from L–theory to the Tate cohomology of $\mathbb{Z}/2$ acting on K–theory (constructed in [WW2] and [WW3]) commutes with external products. Corollary: The Tate cohomology of $\mathbb{Z}/2$ acting on the K–theory of any ring with involution is a generalized Eilenberg–MacLane spectrum, and it is 4–periodic.

0. Introduction

Categories with cofibrations and weak equivalences were introduced in [Wald]. Generalizing earlier constructions due to Quillen and others, Waldhausen defined the K–theory spectrum $\boldsymbol{K}(\mathcal{C})$ of a category with cofibrations and weak equivalences $\mathcal{C}$ (also in [Wald]). It was realized early on [Vo] that suitable notions of Spanier–Whitehead duality in $\mathcal{C}$ give rise to involutions on $\boldsymbol{K}(\mathcal{C})$. We formalized this idea in [WW3, §4] by introducing the notion of a Spanier–Whitehead product (SW product for short) in $\mathcal{C}$. To some extent this will be recalled below. When $\mathcal{C}$ is equipped with an SW product, the following are defined (quoting from [WW3]):

(1) a duality involution on $\boldsymbol{K}(\mathcal{C})$ (more precisely, on something homotopy equivalent to $\boldsymbol{K}(\mathcal{C})$;
(2) a quadratic L–theory spectrum $\boldsymbol{L}(\mathcal{C})$;
(3) a symmetric L–theory spectrum $\boldsymbol{L}^{\blacktriangledown}(\mathcal{C})$;
(4) maps of spectra

$$(1+T): \boldsymbol{L}(\mathcal{C}) \longrightarrow \boldsymbol{L}^{\blacktriangledown}(\mathcal{C})$$
$$\Xi: \boldsymbol{L}^{\blacktriangledown}(\mathcal{C}) \longrightarrow \widehat{\boldsymbol{H}}(\mathbb{Z}/2\,;\boldsymbol{K}(\mathcal{C}))\,.$$

Here $\widehat{\boldsymbol{H}}(\mathbb{Z}/2\,;\boldsymbol{K}(\mathcal{C}))$ is the cofiber of the *norm map* from the homotopy orbit spectrum of the $\mathbb{Z}/2$–action on $\boldsymbol{K}(\mathcal{C})$ to the homotopy fixed point spectrum:

$$H_{\blacktriangledown}(\mathbb{Z}/2\,;\boldsymbol{K}(\mathcal{C})) \xrightarrow{\mathcal{N}} \boldsymbol{H}^{\blacktriangledown}(\mathbb{Z}/2\,;\boldsymbol{K}(\mathcal{C}))\,.$$

(See [WW2, §2] and [AdCoDw] or [GreMa].) Our interest in the map Ξ, or rather in the composition $\Xi(1+T)$, stems from the fact that it appears in an approximative algebraic description of *structure spaces* (moduli spaces of closed manifolds equipped with a homotopy equivalence to a fixed manifold M). See [WW3]. Here we show that Ξ commutes with external products—a fact which greatly facilitates calculations.

Adopting a convention from [WW3], *we usually denote spectra by boldface letters, and the corresponding (zero–th) infinite loop spaces by "unbold" letters. For example, $K(\mathcal{C})$ is an infinite loop space, and $\boldsymbol{K}(\mathcal{C})$ is a spectrum.*

1. External Products

We assume that the reader is familiar with the $S_\bullet$ construction and the definition $K(\mathcal{C}) := \Omega|wS_\bullet\mathcal{C}|$, from [Wald]. For reasons given on p. 330 of the same source, $K(\mathcal{C})$ is an infinite loop space ("To pursue the analogy with Segal's version ... "). For external products, we quote directly from p. 342 (same source):

"We digress to indicate in which way the twice de-looped K–theory $wS_\bullet S_\bullet\mathcal{C}$ is used in defining *products*; or better, *external pairings* (products are induced from those). The ingredient that one needs is a bi–exact functor of categories with cofibrations and weak equivalences. This is a functor $\mathcal{A}\times\mathcal{B}\to\mathcal{C}$, $(A,B)\mapsto A\wedge B$, having the property that for every $A\in\mathcal{A}$ and $B\in\mathcal{B}$ the partial functors $A\wedge ?$ and $?\wedge B$ are exact, and where in addition the following more technical condition must also be satisfied; namely, for every pair of cofibrations $A\rightarrowtail A'$ and $B\rightarrowtail B'$ in $\mathcal{A}$ and $\mathcal{B}$, respectively, the induced square of cofibrations in $\mathcal{C}$ must be admissible in the sense that the map $A'\wedge B\cup_{A\wedge B}A\wedge B' \to A'\wedge B'$ is a cofibration. A bi–exact functor induces a map, of bisimplicial bicategories

$$wS_\bullet\mathcal{A}\times wS_\bullet\mathcal{B}\longrightarrow wwS_\bullet S_\bullet\mathcal{C}$$

which upon passage to geometric realization factors through the smash product

$$|wS_\bullet\mathcal{A}|\wedge|wS_\bullet\mathcal{B}| \longrightarrow |wwS_\bullet S_\bullet\mathcal{C}|$$

and in turn induces

$$\Omega|wS_\bullet\mathcal{A}|\wedge\Omega|wS_\bullet\mathcal{B}| \longrightarrow \Omega\Omega|wwS_\bullet S_\bullet\mathcal{C}|\,.$$

This is the desired pairing in K–theory in view of the homotopy equivalence of $|wS_\bullet\mathcal{C}|$ with $\Omega|wS_\bullet S_\bullet\mathcal{C}|$, and a (much more innocent) homotopy equivalence of $wS_\bullet S_\bullet\mathcal{C}$ with $wwS_\bullet S_\bullet\mathcal{C}$ which we will have occasion later on to consider in detail (the 'swallowing lemma' ...)."

2. External products and SW products

Here we recall as briefly as possible the notion of an SW product and combine it with external products as in the preceding section. As usual, $\mathcal{C}$ is a category with cofibrations and weak equivalences. In addition we assume that the weak equivalences in $\mathcal{C}$ satisfy the saturation axiom [Wald], and that the category $\mathcal{Y}$ of compact pointed CW–spaces and pointed cellular maps acts on $\mathcal{C}$. The action map

$$\mathcal{Y}\times\mathcal{C}\to\mathcal{C}\quad;\quad(Y,C)\mapsto Y\wedge C$$

must be bi–exact, and associative up to natural coherence:

$$X\wedge(Y\wedge C)\cong(X\wedge Y)\wedge C\,. \tag{2–1}$$

See [WW3, §4] for details and examples (categories of chain complexes qualify as a rule). A Spanier–Whitehead product on $\mathcal{C}$ is a covariant functor

$$(C,D)\mapsto C\odot D$$

from $\mathcal{C}\times\mathcal{C}$ to pointed spaces (meaning: well pointed spaces homotopy equivalent to CW–spaces) which takes pairs of weak equivalences to homotopy equivalences, and which is *symmetric, bilinear*, and *nondegenerate.* Here *symmetry* means that $C \odot D$ is homeomorphic to $D \odot C$ by a natural homeomorphism whose square is the identity. *Bilinearity* means, informally and in the presence of symmetry, that for any fixed D in $\mathcal{C}$ the functor $C \mapsto C \odot D$ takes homotopy pushout squares in $\mathcal{C}$ to homotopy pullback squares of pointed spaces, and that it takes the zero object to a contractible space. *Nondegenerate* means that, for any C in $\mathcal{C}$, there exists a B in $\mathcal{C}$, an $n \geq 0$ and a class in $\pi_n(B \odot C)$ having properties similar to those of a Spanier–Whitehead n–duality. (Again, see [WW3, §4] for details). The mother/father of all SW products is in fact the following: $\mathcal{C} = \mathcal{Y}$, the category of compact pointed CW spaces, with

$$X \odot Y = \Omega^\infty\Sigma^\infty(X \wedge Y)\,.$$

To study external products and SW products, jointly, we now make the following assumptions. $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are categories with cofibrations and weak equivalences, equipped with SW products $\odot_1$, $\odot_2$ and $\odot_3$, respectively (and with the additional features required, such as an action of $\mathcal{Y}$). As in §1, we assume that

$$(A, B) \mapsto A \wedge B$$

is a bi–exact functor from $\mathcal{A}\times\mathcal{B}$ to $\mathcal{C}$. This must be compatible with the actions of $\mathcal{Y}$ (a big mouthful):

$$(X \wedge A) \wedge B \cong X \wedge (A \wedge B) \cong A \wedge (X \wedge B) \tag{2–2}$$

for $X \in \mathcal{Y}$, and these isomorphisms must be natural in X, A, B and consistent with the isomorphisms (2–1). (This means that any automorphism of $X \wedge (Y \wedge (A \wedge B))$ obtained by combining (2–2) and (2–1) for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is the identity, for all X, Y in $\mathcal{Y}$ and A in $\mathcal{A}$ and B in $\mathcal{B}$. Equivalently, expressions like $X \wedge Y \wedge A \wedge B$ are well defined in the sense that different ways of inserting two pairs of brackets give canonically isomorphic results.) Also, when $X = \mathbb{S}^0$ in (2–2), there are canonical isomorphisms $X \wedge A \cong A$, $X \wedge (A \wedge B) \cong A \wedge B$, $X \wedge B \cong B$ by [WW3, 4.3], and we require commutativity of the diagram (lower row as in (2–2))

$$\begin{array}{ccccc} A \wedge B & \xrightarrow{\text{id}} & A \wedge B & \xrightarrow{\text{id}} & A \wedge B \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ (\mathbb{S}^0 \wedge A) \wedge B & \xrightarrow{\cong} & \mathbb{S}^0 \wedge (A \wedge B) & \xrightarrow{\cong} & A \wedge (\mathbb{S}^0 \wedge B)\,. \end{array}$$

Finally we need a natural map of pointed spaces

$$(A \odot_1 A') \wedge (B \odot_2 B') \xrightarrow{\psi} \big(A \wedge B) \odot_3 (A' \wedge B') \tag{2–3}$$

(natural in A, B, A', B'). We impose two conditions on ψ. Firstly, the square

$$\begin{array}{ccc} (A \odot_1 A') \wedge (B \odot_2 B') & \xrightarrow{\psi} & \big(A \wedge B) \odot_3 (A' \wedge B') \\ \cong\downarrow & & \cong\downarrow \\ (A' \odot_1 A) \wedge (B' \odot_2 B) & \xrightarrow{\psi} & \big(A' \wedge B') \odot_3 (A \wedge B) \end{array}$$

must commute (for the vertical arrows, use the symmetry property of SW products). Secondly, suppose that $\eta_1 : \mathbb{S}^n \to A \odot_1 A'$ is an n–duality and that $\eta_2 : \mathbb{S}^m \to B \odot B'$ is an m–duality. Then the following composition must be an $m+n$–duality in $\mathcal{C}$:

$$\mathbb{S}^n \wedge \mathbb{S}^m \xrightarrow{\eta_1 \wedge \eta_2} (A \odot_1 A') \wedge (B \odot_2 B') \xrightarrow{\psi} \big(A \wedge B\big) \odot_3 (A' \wedge B')\,.$$

Assuming all this, we shall construct external products

$$\boldsymbol{L}^\blacktriangledown(\mathcal{A}) \wedge \boldsymbol{L}^\blacktriangledown(\mathcal{B}) \longrightarrow \boldsymbol{L}^\blacktriangledown(\mathcal{C})$$
$$\widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{A})) \wedge \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{B})) \longrightarrow \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{C}))$$

and we shall establish the commutativity of the diagram

$$\begin{array}{ccc}
\boldsymbol{L}^\blacktriangledown(\mathcal{A}) \wedge \boldsymbol{L}^\blacktriangledown(\mathcal{B}) & \longrightarrow & \boldsymbol{L}^\blacktriangledown(\mathcal{C}) \\
\big\downarrow{\scriptstyle \Xi\wedge\Xi} & & \big\downarrow{\scriptstyle \Xi} \\
\widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{A})) \wedge \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{B})) & \longrightarrow & \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{C}))\,.
\end{array}$$

3. Space level work

At this point we must refer to [WW3, §4] for notation and terminology. We make two hypotheses on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The first is identical with [WW3, 4.7].

3.1. Hypothesis. The suspension map $[C, D] \to [\Sigma C, \Sigma D]$ is a bijection for all C, D (in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ as appropriate).

Here $[C, D]$ is the set of equivalence classes of *weak morphisms* from C to D. A weak morphism from C to D is a diagram $C \longrightarrow D_1 \leftarrow D$ (in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$, as appropriate) where the arrow from D to D_1 is a weak equivalence.

3.2. Hypothesis. The SW products $\odot_1$, $\odot_2$ on $\mathcal{A}$ and $\mathcal{B}$, respectively, are bispectrum valued (two suspension directions) ; the SW product $\odot_3$ on $\mathcal{C}$ is 4–spectrum valued (four suspension directions), and

$$\psi : (A \odot_1 A') \wedge (B \odot_2 B') \quad\longrightarrow\quad (A \wedge B) \odot_3 (A' \wedge B')$$

is a natural map of 4–spectra.

Explanation. . We allow ourselves to modify the definition of an SW product. Instead of a detailed list of axioms, we give an example. Let $\mathcal{A} = \mathcal{B}$ be the category of CW–spectra with finitely many cells ; let $\mathcal{C}$ be the category of CW–bispectra (two suspension coordinates) with finitely many cells. We define

$$A \odot_1 A' := A \wedge A'\,, \quad B \odot_2 B' := B \wedge B'\,, \quad C \odot_3 C' := C \wedge C'$$

for A, A' in $\mathcal{A}$ and B, B' in $\mathcal{B}$ and C, C' in $\mathcal{C}$. Then $\odot_i$ is bispectrum valued for $i = 1, 2$, and 4–spectrum valued for $i = 3$. We give the usual meaning to $A \wedge B$, so that

$$(A, B) \mapsto A \wedge B$$

is a bi–exact functor from $\mathcal{A}\times\mathcal{B}$ to $\mathcal{C}$. The definition of ψ is obvious. The linearity and nondegeneracy properties of $\odot_i$ for $i = 1, 2, 3$ are well known. As for the symmetry, note that the "usual" isomorphism from $A \wedge A'$ to $A' \wedge A$ involves a permutation of the suspension coordinates. The permutation is part of the list of axioms that we suppressed.

In §5, we will justify these hypothesis by showing: it can be arranged. Recall now that the maps of *infinite loop spaces*

$$\begin{aligned} \Xi : L^\blacktriangledown(\mathcal{A}) &\longrightarrow \widehat{H}(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{A})) \\ \Xi : L^\blacktriangledown(\mathcal{B}) &\longrightarrow \widehat{H}(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{B})) \end{aligned} \tag{3–1}$$

have been constructed (in [WW3, 4.13] and sequel) as simplicial maps from simplicial *sets* to simplicial *spaces.* (Here and in the following, *simplicial* often means: *incomplete simplicial*, i.e. simplicial without degeneracy operators.) If we adopt this approach for $\mathcal{A}$ and $\mathcal{B}$, which we will do, then we have to use a slightly different idea (more *bi*simplicial) for $\mathcal{C}$, since the product of two (incomplete) simplicial sets or spaces is (incomplete) bisimplicial. To start with, however, we recall (very briefly again) the details in (3–1), concentrating on the category $\mathcal{A}$.

Let $sp(\mathcal{A})$ be the set or class of symmetric Poincaré objects in $\mathcal{A}$. Such a thing is an object C in $\mathcal{A}$ together with a $\mathbb{Z}/2$–map ϕ from $(\mathcal{E}\mathbb{Z}/2)_+$ to $C \odot_1 C$ which is a 0–duality (when composed with any nonzero pointed map from $\mathbb{S}^0$ to $(\mathcal{E}\mathbb{Z}/2)_+$). Using more precise language, one could speak of a symmetric Poincaré object of formal dimension 0.

Let $\mathcal{A}[m]$ be the category of covariant functors from the poset $po[m]$ (set of nonempty faces of Δ^m) to $\mathcal{A}$. This comes with the structure of a category with cofibrations and weak equivalences, and with an SW product, lifted from $\mathcal{A}$. See [WW3, 4.13]. Then

$$L^\blacktriangledown(\mathcal{A}) := \big|[m] \mapsto sp(\mathcal{A}[m])\big|\,.$$

3.3. Remark. The n–th homotopy group $\pi_n(L^\blacktriangledown(\mathcal{A}))$ is isomorphic to the bordism group of formally n-dimensional symmetric Poincaré objects in $\mathcal{A}$. The proof is a little tricky, because as it stands $L^\blacktriangledown(\mathcal{A})$ does not have the Kan property (for incomplete simplicial sets, [RoSa]). Here is a "hint": Call an object C in $\mathcal{A}[m]$ *well behaved* (compare [WW2, §3]) if, as a functor from $po[m]$ to $\mathcal{A}$, it can be extended to an *intersection preserving functor* $\hat{C}$ from the category of all CW–subspaces of Δ^m to $\mathcal{A}$. Here *intersection preserving* means that, for any two CW–subspaces $X, Y \subset \Delta^n$, the square

$$\begin{array}{ccc} \hat{C}(X\cap Y) & \longrightarrow & \hat{C}(Y) \\ \big\downarrow & & \big\downarrow \\ \hat{C}(X) & \longrightarrow & \hat{C}(X\cup Y) \end{array}$$

consists of cofibrations and is a pushout square. $L^\blacktriangledown(\mathcal{A})$ has an (incomplete) simplicial subset whose m–simplices are the symmetric Poincaré objects in $\mathcal{A}[m]$ which are well behaved. The inclusion of this simplicial subset is a homotopy equivalence (see [WW2, 3.2, 3.3], and the simplicial subset has the Kan property.

A symmetric Poincaré object (formal dimension 0) in $\mathcal{A}$ determines a homotopy fixed point in the involutive model of $\boldsymbol{K}(\mathcal{A})$, that is, a point in $H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}))$. Generalizing this from $\mathcal{A}$ to $\mathcal{A}[m]$, where $m\ge 0$, we have

$$\big|[m]\mapsto sp(\mathcal{A}[m])\big| \quad\longrightarrow\quad \big|[m]\mapsto H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}[m]))\big|\,. \tag{3–2}$$

This is the map Ξ, in view of an identification of its codomain with the Tate cohomology $\widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}[m]))$. The identification can be made because the simplicial spectrum $[m]\mapsto \boldsymbol{K}(\mathcal{A}[m])$ is an *augmented* $\mathbb{Z}_2$*–resolution* of $\boldsymbol{K}(\mathcal{A})$.

Turning to $\mathcal{C}$ now, we choose a description of Ξ which is more bisimplicial. Let $\mathcal{C}[m,n]$ be the category of covariant functors from the poset $po[m,n]$ (set of nonempty faces of $\Delta^m\times\Delta^n$) to $\mathcal{C}$. This comes with the structure of a category with cofibrations and weak equivalences, and with an SW product, lifted from $\mathcal{C}$. Much as in (3–2), we have

$$\big|[m,n]\mapsto sp(\mathcal{A}[m,n])\big| \quad\longrightarrow\quad \big|[m,n]\mapsto H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,n]))\big| \tag{3–3}$$

and we may hope that this is also a correct description of Ξ. Restricting (3–3) to horizontal or vertical 0–skeletons, what we have is in fact exactly Ξ in the description (3–2). But then we only have to prove the following:

3.4. Lemma. *The inclusions of horizontal zero–skeletons*

$$\begin{array}{rcl}\big|[m]\mapsto sp(\mathcal{A}[m,0])\big| & \hookrightarrow & \big|[m,n]\mapsto sp(\mathcal{A}[m,n])\big| \\ \big|[m]\mapsto H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,0]))\big| & \hookrightarrow & \big|[m,n]\mapsto H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,n]))\big|\end{array}$$

are homotopy equivalences.

Proof. Let $\mathfrak{Y}$ be any bisimplicial space (complete or incomplete). The standard procedure for showing that the inclusion of the horizontal 0–skeleton in $|\mathfrak{Y}|$ is a homotopy equivalence is to show that the appropriate degeneracy operators induce homotopy equivalences

$$\big|[m]\mapsto\mathfrak{Y}[m,0]\big| \quad\longrightarrow\quad \big|[m]\mapsto\mathfrak{Y}[m,n]\big|$$

for all $n\ge 0$. In the case $\mathfrak{Y}[m,n]=sp(\mathcal{A}[m,n])$ we can verify this directly by comparing homotopy groups. In the case $\mathfrak{Y}[m,n]=H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,n]))$ we note that, for fixed n, the simplical spectrum with $\mathbb{Z}/2$–action

$$[m]\mapsto\boldsymbol{K}(\mathcal{C}[m,n])$$

is an augmented $\mathbb{Z}/2$–resolution of $\boldsymbol{K}(\mathcal{C}[0,n])$, since

$$\mathcal{C}[m,n]\cong\mathcal{C}[0,n][m].$$

Therefore

$$\big|[m]\mapsto H(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,n]))\big| \quad\simeq\quad \widehat{H}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{C}[0,n]))\,.$$

Hence it is sufficient to verify that the degeneracy operator which maps $\boldsymbol{K}(\mathcal{C}[0,0])$ to $\boldsymbol{K}(\mathcal{C}[0,n])$ induces a homotopy equivalence of the Tate cohomology spectra,

$$\widehat{\boldsymbol{H}}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{C}[0,0])) \quad \xrightarrow{\cong} \quad \widehat{\boldsymbol{H}}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{C}[0,n]))\,.$$

Since $\widehat{\boldsymbol{H}}(\mathbb{Z}_2;\text{—})$ preserves cofibration sequences up to homotopy and annihilates spectra with $\mathbb{Z}_2$–action which are *induced* [WW2, 2.5, 2.6], we need only show that the cofiber of the degeneracy map

$$\boldsymbol{K}(\mathcal{C}[0,0]) \longrightarrow \boldsymbol{K}(\mathcal{C}[0,n]) \quad \text{or equivalently} \quad \boldsymbol{K}(\mathcal{C}[0]) \longrightarrow \boldsymbol{K}(\mathcal{C}[n])$$

is induced. But we know this already, as part of the statement that $[n]\mapsto \boldsymbol{K}(\mathcal{C}[n])$ is an augmented $\mathbb{Z}/2$–resolution. □

Now it is easy to write down the required multiplications. From the assumptions in §2, we have external products

$$\begin{gathered} sp(\mathcal{A}[m])\times sp(\mathcal{B}[n]) \longrightarrow sp(\mathcal{C}[m\times n])\,, \\ H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}[m]))\times H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{B}[n])) \longrightarrow H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}[m,n]))\,. \end{gathered} \tag{3–4}$$

The first is obvious ; the second needs some care. For any spectrum $\boldsymbol{X}$ with $\mathbb{Z}/2$–action, we can think of $H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{X})$ as the space of $\mathbb{Z}/2$–maps from $\mathcal{E}\mathbb{Z}/2$ to X, the zero–th infinite loop space of $\boldsymbol{X}$. Applying this with $\boldsymbol{X}$ equal to $\boldsymbol{K}(\mathcal{A}[m])$ or $\boldsymbol{K}(\mathcal{B}[n])$ or $\boldsymbol{K}(\mathcal{C}[m,n])$, we see that what we need is a $\mathbb{Z}/2$–map of *spaces*

$$K(\mathcal{A}[m])\times K(\mathcal{B}[n]) \longrightarrow K(\mathcal{C}[m,n])$$

to induce the second map in (3–4). We have such a map, using external products as in §1. Of course, to make it into a $\mathbb{Z}/2$–map, we have to use the appropriate models with involution of $K(\mathcal{A}[m])$, $K([\mathcal{B}[n])$ and $K(\mathcal{C}[m,n])$, respectively. This is not difficult. The important thing is to be *flexible* and describe $K(\mathcal{C}[m,n])$ as the double loop space of a fattened version of $|wwS_\bullet S_\bullet\mathcal{C}[m,n]|$, in contrast to $K(\mathcal{A}[m])$ and $K(\mathcal{B}[n])$ which should be described in the usual way, as single loop spaces of fattened versions of $|wS_\bullet\mathcal{A}[m]|$ and $|wS_\bullet\mathcal{B}[n]|$, respectively. Having said this, we can take the geometric realizations of (3–4) to obtain a commutative diagram of *spaces* and maps

$$\begin{array}{ccc} L^{\blacktriangledown}(\mathcal{A})\wedge L^{\blacktriangledown}(\mathcal{B}) & \xrightarrow{\times} & L^{\blacktriangledown}(\mathcal{C}) \\ \big\downarrow{\scriptstyle \Xi\wedge\Xi} & & \big\downarrow{\scriptstyle \Xi} \\ \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{A}))\wedge \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{A})) & \xrightarrow{\times} & \widehat{H}^{\blacktriangledown}(\mathbb{Z}_2;\boldsymbol{K}(\mathcal{A}))\;. \end{array}$$

4. Spectrum level work

In this section we need to allow some variation of the SW products, which forces us to write $\boldsymbol{L}^{\blacktriangledown}(\mathcal{A},\odot_1)$ and $\boldsymbol{K}(\mathcal{A},\odot_1)$ and so on for what was previously $\boldsymbol{L}^{\blacktriangledown}(\mathcal{A})$ and $\boldsymbol{K}(\mathcal{A})$ and so on. (It may seem that $\boldsymbol{K}(\mathcal{A},\odot_1)$ does not depend on $\odot_1$, but as a spectrum *with involution* it does of course.) In particular, let $\sigma\odot_1$ be the spectrum–valued SW product on $\mathcal{A}$ given by $A\,\sigma\odot_1 A' := \mathbb{S}^1 \wedge (A \odot_1 A')$. Define similarly $\sigma\odot_3$, a bispectrum–valued SW product on $\mathcal{C}$. We shall construct maps

$$\begin{aligned}
\Sigma L^\blacktriangledown(\mathcal{A},\odot_1) &\xrightarrow{\varepsilon} L^\blacktriangledown(\mathcal{A},\sigma\odot_1)\\
\Sigma L^\blacktriangledown(\mathcal{C},\odot_3) &\xrightarrow{\varepsilon} L^\blacktriangledown(\mathcal{C},\sigma\odot_3)\\
\Sigma \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\odot_1)) &\xrightarrow{\varepsilon} \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\sigma\odot_1))\\
\Sigma \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\odot_3)) &\xrightarrow{\varepsilon} \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\sigma\odot_3))
\end{aligned} \tag{4–1}$$

with a number of useful properties most of which can be expressed through commutative diagrams. The commutative diagrams are

$$\begin{array}{ccc}
\Sigma L^\blacktriangledown(\mathcal{A},\odot_1) & \xrightarrow{\Sigma\Xi} & \Sigma\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\odot_1))\\
\big\downarrow{\scriptstyle\varepsilon} & & \big\downarrow{\scriptstyle\varepsilon}\\
L^\blacktriangledown(\mathcal{A},\sigma\odot_1) & \xrightarrow{\Xi} & \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\sigma\odot_1))
\end{array} \tag{4–2}$$

$$\begin{array}{ccc}
\Sigma L^\blacktriangledown(\mathcal{C},\odot_3) & \xrightarrow{\Sigma\Xi} & \Sigma\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\odot_3))\\
\big\downarrow{\scriptstyle\varepsilon} & & \big\downarrow{\scriptstyle\varepsilon}\\
L^\blacktriangledown(\mathcal{C},\sigma\odot_3) & \xrightarrow{\Xi} & \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\sigma\odot_3))
\end{array} \tag{4–3}$$

$$\begin{array}{ccc}
\Sigma L^\blacktriangledown(\mathcal{A},\odot_1)\wedge L^\blacktriangledown(\mathcal{B},\odot_2) & \xrightarrow{\Sigma\times} & \Sigma L^\blacktriangledown(\mathcal{C},\odot_3)\\
\big\downarrow{\scriptstyle\varepsilon\wedge\mathrm{id}} & & \big\downarrow{\scriptstyle\varepsilon}\\
L^\blacktriangledown(\mathcal{A},\sigma\odot_1)\wedge L^\blacktriangledown(\mathcal{B},\odot_2) & \xrightarrow{\times} & L^\blacktriangledown(\mathcal{C},\sigma\odot_3)
\end{array} \tag{4–4}$$

$$\begin{array}{ccc}
\Sigma\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\odot_1))\wedge\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{B},\odot_2)) & \xrightarrow{\Sigma\times} & \Sigma\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\odot_3))\\
\big\downarrow{\scriptstyle\varepsilon\wedge\mathrm{id}} & & \big\downarrow{\scriptstyle\varepsilon}\\
\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\sigma\odot_1))\wedge\widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{B},\odot_1)) & \xrightarrow{\times} & \widehat{H}^\blacktriangledown(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\sigma\odot_3))\ .
\end{array} \tag{4–5}$$

Once the maps in (4–1) have been constructed, commutativity of the diagrams will be obvious. The construction of the maps is not difficult, but we need to be careful about the meaning of *suspension*. Suppose that $\mathfrak{Y}$ is a simplicial pointed space (incomplete to begin with). The geometric realization of $\mathfrak{Y}$ should be defined as the quotient of $\amalg_m \Delta^m_+ \wedge \mathfrak{Y}[m]$ by the usual relations. Define another (incomplete) simplicial pointed space $\Sigma_!\mathfrak{Y}$ by

$$\Sigma_!\mathfrak{Y}[m] = \begin{cases} \mathfrak{Y}[m-1] & m>0\\ \qquad * & m=0\end{cases}$$

where $f^*: \Sigma_!\mathfrak{Y}[n] \to \Sigma_!\mathfrak{Y}[m]$ equals $g^*: \mathfrak{Y}[n{-}1] \to \mathfrak{Y}[m{-}1]$ provided $f:[m]\to[n]$ extends $g:[m{-}1]\to[n{-}1]$ and $f(m)=n$; when $f(m)\neq n$, then f^* is the zero map. It is easy to verify that

$$\big|\Sigma_!\mathfrak{Y}\big| \cong \Sigma|\mathfrak{Y}|\,.$$

For a *complete* simplicial space $\mathfrak{Y}$, we must define $\Sigma_!\mathfrak{Y}$ in terms of generators and relations: take the free simplicial pointed space generated by the spaces $\mathfrak{Y}[m-1]$ in degree $[m]$, for $m>0$, and introduce relations to ensure that

$$f^*(x^\sharp) = \big(g^*(x)\big)^\sharp$$

where $x^\sharp$ is the n–dimensional generator corresponding to some $x\in\mathfrak{Y}[n-1]$, and $f:[m]\to[n]$ extends a monotone $g:[m{-}1]\to[n{-}1]$, with $f(m)=n$. Also, introduce relations

$$f^*(x^\sharp) = \text{ base point}$$

for any monotone $f:[m]\to[n]$ such that $f(m)\neq n$. It may be best to insist that *space* means *simplicial set* in this construction, so that simplicial space means bisimplicial set.

In the following, we shall not distinguish too much between a simplicial pointed space and its realization. In this spirit, we may substitute $\Sigma_!$ for Σ in (4–1). We can then construct the required maps by constructing suitable functors $\mathcal{A}[m–1]\to\mathcal{A}[m]$ and $\mathcal{C}[m{-}1]\to\mathcal{C}[m]$.

4.1. Construction. Identify nonempty faces of Δ^m with nonempty subsets s of $[m]=\{0,1,\dots,m\}$. For a covariant functor A from *po*$[m{-}1]$ (poset of nonempty faces of Δ^{m-1}) to $\mathcal{A}$, define another covariant functor εA from *po*$[m]$ to $\mathcal{A}$ by

$$s \quad\mapsto\quad \begin{cases} A(s\smallsetminus\{m\}) & \text{if } m\in s \text{ and } \{m\}\neq s\\ \quad * & \text{if not.}\end{cases}$$

Induced maps are obvious. We now proceed to show that the functor $A\mapsto\varepsilon A$ from $\mathcal{A}[m{-}1]$ to $\mathcal{A}[m]$ respects SW products up to a shift. In more detail: For A and A' in $\mathcal{A}[m{-}1]$, there is a canonical inclusion map

$$A\odot_1 A' \longrightarrow \varepsilon A\ \sigma\!\odot_1 \varepsilon A' \tag{4–6}$$

which is natural in A and A', makes the diagram

$$\begin{array}{ccc} A\odot_1 A' & \longrightarrow & \varepsilon A\ \sigma\!\odot_1 \varepsilon A' \\ \cong\big\downarrow & & \cong\big\downarrow \\ A'\odot_1 A & \longrightarrow & \varepsilon A'\ \sigma\!\odot_1 \varepsilon A \end{array}$$

commutative, and takes n–dualities to n–dualities for all $n\geq 0$. In order to make (4–6) explicit, we must recall the way in which $\odot_1$ is lifted from $\mathcal{A}$ to $\mathcal{A}[m{-}1]$; namely,

$$A\odot_1 A' = \operatorname*{holim}_{s\subset[m-1]} A(s)\odot_1 A'(s)$$

for A and A' in $\mathcal{A}[m-1]$. Then, by inspection,

$$\varepsilon A\ \sigma\odot_1\ \varepsilon A'\ \cong\ \Omega\,\underset{s}{\operatorname{holim}}\left(\mathbb{S}^1\wedge(A(s)\odot_1 A'(s))\right)\ \supset\ \Omega(\mathbb{S}^1\wedge(A\odot_1 A'))\ \supset\ A\odot_1 A'.$$

(Strictly speaking, the left–hand isomorphism is valid only if $*\odot_1 *$ is strictly zero—not just contractible. In general, $\cong$ should be replaced by $\supset$, but the inclusion is still a homotopy equivalence.) In any case, we have established (4–6), and we conclude that ε induces maps

$$\begin{aligned} sp(\mathcal{A}[m-1],\odot_1) &\longrightarrow sp(\mathcal{A}[m],\sigma\odot_1)\\ H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}[m-1],\odot_1)) &\longrightarrow H^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A}[m],\sigma\odot_1))\end{aligned}$$

of which the first is a map of pointed sets (or classes), whereas the second is a map of pointed spaces. Varying m now and taking geometric realizations, we get two of the four maps in (4–1):

$$\begin{aligned} \Sigma L^{\blacktriangledown}(\mathcal{A},\odot_1) &\xrightarrow{\varepsilon} L^{\blacktriangledown}(\mathcal{A},\sigma\odot_1)\\ \Sigma\widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\odot_1)) &\xrightarrow{\varepsilon} \widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\sigma\odot_1))\,.\end{aligned}$$

The other two are constructed in exactly the same way.

4.2. Proposition. *The adjoints of the maps in* (4–1) *are homotopy equivalences:*

$$\begin{aligned} L^{\blacktriangledown}(\mathcal{A},\odot_1) &\xrightarrow{\simeq} \Omega L^{\blacktriangledown}(\mathcal{A},\sigma\odot_1)\\ L^{\blacktriangledown}(\mathcal{C},\odot_3) &\xrightarrow{\simeq} \Omega L^{\blacktriangledown}(\mathcal{C},\sigma\odot_3)\\ \widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\odot_1)) &\xrightarrow{\simeq} \Omega\widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{A},\sigma\odot_1))\\ \widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\odot_3)) &\xrightarrow{\simeq} \Omega\widehat{H}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C},\sigma\odot_3))\ .\end{aligned}$$

4.3. Remark. The meaning of (4–2) and (4–3) is that Ξ is a map of spectra. The meaning of (4–4) and (4–5) is that certain external product maps are maps of bispectra. The meaning of 4.2 is that all spectra and bispectra in sight are Ω–spectra.

Proof of 4.2. The first map in the list is a homotopy equivalence by a check on homotopy groups, so we concentrate on the third map in the list. This is adjoint to the geometric realization of a simplicial map (third item in (4–1)) between simplicial infinite loop spaces, induced by a simplicial map between simplicial spectra. Since the relevant spectra are all (-1)–connected by construction, geometric realization commutes with passage from spectra to (zero–th) infinite loop spaces up to homotopy equivalence [WW2, 1.3].

5. Technical Hypotheses

Note first of all that in most cases of interest, hypotheses 3.1 and 3.2 are satisfied or can easily be arranged—one good reason to skip this section. For the reader who does not want to skip it, we quote from [WW3, 4.11±ε] with minor changes in notation and numbering:

" ... what if $\mathcal{A}$ does not satisfy hypothesis 3.1 ? We then force it by creating a suitable *stable* category $\mathcal{A}_\omega$. The objects of $\mathcal{A}_\omega$ are the same as those of $\mathcal{A}$, and a morphism from C to D in $\mathcal{A}_\omega$ is an element of

$$\operatorname*{colim}_n \operatorname{mor}_{\mathcal{A}}(\mathbb{S}^n \wedge C, \mathbb{S}^n \wedge D)\,.$$

Such a morphism is a cofibration {weak equivalence} if it can be represented by a cofibration {weak equivalence} $f : \mathbb{S}^n \wedge C \to \mathbb{S}^n \wedge D$, for some n. The next lemma is an easy consequence of [Wald, 1.6.2].

5.1. Lemma. *The inclusion $\mathcal{A} \subset \mathcal{A}_\omega$ induces a homotopy equivalence from $\boldsymbol{K}(\mathcal{A})$ to $\boldsymbol{K}(\mathcal{A}_\omega)$.*

It is not difficult to extend the action of $\mathcal{Y}$ on $\mathcal{A}$ to one of $\mathcal{Y}$ on $\mathcal{A}_\omega$. By contrast, it is a little hairy to extend or lift the $\odot$ product from $\mathcal{A}$ to $\mathcal{A}_\omega$. (For better distinction, write $\odot_\omega$ for the new product.) Let $\mathcal{I}$ be the subcategory of the category of pointed spaces generated by the diagram of inclusion maps

$$\begin{array}{ccccccccc}
S^0 & \longrightarrow & E^1_u & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
E^1_\ell & \longrightarrow & S^1 & \longrightarrow & E^2_u & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E^2_\ell & \longrightarrow & S^2 & \longrightarrow & E^3_u & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E^3_\ell & \longrightarrow & S^3 & \longrightarrow & \dots \\
 & & & & & & \downarrow & & \\
 & & & & & & \dots & &
\end{array}$$

where E^n_u and E^n_ℓ are the upper and lower hemispheres of S^n, respectively. Let $C \odot_\omega D$ be the space (better, simplicial set) of *almost* natural transformations

$$X \wedge Y \longrightarrow (X \wedge C) \odot (Y \wedge D) \qquad\qquad (X, Y \text{ in } \mathcal{I}).$$

Here an *almost* natural transformation need only be defined for all X and Y in $\mathcal{I}$ whose (naive) dimension is sufficiently large, and two almost natural transformations are considered equal if they agree for X and Y whose dimension is sufficiently large. Thanks to this little precaution, the rule $(C, D) \mapsto C \odot_\omega D$ is a functor on $\mathcal{A}_\omega \times \mathcal{A}_\omega$. Finally, we need to know that the functors

$$(C, D) \mapsto C \odot D \qquad \text{and} \qquad (C, D) \mapsto C \odot_\omega D$$

on $\mathcal{A} \times \mathcal{A}$ are related by a chain of natural homotopy equivalences. This is easy (the chain has length two).

5.2. Lemma. *The functor $(C,D) \mapsto C \odot_\omega D$ is an SW–product on $\mathcal{A}_\omega$. Furthermore, $\mathcal{A}_\omega$ satisfies (hypothesis 3.1). "*

(End of quotation.)

Now remember that we are dealing with three categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and a bi–exact functor from $\mathcal{A}\times\mathcal{B}$ to $\mathcal{C}$, among other things. If we construct "stabilized" categories $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$ along the lines suggested above, then we should *not* construct $\mathcal{C}_\omega$ in the same way—because that would make it impossible to extend the bi–exact functor to one of the form

$$\mathcal{A}_\omega \times \mathcal{B}_\omega \longrightarrow \mathcal{C}_\omega\,. \tag{5–1}$$

Instead, we construct $\mathcal{C}_\omega$ as follows: $\mathcal{C}_\omega$ has the same objects as $\mathcal{C}$, and a morphism from C to D in $\mathcal{C}_\omega$ is an element in

$$\operatorname*{colim}_{m,n} \operatorname{mor}_{\mathcal{C}}(\mathbb{S}^m \wedge \mathbb{S}^n \wedge C\,,\, \mathbb{S}^m \wedge \mathbb{S}^n \wedge D)\,.$$

Then the extension (5–1) is obvious.

We construct "stabilized" SW products $(\odot_1)_\omega$ and $(\odot_2)_\omega$ on $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$, respectively, along the lines suggested above, but again we adopt a slightly different approach to construct $(\odot_3)_\omega$ on $\mathcal{C}_\omega$. Namely, for C and D in $\mathcal{C}_\omega$, we define $C(\odot_3)_\omega D$ as the space of almost natural transformations

$$W\times X\times Y\times Z \longrightarrow (W\wedge Y\wedge C)\odot_3(X\wedge Z\wedge D)$$

where W,X,Y,Z are objects in $\mathcal{I}$. Then it is easy to extend the natural transformation (2–3) to one of the form

$$(A(\odot_1)_\omega A')\wedge(B(\odot_2)_\omega B') \xrightarrow{\psi} \big(A\wedge B)(\odot_3)_\omega(A'\wedge B') \tag{5–2}$$

where A, A' are in $\mathcal{A}_\omega$ and B, B' are in $\mathcal{B}_\omega$. Summarizing, we have forced hypothesis 3.1 without losing anything substantial.

The SW products $(\odot_1)_\omega$, $(\odot_2)_\omega$ and $(\odot_3)_\omega$ have some special properties which we can put to good use. Working in $\mathcal{A}_\omega$, and writing $\odot_\omega$ instead of $(\odot_1)_\omega$, we have natural homotopy eqivalences

$$\begin{aligned} A\odot_\omega A' &\xrightarrow{\simeq} \Omega\big((\mathbb{S}^1\wedge A)\odot_\omega A'\big)\\ A\odot_\omega A' &\xrightarrow{\simeq} \Omega\big(A\odot_\omega(\mathbb{S}^1\wedge A')\big)\end{aligned} \tag{5–3}$$

for A, A' in $\mathcal{A}_\omega$. They are defined as follows: A point in $A\odot_\omega A'$ is an almost natural transformation, defined for "most" X, Y in $\mathcal{I}$:

$$f: X\wedge Y \longrightarrow (X\wedge A)\odot(Y\wedge A')\,.$$

For $t \in \mathbb{S}^1$, let $\bar{f}_t$ be the natural transformation making

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\bar{f}_t} & X \wedge (\mathbb{S}^1 \wedge A) \odot (Y \wedge A') \\ {\scriptstyle (x,y)\mapsto(x,t,y)}\downarrow & & \cong\downarrow \\ (X \wedge \mathbb{S}^1) \wedge Y & \xrightarrow{f} & ((X \wedge \mathbb{S}^1) \wedge A) \odot (Y \wedge A') \end{array}$$

commutative. Then $t \mapsto \bar{f}_t$ is a loop in $(\mathbb{S}^1 \wedge A) \odot_\omega A'$, which establishes one half of (5–3). The other half is similar, and we conclude that

$$\{(\mathbb{S}^m \wedge A) \odot_\omega (\mathbb{S}^n \wedge A') \mid m, n \geq 0\} \tag{5–4}$$

is an Ω–bispectrum. Similarly, for B, B' in $\mathcal{B}_\omega$,

$$\{(\mathbb{S}^i \wedge B) \odot_\omega (\mathbb{S}^j \wedge B') \mid i, j \geq 0\} \tag{5–5}$$

is an Ω–bispectrum, and for C, C' in $\mathcal{C}_\omega$,

$$\{(\mathbb{S}^m \wedge \mathbb{S}^i \wedge C) \odot_\omega (\mathbb{S}^n \wedge \mathbb{S}^j \wedge C') \mid m, n, i, j \geq 0\} \tag{5–6}$$

is an Ω–4–spectrum (four suspension directions). If $C = A \wedge B$ and $C' = A' \wedge B'$ in (5–6), then we can use (5–2) to make a map of 4–spectra:

$$(5\text{–}4) \wedge (5\text{–}5) \longrightarrow (5\text{–}6)\,.$$

We have forced hypothesis 3.2 now, taking (5–4), (5–5) and (5–6) as the definition of the bispectrum valued (or 4–spectrum valued) SW products in $\mathcal{A}_\omega$, $\mathcal{B}_\omega$ and $\mathcal{C}_\omega$, respectively.

6. Associativity and Commutativity

Two important examples of bi–exact functors are: the external product of retractive spaces, and the tensor product of chain complexes. Both of these can be made strictly associative with some effort, and one may wonder whether the induced external multiplication maps in K–theory are also associative in some sense. The same question can be asked for symmetric L–theory, and for the Tate cohomology of $\mathbb{Z}/2$ acting on K–theory, when defined. For simplicity, we consider categories of chain complexes, and for honesty, we consider symmetric L-theory and Tate cohomology of $\mathbb{Z}/2$ acting on K–theory. We assume or pretend that tensor products (of rings, or of chain complexes) are *strictly* associative. This is one of the things that “can be arranged”, given that we are willing to replace the category of rings, or the category of chain complexes for that matter, by any equivalent category. See [May] and further references given there.

Suppose that F is a covariant functor from rings with involution to spectra. We shall try to say what an associative *external product* on F is. Naively, we want maps from $F(R_1) \wedge F(R_2)$ to $F(R_1 \otimes R_2)$ for rings with involution R_1 and R_2, but we cannot have them because $F(R_1) \wedge F(R_2)$ is a bispectrum and $F(R_1 \otimes R_2)$ is an ordinary spectrum. In the words of [Ad, p.60], we have to allow flab. It seems that we have to introduce other functors, from rings with involution to bispectra, trispectra, ... , n–spectra (indexed by $\mathbb{Z}^n$) for any $n > 0$.

6.1. Definition. An *associative external product* on F consists of

- a sequence of functors F_n, from rings with involution to n–spectra ($n > 0$), such that $F_1 = F$;
- maps of $m+n$–spectra
$$\mu : F_m(R_1) \wedge F_n(R_2) \longrightarrow F_{m+n}(R_1 \otimes R_2)\,,$$
for all $m, n > 0$, which are associative ;
- a map $\eta : \boldsymbol{S}^0 \longrightarrow F_1(\mathbb{Z})$ such that the following compositions are homotopy equivalences, for every R and every $n > 0$:
$$\boldsymbol{S}^0 \wedge F_n(R) \xrightarrow{\eta\wedge\mathrm{id}} F_1(\mathbb{Z}) \wedge F_n(R) \xrightarrow{\mu} F_{1+n}(\mathbb{Z}\otimes R)$$
$$F_n(R) \wedge \boldsymbol{S}^0 \xrightarrow{\mathrm{id}\wedge\eta} F_n(R) \wedge F_1(\mathbb{Z}) \xrightarrow{\mu} F_{n+1}(R\otimes \mathbb{Z})\,.$$

If, in addition, each $F_n(R)$ comes with a *crossed* action (explained below) of the symmetric group Σ_n, natural in R, and
$$\mu : F_m(R_1) \wedge F_n(R_2) \longrightarrow F_{m+n}(R_1 \otimes R_2)$$
is a $\Sigma_m \times \Sigma_n$–map for arbitrary R_1 and R_2, then we speak of a *commutative and associative external product.*

Explanation. Let $\boldsymbol{X}$ be an n–spectrum (indexed by $\mathbb{Z}^n$). Any $\sigma \in \Sigma_n$ determines an automorphism of $\mathbb{Z}^n$ by permuting axes ; we can use this to re–index $\boldsymbol{X}$. Call the resulting n–spectrum $\boldsymbol{X}^\sigma$. Note that $(\text{—})^\sigma$ is a functor. A *crossed* action of Σ_n on $\boldsymbol{X}$ consists of maps $\sigma_* : \boldsymbol{X} \longrightarrow \boldsymbol{X}^\sigma$, for $\sigma \in \Sigma_n$, such that $(\sigma\tau)_* = (\sigma_*)^\tau(\tau_*)$.

6.2. Remark. If F is equipped with a commutative and associative external product, then $F(\mathbb{Z})$ is a homotopy associative and homotopy commutative ring spectrum (since $\mathbb{Z}\otimes\mathbb{Z} \cong \mathbb{Z}$). Is it an E_∞ ring spectrum ? We suspect it is.

Essentially by interpreting the results of the previous sections, we are going to show that the functors
$$R \mapsto \boldsymbol{L}^\blacktriangledown(R)\,, \qquad R \mapsto \widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(R))$$
can be equipped with commutative and associative external products such that Ξ from $\boldsymbol{L}^\blacktriangledown(R)$ to $\widehat{\boldsymbol{H}}^\blacktriangledown(\mathbb{Z}/2; \boldsymbol{K}(R))$ respects external products. Here we have abbreviated $\boldsymbol{L}^\blacktriangledown(R) = \boldsymbol{L}^\blacktriangledown(\mathcal{C}(R))$ and $\boldsymbol{K}(R) = \boldsymbol{K}(\mathcal{C}(R))$, where $\mathcal{C}(R)$ is the category of finitely generated chain complexes (graded over $\mathbb{Z}$) of left projective R–modules. We must recall the (standard) SW product in $\mathcal{C}(R)$, and to do so we start with a review of the Kan–Dold construction.

6.3. Reminder. The Kan–Dold construction associates to a chain complex C, graded over the integers, a simplicial abelian group C^Δ such that $C^\Delta[n]$ is the abelian group of chain maps from the cellular chain complex of the standard simplex Δ^n to C. The Kan–Dold construction, restricted to chain complexes graded over the *positive* integers, is an equivalence of categories. For chain complexes with nontrivial chain groups in negative degrees, it is useful to know how the Kan–Dold construction behaves with respect to suspension: there is a canonical map of simplicial sets

$$\Sigma(C^{\Delta}) \longrightarrow (\Sigma C)^{\Delta}\,.$$

(The suspension of a simplicial pointed set or simplicial pointed space has been defined in §4. In particular, if $X = \Delta^n$, then we have a unique nondegenerate simplex of dimension $n+1$ in $\Sigma\Delta^n$, with characteristic map $\Delta^{n+1} \to \Sigma\Delta^n$. Composing with the induced map on cellular chains, we have a way to map the n–chains in $\Sigma(C^{\Delta})$ to the $(n+1)$–chains in $(\Sigma C)^{\Delta}$.)

The Kan–Dold construction *does not* commute with tensor products up to isomorphism. However, there is a natural homotopy equivalence from $C^{\Delta} \otimes D^{\Delta}$ to $(C \otimes D)^{\Delta}$, and it can be constructed using Eilenberg–Zilber ideas.

Returning to our ring with involution R and the chain complex category $\mathcal{C}(R)$, we note that $\mathcal{C}(R)$ is a category with cofibrations (=dimensionwise split injections) and weak equivalences. For C, D in $\mathcal{C}(R)$ we let $C \odot D$ be the bispectrum

$$\big\{(\Sigma^m C^t)^{\Delta} \otimes_R (\Sigma^n D)^{\Delta} \bigm| m, n \geq 0\big\}\,.$$

(The tensor product of a simplicial right R–module with a simplicial left R–module is a simplicial abelian group.) Then $\odot$ is an SW product, and we can define

$$F(R) = F_1(R) := \boldsymbol{L}^{\blacktriangledown}(\mathcal{C}(R))\,, \qquad G(R) = G_1(R) := \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{C}(R)))\,. \tag{6–1}$$

Here we use the definitions from around (3–2), so that $F(R)$ and $G(R)$ are spectra made up of simplicial sets and simplicial spaces, respectively. Now for the external products: We have already seen (around (3–3)) competing definitions of $\boldsymbol{L}^{\blacktriangledown}(\mathcal{C}(R))$ and $\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2; \boldsymbol{K}(\mathcal{C}(R)))$ as *bi*spectra made up of *bi*simplicial sets and *bi*simplicial spaces, respectively. For better distinction, we denote these constructions by $F_2(R)$ and $G_2(R)$, respectively. By a straightforward generalization, we can also construct n–spectrum versions, made up of n–simplicial sets and n–simplicial spaces, respectively: denote these by $F_n(R)$ and $G_n(R)$. In §3 and §4, we constructed external products

$$\mu : F_m(R) \wedge F_n(R) \longrightarrow F_{m+n}(R)$$
$$\mu : G_m(R) \wedge G_n(R) \longrightarrow F_{m+n}(R)$$

when $m = n = 1$, but now it is clear that it will work for $m, n > 0$. We can describe $\eta : \boldsymbol{S}^0 \to F_1(\mathbb{Z})$ by specifying a point (0–simplex) in the zero–th space of $F_1(\mathbb{Z})$, which is $L^{\blacktriangledown}(\mathbb{Z})$. We choose the point corresponding to $\mathbb{Z}$, regarded as a 0–dimensional symmetric Poincaré object. We *define* the unit map $\eta : \boldsymbol{S}^0 \to G_1(\mathbb{Z})$ as the composition of $\eta : \boldsymbol{S}^0 \to F_1(\mathbb{Z})$ with $\Xi : F_1(\mathbb{Z}) \to G_1(\mathbb{Z})$. It is then easy to verify that we have constructed commutative and associative external products on $F = F_1$ and $G = G_1$, and that Ξ respects the external products. That is, $\Xi : F_1 \to G_1$ extends to $\Xi : F_n \to G_n$ for all $n > 0$, commuting with μ and η.

6.4. Proposition. *The spectrum* $\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(R))$ *is a generalized Eilenberg–MacLane spectrum, for every ring with involution* R*. It is also 4–periodic:*

$$\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(R)) \simeq \Omega^4\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(R)).$$

Proof. $\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(R))$ is a 2–local spectrum, and a module spectrum over the ring spectrum $\boldsymbol{L}^{\blacktriangledown}(\mathbb{Z})$. Now $\boldsymbol{L}^{\blacktriangledown}(\mathbb{Z})$ is 4–periodic (with the definition we use, which agrees with [Ra2] but not with [Ra1]). Further, $\boldsymbol{L}^{\blacktriangledown}(\mathbb{Z})$ is a generalized Eilenberg–MacLane spectrum when localized at the prime 2. □

6.5. Remark. For the nonlinear version of 6.1, assume that F is a functor from *spaces* with spherical fibration to CW–spectra. (The "spaces" could be CW–spaces or ENR's, and the spherical fibrations should be equipped with a distinguished section.) The idea is that spaces replace rings, and spherical fibrations replace involutions ; instead of the tensor product of rings with involution, we have the construction making $(X\times Y,\gamma\times\zeta)$ out of (X,γ) and (Y,ζ), where $\gamma\times\zeta$ is our (bad) notation for the fiberwise external smash product of γ and ζ.

An associative external product on F consists of a sequence of functors F_n from spaces with spherical fibration to n–spectra, $n>0$, and natural transformations

$$\mu : F_m(X,\gamma)\wedge F_n(Y,\zeta) \longrightarrow F_{m+n}(X\times Y,\gamma\times\zeta)$$

which are associative, and so on as in 6.1. For example, the functors

$$F(X,\gamma) = \boldsymbol{L}^{\blacktriangledown}(\mathcal{C}_X)$$
$$G(X,\gamma) = \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}_X))$$

admit such external products (where $\mathcal{C}_X$ is the category of finite retractive CW–spaces over X, with an SW product depending on γ, see [WW3 4.17]). The map

$$\Xi : \boldsymbol{L}^{\blacktriangledown}(\mathcal{C}_X) \longrightarrow \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2;\boldsymbol{K}(\mathcal{C}_X))$$

constructed in [WW2] or [WW3, §4] respects external products. None of this is surprising. But proposition 6.4 does not generalize to the nonlinear setting: $\boldsymbol{L}^{\blacktriangledown}(\mathcal{C}_X)$ is not 4–periodic, and it is not a generalized Eilenberg–MacLane spectrum localized at 2, when $X = *$.

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