Hodge decompositions of Loday Symbols in K-theory and cyclic homology

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Abstract

In this paper we study the Hodge decompositions of K-theory and cyclic homology induced by the operations ψ^k and λ^k, and in particular the decomposition of the Loday symbols $\ll x, y, ..., z \gg$. Except in special cases, these Loday symbols do not have pure Hodge index. In $K_n(A)$ they can project into every component $K_n^{(i)}$ for $2 \leq i \leq n$, and the projection of the Loday symbol $\ll x, y, ..., z \gg$ into $K_n^{(n)}$ is a multiple of the generalized Dennis-Stein symbol $< x, y, ..., z >$. Our calculations disprove conjectures of Beilinson and Soulé in K-theory, and of Gerstenhaber and Schack in Hochschild homology.

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The purpose of this paper is to improve our understanding of the Hodge decompositions of the higher K-theory $K_*(A)$ and cyclic homology $HC_*(A)$ of a commutative ring with unit A. To do this, we study the action of the operators ψ^k and λ^k on Loday symbols, which are elements in higher K-theory and/or cyclic homology. For simplicity, we shall assume A is of finite type over a field ℓ of characteristic 0.

For Hochschild homology and cyclic homology, the Hodge decomposition means the direct sum decompositions $HH_n(A) = \oplus_{i=0}^n HH_n^{(i)}(A)$ and $HC_n(A) = \oplus_{i=0}^n HC_n^{(i)}(A)$ constructed in [GS], [FT], and [Lop] (see §1). They are given by the Eulerian idempotents in the group ring $\mathbf{Q}\Sigma_n$ and take their name from the fact, noted in [GS], that in a special case they yield the classical Hodge decomposition of complex analytical cohomology. The operations ψ^k and λ^k on $HH_*(A)$ and $HC_*(A)$ are defined to be scalar multiplication on each of the Hodge components:

$$\psi^k(x) = k^{i+1}x,\ \lambda^k(x) = (-1)^k k^i \text{ for } x \in HH_*^{(i)}(A) \text{ or } HC_*^{(i)}(A).$$

By the *Hodge decomposition* of the K-theory $K_*(A)$ of a commutative ring A we mean the analogous decomposition of $K_*(A)_{\mathbf{Q}} = K_*(A) \otimes \mathbf{Q}$ according to the eigenvalues of the operators ψ^k and λ^k:

$$K_*(A)_{\mathbf{Q}} \cong \bigoplus_{i=0}^{\infty} K_*^{(i)}(A).$$

Elements of $K_*^{(i)}(A)$ are eigenvectors for each ψ^k and λ^k, the eigenvalues being k^i and $(-1)^k k^{i-1}$, respectively ([Hil, 4.7]). The groups $K_*^{(i)}(A)$ are also the graded quotients for the γ-filtration of $K_*(A)_{\mathbf{Q}}$. A third interpretation is given by Beilinson in [B, 2.2.1]; for any scheme X (such as $X = Spec(A)$) he writes $H_{\mathcal{A}}^{2i-n}(X, \mathbf{Q}(i))$ for $K_n^{(i)}(X)$, and calls these the *absolute (motivic) cohomology groups* of X. Soulé proved in [S, 2.1] that $K_n^{(i)}(A) = 0$ for $i > dim(A) + n$, so that the Hodge decomposition of each $K_n(A)$ is finite.

The Hodge decomposition of $K_0(A)$ is classically part of Grothendieck's formulation of the Riemann-Roch Theorem. $K_0^{(0)}(A)$ is the "rank" $H^0(Spec(A), \mathbf{Q})$; the determinant map induces an isomorphism $K_0^{(1)}(A) \cong Pic(A)_{\mathbf{Q}}$, and $K_0^{(i)}(A) = 0$ for $i > dim(A)$. If A is regular then $K_0^{(i)}(A)$ is rationally the Chow group of codimension i cycles on $Spec(A)$. The Hodge decomposition of $K_1(A)$ is less well understood. We know that $K_1^{(0)}(A) = 0$, $K_1^{(1)}(A)$ is (rationally) the group A^* of units of A, and that $K_1^{(i)}(A) = 0$ if $i > dim(A) + 1$.

For $n \geq 2$, the Hodge decomposition is not well-understood. Kratzer observed in [K, 6.8] that $K_n^{(0)}(A) = K_n^{(1)}(A) = 0$, so the only possible Hodge indices i with $K_n^{(i)}(A) \neq 0$ are $i = 2, ..., dim(A) + n$. Some individual elements of $K_n(A)$ are known to have pure Hodge index. For example, the Steinberg symbols $\{x_1, ..., x_n\}$ and generalized Dennis-Stein symbols $< x_1, ..., x_n >$ are known to have pure Hodge index n, i.e., to lie in $K_n^{(n)}(A)$. (This follows from [Hil, 8.1] and [Lsym, 1.6]; see 2.2 below.)

In this paper we study the Hodge decomposition of Loday symbols, and show that they usually don't have pure Hodge index. Recall from [Lsym] that the K-theory *Loday symbol* $\ll x_1, ..., x_n \gg$ is an element of $K_n(A)$, defined whenever $x_1x_2 = \ldots = x_ix_{i+1} = \ldots = x_nx_1 = 0$. There are corresponding Loday symbols in $HH_{n-1}(A)$ and $HC_{n-1}(A)$ introduced in (2.5) below. For clarity, we shall only describe our results for the K-theory Loday symbols; our results for HC-theory Loday symbols differs only by a shift in indices.

It turns out that the Hodge indices of the K-theory Loday symbol $\ll x_1, ..., x_n \gg$ are restricted to the range $2 \leq i \leq n$. We prove in Theorem 2.10 that the top Hodge component of this Loday symbol (lying in $K_n^{(n)}(A)$) is $\pm 1/(n-1)!$ times the corresponding generalized Dennis-Stein symbol $< x_1, ..., x_n >$. This generalizes Loday's observation in [Lsym] that $\ll x, y \gg = \pm < x, y >$ in $K_2^{(2)}(A)$. More interesting results occur in K_3, K_4 and K_5.

Example 0.1 (Fat point). (See §4.) Consider $R = \ell[x, y, z]/(x, y, z)^2$, an Artinian ring whose maximal ideal has square zero. The Loday symbol $s = \ll x, y, z \gg \in K_3(R)$ does not have pure Hodge index. The Hodge components of s are $\frac{1}{2}(s + \overline{s}) \in K_3^{(2)}(R)$ and $\frac{1}{2}(s - \overline{s}) \in K_3^{(3)}(R)$, where $\overline{s} = \ll x, z, y \gg$. Note that $\overline{s} - s$ is the generalized Dennis-Stein symbol $< x, y, z >$ by [Lsym, 2.6]. On the other hand, in $K_4(R)$ the Loday symbols $\ll x, y, z, y \gg$ and $\ll x, y, x, y \gg$ have pure Hodge index 3, but $\ll x, x, y, y \gg$ has pure Hodge index 2. In $K_5(R)$ all the Loday symbols project nontrivially into $K_5^{(3)}(R)$, but not all have pure Hodge index 3. In particular, $\ll x, x, y, y, z \gg$ has Hodge indices 2 and 3. The difference of Loday symbols

$$\ll x, x, y, y, z \gg - \ll y, y, x, x, z \gg$$

has pure Hodge index 2, i.e., it is a nonzero element contained in $K_5^{(2)}(R)$.

This nonzero element of $K_5^{(2)}(R)$ provides a counterexample to the conjecture, formulated independently by Beilinson ([B, 2.2.2]; see [Sch, p.12]) and Soulé ([S, 2.9]), that $K_n^{(i)}(A) = 0$ for $i < n/2$, or equivalently that $H_{\mathcal{A}}^{(j)} = 0$ for $j < 0$. Here is a more geometric counterexample:

Example 0.2 (Coordinate axes). (See 2.9.) Let $R_{m-1} = \ell[x_1, ..., x_m]/(x_i x_j, i \neq j)$ be the coordinate ring of the coordinate axes in affine m-space over ℓ. Then the Loday symbol $\ll x_1, ..., x_m \gg$ in $K_m(R_{m-1})$ projects nontrivially into $K_m^{(i)}(R_{m-1})$ for all i in the range $2 \leq i \leq m$.

Of course our rings are necessarily singular, so the conjecture may still hold for regular rings. The following example shows that in the Beilinson–Soulé conjecture we cannot replace $b = n/2$ by any lower bound b(n,A) which increases with n.

Example 0.3. (See 8.2.) When $A = \ell[x, y, z]/(x, y, z)^2$ every single one of the $n-1$ symbols in $K_n(A)$ having $n-2$ x's, one y and one z,

$$\ll x, x, ..., x, y, z \gg, \ \ll x, x, ..., x, y, x, z \gg, \text{ etc.},$$

projects nontrivially into $K_n^{(i)}(A)$ for all i in the range $2 \leq i \leq (n+1)/2$. If i is outside this range, the projections into $K_n^{(i)}(A)$ are zero.

The corresponding Loday symbols in $HH_{n-1}(A)$ and $HC_{n-1}(A)$ have Hodge indices in the same range, and also project nontrivially into the i^{th} component of the Hodge decomposition for all i in the range $2 \leq i \leq (n+1)/2$. This answers negatively a question posed in [GS2, p.268] by showing that $HH_*^{(i)}(A)$ does not vanish for $i >> 0$.

After we had found these examples, Soulé pointed out to us that Feigin and Tsygan had already observed in [FT, 7.5.6] that the Beilinson-Soulé conjecture fails for $A = F \oplus I$, $I^2 = 0$, when I is an infinite-dimensional vector space over F. Our example 0.1 is a finite-dimensional version of their counterexample, and our results contain more information than theirs because we use the Eulerian idempotents $e_n^{(i)}$ of [Barr] and [GS] as well as the generating functions of Hanlon ([H]) for Hochschild and cyclic homology.

Our basic method is to compare K-theory with Hochschild and cyclic homology via the Dennis trace map $D : K_n(A) \to HH_n(A)$, where the Hochschild homology of A is taken over the ground field $k = \mathbf{Q}$. Suppose that $A = A_0 \oplus A_1 \oplus \cdots$ is a graded ℓ-algebra. We shall use the following notation: if F is any functor from ℓ-algebras to abelian groups, we write $\widetilde{F}(A)$ for the kernel of the augmentation $F(A) \to F(A_0)$. A result in [Gw] states that the map $B : \widetilde{HC}_{m-1}(A) \to \widetilde{HH}_m(A)$ is an injection. We know from [Wnil] that the Dennis trace map factors through cyclic homology:

$$\widetilde{K}_m(A) \xrightarrow{\nu} \widetilde{HC}_{m-1}(A) \xrightarrow{B} \widetilde{HH}_m(A).$$

The following two results are proven in (2.3) and (3.1) below. Theorem 0.4 allows us to detect nontrivial elements in the Hodge decomposition of $\widetilde{K}_*^{(i)}$; Theorem 0.5 tells us when an element projects nontrivially into $K_*^{(i)}(A)$.

Theorem 0.4. *If $A = A_0 \oplus A_1 \oplus \cdots$ is a graded ℓ-algebra, then ν commutes with the operations ψ^k and λ^k. The Dennis trace map only commutes with ψ^k and λ^k up to a factor of k, in the sense that for $x \in \widetilde{K}_*(A)$ we have:*

$$\psi^k D(x) = kD(\psi^k(x)) \text{ and } \lambda^k D(x) = kD(\lambda^k(x)).$$

In other words, the Dennis trace map preserves Hodge filtrations, factoring as:

$$\widetilde{K}_m^{(i)}(A) \xrightarrow{\nu} \widetilde{HC}_{m-1}^{(i-1)}(A) \xrightarrow{B} \widetilde{HH}_m^{(i)}(A).$$

Remark 0.4.1. We believe that for every commutative ring A the Dennis trace map as well as its lift $D : K_m(A) \to HC_m^-(A)$ satisfy $\psi^k D(x) = kD(\psi^k(x))$ modulo torsion, but we have no proof. If the augmentation ideal is nilpotent, Theorem 0.4 is a consequence of a theorem of Cathelineau ([C]), and was also asserted in [FT, 7.5.5].

Theorem 0.5. *Let A be a discrete Hodge algebra over a smooth $\mathbf{Q}$-algebra ℓ, i.e., an algebra of the form $A = \ell[x_1, ..., x_m]/I$, where I is an ideal generated by monomials. Then the main result of [OW] implies that $\nu : \widetilde{K}_n(A) \to \widetilde{HC}_{n-1}(A)$ is an injection, the cyclic homology being taken over $\mathbf{Q}$. In particular, if $i > n$ then $\widetilde{K}_n^{(i)}(A) = 0$. Moreover, if $i < n$ then*

$$\widetilde{K}_n^{(i)}(A) \cong \widetilde{HC}_{n-1}^{(i-1)}(A).$$

The curious phrasing in 0.5 is due to the presence of a gap in the proof of the main result of [OW]. In order to circumvent this gap, we have included an appendix proving the following weaker assertion which is adequate for our purposes.

Proposition 0.6. *Let $x_0, ..., x_n$ be elements of an ℓ-algebra A such that $x_i x_{i+1} = 0$ for $0 \leq i < n$ and $x_n x_0 = 0$. Then there is a map from the multiple-relative cyclic homology group $HC_n(A; x_0, ..., x_n)$ to $K_{n+1}(A; x_0, ..., x_n)$ sending the Loday symbol $\ll x_0, ..., x_n \gg$ to the Loday symbol $\ll x_0, ..., x_n \gg$. Moreover, this map commutes with the Adams operations ψ^k.*

We have organized this paper as follows. In §1, we describe the Hodge decompositions of Hochschild and cyclic homology, mentioned above. In §2, we discuss the Dennis trace map, prove Theorem 0.4, establish Example 0.2, and show that the top Hodge component of a homology Loday symbol is a multiple of the corresponding Dennis-Stein symbol. In §3, we use the main result of [OW] to prove Theorem 0.5 for discrete Hodge algebras. In §4, we grind out explicit calculations of the Hodge decomposition of Loday symbols in low dimensions (up to K_5) and establish Example 0.1. In §5, we strengthen a result of Burghelea and Vigué by showing that, if $A = \ell[x_0, ..., x_m]/(f)$ is the homogeneous coordinate ring of a smooth hypersurface in projective space, then $\widetilde{HC}_p(A)$ has pure Hodge index (p if $p \leq m$, $(p+m)/2$ if $p > m$). We also calculate these groups.

In the last three sections, we shift gears in order to calculate the dimensions of the groups $\widetilde{HC}_n^{(i)}(A)$. This uses a new tool, which we introduce in §6 – Hanlon's generating functions. In §7, we use this tool to calculate the Hodge decomposition of Loday symbols involving only two variables, such as $\ll x, x, y, ..., y \gg$. In §8, we use Hanlon's generating functions to analyze the Hodge decomposition of Loday symbols involving only three variables, and verify Example 0.3.

§1. Hodge decomposition of HH and HC

1.1. $HH_*^{(i)}$ If A is a commutative ℓ-algebra with identity and $\mathbf{Q} \subseteq \ell$, the Hochschild homology $HH_*(A)$ of A is the homology of the reduced bar complex $C_*(A)$ which has $C_n(A) = A \otimes_\ell \overline{A}^{\otimes n}$ ($\overline{A}$ = cokernel of $\ell \to A$). The symmetric group Σ_n acts on $C_*(A)$ on the left by

$$\sigma(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}, \ \sigma \in \Sigma_n.$$

The Eulerian idempotents $e_n^{(i)}$, $i = 1, \ldots, n$ of [GS, §1] are mutually orthogonal (non-central) idempotents in $\mathbf{Q}[\Sigma_n]$ whose sum is 1. They commute with the boundary map b in the bar complex in the sense that $be_n^{(i)} = e_{n-1}^{(i)}b$, so C_* is the direct sum of the complexes $C_*^{(i)}$, where $C_n^{(i)} = e_n^{(i)} C_n$. Writing $HH_n^{(i)}(A)$ for $H_n(C_*^{(i)}(A))$, we obtain the Hodge decomposition for Hochschild homology:

$$HH_n(A) = HH_n^{(1)}(A) \oplus \cdots \oplus HH_n^{(n)}(A), \ n \neq 0.$$

When $n = 0$, we take $HH_0(A) = HH_0^{(0)}(A) = A$. This choice is dictated by [LP, 2.4].

The idempotent $e_n^{(n)}$ is the signature idempotent $\frac{1}{n!}\Sigma(-1)^\sigma \sigma$ of $\mathbf{Q}[\Sigma_n]$ corresponding to the sign representation ([GS, 1.3.iv]), and $HH_n^{(n)}(A) \cong \Omega^n_{A/\ell}$, the A-module of Kähler differentials of A over ℓ ([Lop, 3.5.d]). At the other extreme, M. Barr discovered $e_n^{(1)}$ in [Barr] and proved that $HH_n^{(1)}(A)$ is Harrison homology (also known as André-Quillen homology) for commutative ℓ-algebras. (R. Hain rediscovered $e_n^{(1)}$ in [Hain].)

For $1 < i < n$, little is known about the idempotents $e_n^{(i)}$. The character table for the representations $e_n^{(i)}\mathbf{Q}[\Sigma_n]$, $n \leq 4$ is given on p.117 of [H]. Hanlon also proves ([H, 5.13])that the dimension of $e_n^{(i)}\mathbf{Q}[\Sigma_n]$ equals the ith Stirling number (the number of permutations in Σ_n with exactly i cycles). In particular, $e_n^{(i)} \neq 0$ for $1 \leq i \leq n$. We refer the reader to [H] for more information.

Examples 1.2. In $\mathbf{Q}[\Sigma_2] \cong \mathbf{Q} \times \mathbf{Q}$ we have:

$$e_2^{(1)} = \frac{1}{2}(1 + (12)); \quad e_2^{(2)} = \frac{1}{2}(1 - (12)).$$

In $\mathbf{Q}[\Sigma_3]$ we have:

$$\begin{aligned}
e_3^{(1)} &= \frac{1}{6}\{2-(123)-(132)+(12)+(23)-2(13)\};\\
e_3^{(2)} &= \frac{1}{2}\{1+(13)\};\\
e_3^{(3)} &= \frac{1}{6}\Sigma(-1)^{\sigma}\sigma = 1-e_3^{(1)}-e_3^{(2)}.
\end{aligned}$$

In $\mathbf{Q}[\Sigma_4]$ each idempotent has 24 terms:

$$\begin{aligned}
e_4^{(1)} &= \frac{3}{12}\{1-(14)(23)\}+\frac{1}{12}\{-\sum_{i<j}(-1)^{i+j}(ij)+\sum_{\substack{i,j,k\\ i<j,k}}(-1)^{[\frac{i+j+k+1}{2}]}(ijk)\\
&\qquad +(1234)+(1432)+(1243)-(1423)-(1324)-(1342)-(13)(24)+(12)(34)\};\\
e_4^{(2)} &= -e_4^{(4)}+\frac{1}{2}\{1+(14)(23)\};\\
e_4^{(3)} &= -e_4^{(1)}+\frac{1}{2}\{1-(14)(23)\};\\
e_4^{(4)} &= \frac{1}{24}\Sigma(-1)^{\sigma}\sigma.
\end{aligned}$$

Since Σ_n is large for $n \geq 5$, a more compact notation is necessary. Let $i(\sigma)$ be the number of times $i-1$ appears after i in the sequence $\{\sigma^{-1}(1), ..., \sigma^{-1}(n)\}$, i.e., the number of twists in the inverse permutation or the number of descents of the permutation. Let $B_j = \Sigma_{i(\sigma)=j-1}(-1)^{\sigma}\sigma$. Thus $B_j = (-1)^{j-1}\ell_n^j$ where ℓ_n^j is defined in [Lop, 1.5a]. Les Reid (private communication) has shown that the $e_n^{(i)}$ can be written in terms of the B_j as follows (cf. [Lop,2.8d]).

$$e_4^{(1)} = \frac{1}{12}(3B_1-B_2+B_3-3B_4);$$

$$\begin{aligned}
e_5^{(1)} &= \frac{1}{120}(24B_1-6B_2+4B_3-6B_4+24B_5);\\
e_5^{(2)} &= \frac{1}{120}(50B_1-5B_2+5B_4-50B_5);\\
e_5^{(3)} &= \frac{1}{120}(35B_1+5B_2-5B_3+5B_4+35B_5);\\
e_5^{(4)} &= \frac{1}{120}(10B_1+5B_2-5B_4-10B_5);\\
e_5^{(5)} &= \frac{1}{120}(B_1+B_2+B_3+B_4+B_5);
\end{aligned}$$

$$
\begin{aligned}
e_6^{(1)} &= \frac{1}{720}(120B_1 - 24B_2 + 12B_3 - 12B_4 + 24B_5 - 120B_6);\\
e_6^{(2)} &= \frac{1}{720}(274B_1 - 26B_2 + 4B_3 + 4B_4 - 26B_5 + 274B_6);\\
e_6^{(3)} &= \frac{1}{720}(225B_1 + 15B_2 - 15B_3 + 15B_4 - 15B_5 - 225B_6);\\
e_6^{(4)} &= \frac{1}{720}(85B_1 + 25B_2 - 5B_3 - 5B_4 + 25B_5 + 85B_6);\\
e_6^{(5)} &= \frac{1}{720}(15B_1 + 9B_2 + 3B_3 - 3B_4 - 9B_5 - 15B_6);\\
e_6^{(6)} &= \frac{1}{720}(B_1 + B_2 + B_3 + B_4 + B_5 + B_6);
\end{aligned}
$$

Caveat lector 1.2.1. There are two ways of defining the multiplication in the symmetric group, one as composition of functions (i.e.,working right to left), the other in reading order (i.e., working left to right). For the first, the symmetric group acts on the left (i.e. $\sigma(a_1, ..., a_n) = (a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(n)}))$, and for the second it acts on the right (i.e. $\sigma(a_1, ..., a_n) = (a_{\sigma(1)}, ..., a_{\sigma(n)}))$. Our convention is the one used in [GS] and [Lop], namely to use the composition of functions and the attendent left action. The other convention is used in [LP]. Thus our listed $e_n^{(i)}$ are for the left action. In order to find the idempotents for the right action, simply send σ to σ^{-1} in each $e_n^{(i)}$. Many of the Eulerian idempotents are invariant under this substitution, but for example $e_4^{(1)}$ is not.

1.3. Symmetrizing idempotent. The following observation simplifies many calculations in Hochschild Homology. (See §4). Let $\rho_n \in \Sigma_n$ denote the order-reversing involution which interchanges i and $n-i$. The *symmetrizing idempotent* is $\sigma_n = \frac{1}{2}(1+(-1)^{n(n+1)/2}\rho_n)$. For small n, we see from (1.2) that $\sigma_2 = e_2^{(2)}$, $\sigma_3 = e_3^{(2)}$, and $\sigma_4 = e_4^{(2)} + e_4^{(4)} = \frac{1}{2}(1 + (14)(23))$. In general, we know from [GS, 5.2] that $\sigma_n = \Sigma e_n^{(2i)}$. Therefore if an n-cycle is fixed (resp., annihilated) by σ_n, its Hodge decomposition has only even (resp., odd) components.

1.4. $HC_*^{(i)}$ Let $B : C_{n-1}(A) \to C_n(A)$ be the Connes operator ([Co, p.96]) defined by

$$B(a_1 \otimes \cdots \otimes a_n) = \Sigma_{i=1}^{n}(-1)^{(n-1)i} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}.$$

By definition ([LQ]), the cyclic homology $HC_*(A)$ is the homology of Connes' double complex ([Co, p.119]) $C_{pq} = C_{q-p}$ with vertical differential b and horizontal differential B.

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
A\otimes\overline{A}^{\otimes 2} \arrow[d] & A\otimes\overline{A} \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & 0 \arrow[l] \\
A\otimes\overline{A} \arrow[d] & A \arrow[l] \arrow[d] & 0 \arrow[l] & \\
A \arrow[d] & 0 & & \\
0 & & &
\end{tikzcd}$$

From [Lop, 2.14] we see that if A is commutative the map B shifts the Hodge decomposition of $C_*(A)$ in that it sends $C_{n-1}^{(i-1)}$ to $C_n^{(i)}$. Thus Connes' double complex is the direct sum of double complexes $C_{**}^{(i)}$, the pth column of $C_{**}^{(i)}$ being the complex $C_*^{(i-p)}$ shifted p places vertically so that $C_{**}^{(i)}$ is zero below the ith row. Writing $HC_n^{(i)}(A)$ for $H_n(C_{**}^{(i)})$, we obtain the Hodge decomposition for cyclic homology:

$$HC_n(A) = HC_n^{(1)}(A) \oplus \cdots \oplus HC_n^{(n)}(A),\ n \neq 0.$$

Again, when $n = 0$, we take $HC_0(A) = HC_0^{(0)}(A) = A$. (This decomposition is from [Lsym] and [NS]. It is also the decomposition of $HC_n(A)$ given in [FT, 5.3] and [LP, 2.4] determined by the embedding $HC_n(A)$ in $H_{n+1}(gl(A))$ and using the operations λ^k on $H_{n+1}(gl(A))$.) The i^{th} row of $C_{**}^{(i)}$ yields an isomorphism $HC_n^{(n)} \cong \Omega_A^n/d\Omega_A^{n-1}$ and the edge maps $HC_n^{(i)}(A) \to H_{dr}^{2i-n}(A)$ for $\frac{n}{2} \leq i < n$. If A is smooth, these edge maps are isomorphisms ([Lop, 4.6]) and $HC_n^{(i)}(A) = 0$ for $i < n/2$. One of the main points of this paper is that for general algebras A, $HC_n^{(i)}(A)$ can be nonzero for $i < n/2$.

Variant 1.4.1. Following Goodwillie ([Gw, II.3]), we can extend Connes' double complex to the entire half-plane above the line $p = q$; the *periodic cyclic homology* $HP_*(A)$ is the homology of its total complex. Similarly we define the *negative cyclic homology* $HC_*^-(A)$ to be the homology of the total complex of the double complex obtained by taking that part of the complex with $p \leq 0$ (but still above the line $p = q$). These double complexes break up into the sum of complexes $C_{**}^{(i)}$ (see [Nuss]), so we obtain Hodge decompositions

$$HP_m(A) \cong \Pi_{i=0}^{\infty} HP_m^{(i)}(A);\quad HC_m^{-\ (i)}(A) \cong \Pi_{i=0}^{\infty} HC_m^{-\ (i)}(A).$$

We remark that $HP^{(i)}_m(A) \cong HP^{(i+1)}_{m+2}$, and that the map B factors as

$$HC^{(i-1)}_m(A) \to HC^{-\ (i)}_m(A) \to HH^{(i)}_m(A).$$

Variant 1.4.2. If I is an ideal in A, the relative Hochschild homology modules $HH_*(A, I)$ are obtained from the chain complex $C_*(A, I)$ which is the kernel of $C_*(A) \to C_*(A/I)$. The symmetric group Σ_n acts on $C_n(A, I)$ and we obtain a Hodge decomposition

$$HH_n(A, I) \cong HH^{(1)}_n(A, I) \oplus \cdots \oplus HH^{(n)}_n(A, I),\ n \neq 0.$$

The argument of 1.4 (and 1.4.1) gives an analogous Hodge decomposition of $HC_*(A, I)$, etc. with the same ranges of Hodge indices as in *loc. cit.*

Similarly, if I and J are two ideals in A with zero intersection, the double relative Hochschild homology modules $HH_*(A, I, J)$ are obtained from the chain complex $C_*(A, I, J)$ which is the kernel of both the maps $C_*(A, I) \to C_*(A/I, J/I)$ and $C_*(A, J) \to C_*(A/J, I/J)$. The symmetric group Σ_n acts on $C_n(A, I, J)$, so the argument of 1.1, 1.4 and 1.4.1 gives Hodge decompositions for $HH_n(A, I, J)$, $HC_*(A, I, J)$, etc. with the same ranges of Hodge indices as in *loc. cit.*

1.5. SBI sequence. The usual SBI sequence of cyclic homology also has a Hodge decomposition; it breaks up as the direct sum of the sequences

$$\cdots HC^{(i)}_{n+1}(A) \xrightarrow{S} HC^{(i-1)}_{n-1}(A) \xrightarrow{B} HH^{(i)}_n(A) \xrightarrow{I} HC^{(i)}_n(A) \xrightarrow{S} HC^{(i-1)}_{n-2}(A) \cdots$$

ending in $HH^{(i)}_i(A) \to HC^{(i)}_i(A) \to 0$. (See [Lop, 4.8] and [NS, Thm. 3].) If $A = A_0 \oplus A_1 \oplus \cdots$ is a graded ℓ-algebra, and H is any functor from ℓ-algebras to abelian groups, we write $\widetilde{H}(A)$ for the kernel of the augmentation map $H(A) \to H(A_0)$, so that $H(A) \cong H(A_0) \oplus \widetilde{H}(A)$. It is well-known ([Gw, 3.4.4]) that the S map is zero on $\widetilde{HC}$, so the SBI sequence breaks up into short exact sequences:

$$0 \to \widetilde{HC}^{(i-1)}_{n-1}(A) \xrightarrow{B} \widetilde{HH}^{(i)}_n(A) \xrightarrow{I} \widetilde{HC}^{(i)}_n(A) \to 0.$$

We shall also use an alternate formulation, namely the epi-monic factorization of the map BI:

$$\widetilde{HH}^{(i)}_n(A) \xrightarrow{onto} \widetilde{HC}^{(i)}_n(A) \hookrightarrow \widetilde{HH}^{(i+1)}_{n+1}(A).$$

1.6. Products. If A is a commutative ℓ-algebra, its Hochschild homology is naturally a graded algebra, via the shuffle product $\#$. In fact, $C_*^{(*)}$ is a bigraded chain algebra ([GS2, p.267], [Kas]): $C_m^{(i)} \# C_n^{(j)} \subseteq C_{m+n}^{(i+j)}$. Consequently, the Hodge decomposition respects products:

$$HH_m^{(i)}(A) \# HH_n^{(j)}(A) \subseteq HH_{m+n}^{(i+j)}(A).$$

This result was originally proven in [BV, 3.1(3)] using differential graded algebras.

1.7. Adams and Lambda operations. For each $k \geq 1$ we define operators ψ^k and λ^k on Hochschild and cyclic homology via the Hodge decomposition (cf. [Lop,2.8.g]), letting

$$\psi^k = \text{multiplication by } k^{i+1} \text{ on } HH_*^{(i)} \text{ and } HC_*^{(i)}$$
$$\lambda^k = \text{multiplication by } (-1)^{k-1}k^i \text{ on } HH_*^{(i)} \text{ and } HC_*^{(i)}.$$

We may therefore think of the Hodge decomposition as the spectral decomposition for the operators ψ^k and λ^k acting on Hochschild and cyclic homology. Note that by (1.5) the map $B : HC_n(A) \to HH_{n+1}(A)$ satisfies $\psi^k B(x) = kB(\psi^k x)$ and $\lambda^k B(x) = kB(\lambda^k x)$.

§2. The Dennis trace map

For any ring A, the Dennis trace map is a map $D : K_m(A) \to HH_m(A)$ from the K-theory of A to the Hochschild homology of A. If A is a $\mathbf{Q}$-algebra, D factors through $K_m(A) \otimes \mathbf{Q}$, allowing us to compare the Hodge decomposition on K-theory with the Hodge decompositions on Hochschild and cyclic homology.

Example 2.1. If $a_1, ..., a_n$ are units of A, the Steinberg symbol $\{a_1, ..., a_n\}$ lies in $K_n^{(n)}(A)$ by [Hil, 8.1]. Since $D(a) = \frac{da}{a}$ lies in $\Omega_A^1 = HH_1^{(1)}(A)$ and D preserves products ([I, 5.c.3]), it follows that $D(\{a_1, ..., a_n\}) = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \in \Omega_A^n = HH_n^{(n)}(A)$. By definition (1.7) of the Adams operations in $HH_*(A)$, we see that

$$\psi^k D(\{a_1, ..., a_n\}) = kD(\psi^k \{a_1, ..., a_n\}) = k^{n+1} D(\{a_1, ..., a_n\}).$$

2.2. Generalized Dennis-Stein symbols Suppose that A is a commutative ring and that $a_1, ..., a_n \in A$ are elements such that $1 - a_1 ... a_n$ is a unit. The *generalized Dennis-Stein symbol* $< a_1, ..., a_n >$ in $K_n(A)$ was constructed by Loday in [Lsym] and satisfies the following skew-symmetry property (see [Lsym]):

$(D1)$ For $\sigma \in \Sigma_n, \ < a_{\sigma(1)}, ..., a_{\sigma(n)} >= (-1)^\sigma < a_1, ..., a_n >$.

If each a_i is a unit, then by [Lsym, 1.6] we have

$$< a_1, ..., a_n >= \{1 - (a_1 \cdots a_n), (-1)^n a_2, (-1)^n a_3, ..., (-1)^n a_n\}.$$

Using this formula, and the ring $\mathbf{Z}[x_1, ..., x_n, (1 - x_1 ... x_n)^{-1}, x_1^{-1}, ..., x_n^{-1}]$, a well-known calculation dating back to [D] shows that $< a_1, ..., a_n >$ lies in $K_n^{(n)}(A)$ and that

$$D(< a_1, ..., a_n >) = \frac{-da_1 \wedge \cdots \wedge da_n}{1 - a_1 ... a_n} \in \Omega_A^n = HH_n^{(n)}(A).$$

Once again, $\psi^k D(< a_1, ..., a_n >) = kD(\psi^k < a_1, ..., a_n >)$.

Remark 2.2.1. The Dennis-Stein symbol $< a_1, a_2 >$ changed meaning circa 1980, the new symbol being $- < -a_1, a_2 >$ in the old notation. When $n = 2$, Loday's generalized symbol agrees with the <u>new</u> notation, i.e., what was once called $- < -a_1, a_2 >$. We include this comment in order to clarify [Lsym] where there are several typographical errors in his explicit description of $< a, b >$.

Here is a third case in which we can say something about D. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded $\mathbf{Q}$-algebra. In the $\widetilde{\ }$ notation of (1.5) we know from [Wms] and [Wnil] that $\widetilde{K}_m(A)$ is a $\mathbf{Q}$-module and that the image of D lands in the subgroup $\widetilde{HC}_{m-1}(A)$ of $\widetilde{HH}_m(A)$. Thus D factors as

$$\widetilde{K}_m(A) \xrightarrow{\nu} \widetilde{HC}_{m-1}(A) \xrightarrow{B} HH_m(A).$$

Theorem 2.3. *If $A = A_0 \oplus A_1 \oplus \cdots$ is a graded $\mathbf{Q}$-algebra, then:*

(i) the map ν commutes with the operations ψ^k and λ^k;

(ii) if $x \in \widetilde{K}_m(A)$ then $\psi^k(D(x)) = kD(\psi^k(x))$;

(iii) the Dennis trace map preserves the Hodge decomposition of $\widetilde{K}_*(A)$ in that it is the direct sum of the component maps

$$\widetilde{K}^{(i)}_m(A) \xrightarrow{\nu} \widetilde{HC}^{(i-1)}_{m-1}(A) \xrightarrow{B} \widetilde{HH}^{(i)}_m(A).$$

Proof. By 1.4 and 1.7, it suffices to prove the assertions about ν. Suppose first that $A_n = 0$ for large n, i.e., that the augmentation ideal A_+ is nilpotent. In this case, the result was asserted in [FT, 7.5.5] and proven by Cathelineau in [C]. In fact, ν is an isomorphism by [GwK]: $\widetilde{K}^{(i)}_m(A) \cong HC^{(i-1)}_{m-1}(A, A_+)$.

In the general case, write $A_{(n)}$ for A modulo the ideal $A_n \oplus A_{n+1} \oplus \cdots$, so that the augmentation ideal of each $A_{(n)}$ is nilpotent. A simple weight argument (as in [GW, 2.2, 2.3]) shows that $\widetilde{HC}_*(A)$ injects into the inverse limit of the $\widetilde{HC}_*(A_{(n)})$; in effect, the weight w parts of $Tot(C(A))$ and $Tot(C(A_{(n)}))$ are the same for all $n > w$. From the diagram

$$\begin{tikzcd}
\widetilde{K}^{(i)}_m(A) \arrow[r] \arrow[d, "\nu"] & \lim_{\leftarrow} \widetilde{K}^{(i)}_m(A_{(n)}) \arrow[d, "\nu"] \\
\widetilde{HC}_{m-1}(A) \arrow[r, hook] & \lim_{\leftarrow} \widetilde{HC}_{m-1}(A_{(n)})
\end{tikzcd}$$

we see that ν sends $\widetilde{K}^{(i)}_m(A)$ into $\widetilde{HC}^{(i-1)}_{m-1}(A)$ as desired. ∎

Question 2.4. Does D respect the Hodge decomposition in the sense that $D(K_m^{(i)}(A))$ is contained in $HH_m^{(i)}(A)$? Alternatively, for every commutative ring A and every $x \in K_m(A)$, do we have $\psi^k D(x) = kD(\psi^k(x))$, at least modulo torsion? It seems likely that this is true even if we replace D by its lift $D^- : K_m(A) \to HC_m^-(A)$ and use the Hodge decomposition on $HC_m^-(A)$ given in (1.4.1). This is the case in (2.1), (2.2) and (2.3) because B factors through $HC_*^-(A)$.

2.5. Loday symbols. Let A be an associative algebra and $x_0, ..., x_n \in A$ be elements such that $x_i x_{i+1} = 0$ for $0 \leq i < n$ and $x_n x_0 = 0$. Attached to this data are three elements:

1) The class of the cycle $x_0 \otimes x_1 \otimes \cdots \otimes x_n$ in $HH_n(A)$, called the *Hochschild homology Loday symbol*;

2) The image $I(x_0 \otimes x_1 \otimes \cdots \otimes x_n)$ of the above symbol in $HC_n(A)$, called the *cyclic homology Loday symbol* and written $\ll x_0, ..., x_n \gg$;

3) The K*-theory Loday symbol* $\ll x_0, ..., x_n \gg$ in $K_{n+1}(A)$ constructed in [Lsym, 2.4] as the Steinberg symbol associated to the pairwise commuting elementary matrices $e_{12}(x_0), ..., e_{n+1,1}(x_n)$ in $M_{n+1}(A)$:
$$\ll x_0, ..., x_n \gg = \{e_{12}(x_0), ..., e_{n+1,1}(x_n)\} \in K_{n+1}(A) \cong K_{n+1}(M_{n+1}(A)).$$
We warn the reader that the K-theory identity $\ll x_0, x_1 \gg = - < x_0, x_1 >$ differs by a minus sign from [Lsym]. In all three cases, note that we have the cyclic relation
$$\ll x_1, ..., x_n, x_0 \gg = (-1)^n \ll x_0, x_1, ..., x_n \gg .$$
For K-theory this is proven in [Lsym, 2.5]; for HH (and HC) it follows from applying the operator b to $1 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_0$.

Corollary 2.5.1. *If A is a commutative* $\mathbf{Q}$*-algebra, the Dennis trace map preserves the Hodge decomposition of Loday symbols, and*
$$\psi^k D(\ll x_0, ..., x_n \gg) = kD(\psi^k \ll x_0, ..., x_n \gg).$$

Proof. By 2.3 this is true for the Loday symbol $\ll x_0 t, ..., x_n t \gg$ in $\widetilde{K}_*(A[t])$. The result follows by naturality upon setting $t = 1$. ■

Remark 2.5.2. If A is a $\mathbf{Q}$-algebra, the Loday symbols live in a $\mathbf{Q}$-vector space; if $q \in \mathbf{Q}$, we have $q \ll x_0, ..., x_n \gg = \ll qx_0, ..., x_n \gg$. This is clear for Hochschild and cyclic homology; for K-theory it follows from [Lsym, 2.5]. As a technical point, the K-theory $\mathbf{Q}$-vector space should be the space "nil$K_n(A)$" of [Wnil]; the subgroup of $K_n(A)$ containing the Loday symbols is divisible but may not be torsion free.

If $n = 1$, the K-theory Loday symbol $\ll x_0, x_1 \gg \in K_2(A)$ is the negative of the (new) Dennis-Stein symbol $< x_0, x_1 >$, and in $HH_1(A) \cong \Omega^1_A$ we have $\ll x_0, x_1 \gg = x_0 dx_1$. For $n > 1$, the following result was derived in C. Ogle's thesis [Ogle T, 3.4]; cf. [Ogle I, p.243]. Because [Ogle T] is not published, we include the proof here.

Proposition 2.6. *If A is an associative $\mathbf{Z}$-algebra, the various Loday symbols are related by the following equation in $HH_{n+1}(A)$:*

$$D(\ll x_0, ..., x_n \gg) = B(\ll x_0, ..., x_n \gg) = BI(x_0 \otimes \cdots \otimes x_n).$$

Proof (Ogle). As in (2.1), $D(\ll x_0, ..., x_n \gg)$ is the shuffle product of the $D(e_{ij}(x)) = e_{ij}(-x) \otimes e_{ij}(x)$ in $C_*(M_{m+1}(A))$ because D preserves products. When we take the trace map to $C_*(A)$, all the terms in the shuffle product vanish except those differing by a cyclic permutation π from the trace of $g \otimes e_{12}(x_0) \otimes \cdots \otimes e_{n+1,1}(x_n)$, where g is an upper triangular matrix, yielding

$$\sum_\pi (-1)^\pi \otimes x_{\pi^{-1}(0)} \otimes x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)} = BI(x_0 \otimes \cdots \otimes x_n) = B(\ll x_0, ..., x_n \gg). \quad \blacksquare$$

Corollary 2.7. *If A is any commutative $\mathbf{Q}$-algebra, then the Adams operations on Loday symbols are related by*

$$D(\psi^k \ll x_0, ..., x_n \gg) = B(\psi^k \ll x_0, ... x_n \gg) = \frac{1}{k} \psi^k B(\ll x_0, ..., x_n \gg).$$

Proof. It suffices to prove this for $A = \mathbf{Q}[x_0, ..., x_n]/(x_0 x_1 = ... = x_n x_0 = 0)$. Since $\ll x_0, ..., x_n \gg \in \widetilde{K}_{n+1}(A)$, we may use (2.3) and (1.7) to get

$$\begin{aligned} D(\psi^k \ll x_0, ..., x_n \gg) &= B\nu(\psi^k \ll x_0, ..., x_n \gg) \\ &= B(\psi^k \ll x_0, ..., x_n \gg) = \frac{1}{k} \psi^k B(\ll x_0, ..., x_n \gg). \blacksquare \end{aligned}$$

Theorem 2.8. *Suppose $a_0, ..., a_n$ are elements in an associative ring A such that $a_i a_j = 0$ for $i \neq j$. Then the symmetric group Σ_n acts on the set of K-theory Loday symbols $\ll a_0, ..., a_n \gg$ by*

$$\sigma \ll a_0, ..., a_n \gg = \ll a_0, a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(n)} \gg \in K_{n+1}(A), \text{ for each } \sigma \in \Sigma_n.$$

If A is a commutative $\mathbf{Q}$-algebra and $e_n^{(i)}$ denotes the Eulerian idempotents in $\mathbf{Q}\Sigma_n$, then the projection of $\ll a_0, ..., a_n \gg$ into $K_{n+1}^{(i+1)}(A)$ is given by the $\mathbf{Q}$-linear combination $e_n^{(i)} \ll x_0, ..., x_n \gg$.

Proof. It suffices to consider the universal case, i.e., to suppose that A is the graded ring $\mathbf{Q}[x_0, ..., x_n]/(x_i x_j = 0,\ i \neq j)$. By (2.3) and (2.6), $e_n^{(i)} \ll x_0, ..., x_n \gg$ and the projection of $\ll x_0, ..., x_n \gg$ into $\widetilde{K}_{n+1}^{(i+1)}(A)$ both map to the linear combination of Loday symbols $e_n^{(i)} \ll x_0, ..., x_n \gg$ in $\widetilde{HC}_n(A)$ under the map $\nu : \widetilde{K}_{n+1}(A) \to \widetilde{HC}_n(A)$. By Theorem 3.1 below or by the appendix, ν is an injection. ∎

Application 2.9. The universal ring in (2.8) is the coordinate ring R_n of the coordinate axes in affine $(n+1)$-space. The $\mathbf{Q}$-vector space $\widetilde{K}_*(R_n)$ is a Σ_n-module under the permutations $\sigma(x_i) = x_{\sigma^{-1}(i)}$ fixing x_0. The Σ_n-invariant submodule of $\widetilde{K}_{n+1}(R_n)$ generated by $\ll x_0, ..., x_n \gg$ is isomorphic to $\mathbf{Q}\Sigma_n$ because its image in $\widetilde{HC}_n(R_n)$ is isomorphic to $\mathbf{Q}\Sigma_n$. The Hodge decomposition of this submodule is the decomposition of $\mathbf{Q}\Sigma_n$ according to the Eulerian idempotents. Since $e_n^{(i)} \neq 0$ for $i = 1, ..., n$, the corresponding projections of $\ll x_0, ..., x_n \gg$ into $\widetilde{K}_{n+1}^{(i+1)}(R_n)$ are nonzero. This establishes Example 0.2 of the introduction.

Theorem 2.10. *Let A be a commutative $\mathbf{Q}$-algebra and $a_i \in A$ be elements such that $a_0 a_1 = ... = a_n a_0 = 0$. Then*

(i) The Hochschild homology Loday symbol $\ll a_0, ..., a_n \gg$ projects into $\Omega_A^n \cong HH_n^{(n)}(A)$ as $\frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n$.

(ii) The composition $K_{n+1}(A) \xrightarrow{D} HH_{n+1}(A) \to \Omega_A^{n+1}$ sends $\ll a_0, ..., a_n \gg$ to the form $\frac{1}{n!} da_0 \wedge \cdots \wedge da_n$.

(iii) The projection $pr_{n+1}^{(n+1)} : K_{n+1}(A) \to K_{n+1}^{(n+1)}(A)$ sends the Loday symbol

$\ll a_0, ..., a_n \gg$ *to* $-1/n!$ *times the generalized Dennis-Stein symbol:*

$$pr_{n+1}^{(n+1)} \ll a_0, ..., a_n \gg = -\frac{1}{n!} < a_0, ..., a_n > .$$

Proof. Recall from (1.1) that $e_n^{(n)}$ is the signature idempotent $\frac{1}{n!}\sum(-1)^\sigma \sigma$. By the definition of the shuffle product,

$$a_0 da_1 \wedge \cdots \wedge da_n = \sum_{\sigma \in \Sigma_n} (-1)^\sigma a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} = n! e_n^{(n)} \ll a_0, ..., a_n \gg .$$

This establishes part (i). By (2.5.1), (2.6), (1.7) and part (i),

$$\begin{aligned} D(pr_{n+1}^{(n+1)} \ll a_0, ..., a_n \gg) &= e_{n+1}^{(n+1)} D(\ll a_0, ..., a_n \gg) \\ &= e_{n+1}^{(n+1)} B(\ll a_0, ..., a_n \gg) = \frac{1}{n!} BI(a_0 da_1 \wedge \cdots \wedge da_n) \\ &= \frac{1}{n!} da_0 \wedge da_1 \wedge \cdots \wedge da_n. \end{aligned}$$

By (2.2) this equals $-\frac{1}{n!}D(< a_0, ..., a_n >)$. For part (iii) we consider the universal ring $A = \mathbf{Q}[x_0, ..., x_n]/(x_0x_1, ..., x_nx_0)$. By Theorem 2.3, the above computation shows that the K-theory element

$$z = n! pr_{n+1}^{(n+1)} \ll a_0, ..., a_n \gg + < a_0, ..., a_n >$$

is in the kernel of $\nu : \widetilde{K}_{n+1}^{(n+1)}(A) \to \widetilde{HC}_n^{(n)}(A)$. By Theorem 3.1 below or by the appendix, ν is an injection. Hence $z = 0$, which is the conclusion of (iii). ∎

§3. Discrete Hodge Algebras

Let ℓ be a commutative ring containing $\mathbf{Q}$. A *discrete Hodge algebra* over ℓ is a commutative ring A of the form $\ell[x_1, \ldots, x_m]/\Im$, where $\Im$ is an ideal generated by monomials. (See [DEP] for more details, including applications of discrete Hodge algebras to deformation theory.) Since A is graded, we know from 2.3 that the map $\nu : \widetilde{K}_n(A) \to \widetilde{HC}_{n-1}(A)$ is compatible with the Hodge decompositions on K-theory and cyclic homology (of $\mathbf{Q}$-algebras).

The following result effectively calculates $\widetilde{K}_n(A)$. However, it uses the main result of [OW] which states that if I and J are ideals of a $\mathbf{Q}$-algebra A such that $I \cap J = 0$, then $\nu : \widetilde{K}_n(A, I, J) \cong \widetilde{HC}_{n-1}(A, I, J)$. At present, there is a gap in the proof of this result. Nevertheless, we feel that the usefulness of the following result as an organizing principle justifies its inclusion here.

Theorem 3.1. *Let A be a discrete Hodge algebra over ℓ, where ℓ is a smooth $\mathbf{Q}$-algebra. Assume that the main result of [OW] holds. Then $\nu : \widetilde{K}_n(A) \to \widetilde{HC}_{n-1}(A)$ is an injection, the cyclic homology being taken over $\mathbf{Q}$. Moreover:*

- *If $i < n$, then $\nu : \widetilde{K}_n^{(i)}(A) \cong \widetilde{HC}_{n-1}^{(i-1)}(A)$;*
- *If $i = n$, then the cokernel of $\nu : \widetilde{K}_n^{(n)}(A) \to \widetilde{HC}_{n-1}^{(n-1)}(A)$ is effectively computeable, and contains $(A_{red}/\ell) \otimes_\ell \Omega_\ell^{n-1}$ where A_{red} is the reduced ring of A;*
- *If $i > n$, then $\widetilde{K}_n^{(i)}(A) = 0$.*

Example 3.2. When $A = \ell[x_0, ..., x_m]/(x_i x_j = 0, i \neq j)$, the ring of the coordinate axes, Theorem 3.1 gives the Hodge decomposition of the conclusion of [GRW, 7.1], where the cokernel is exactly $(A/\ell) \otimes_\ell \Omega_\ell^{n-1}$. Before proving theorem 3.1, we isolate an important special case: polynomial rings.

Lemma 3.3. (Polynomial rings as discrete Hodge algebras) *If $A = \ell[x_1, ..., x_r]$ and ℓ is a smooth* $\mathbf{Q}$*-algebra, then*

$$\begin{aligned} K_*(A) &= K_*(\ell), \\ HC_n^{(i)}(A) &= HC_n^{(i)}(\ell),\ i \neq n,\ \text{and} \\ HC_n^{(n)}(A) &= HC_n^{(n)}(\ell) \oplus \coprod_{p=0}^{n} (d\Omega^p_{A/\ell}) \otimes_\ell \Omega^{n-p}_{\ell/\mathbf{Q}}. \end{aligned}$$

Proof. The K-theory formula is classical. Since A is graded, its de Rham cohomology is trivial. By 1.4, $\widetilde{HC}_p^{(i)}(\mathbf{Q}[x_1, ..., x_n])$ is zero if $i \neq p$ and equals

$$\Omega^p_{A/\ell}/d\Omega^{p-1}_{A/\ell} \cong d\Omega^p_{A/\ell} \subseteq \Omega^{p+1}_{A/\ell}$$

if $i = p$. The result now follows from the explicit form of Kassel's base-change formula given in [GRW, 5.6]. Note that the $p = 0$ term for $\widetilde{HC}_n^{(n)}$ is $(A/\ell) \otimes_\ell \Omega^n_{\ell/\mathbf{Q}}$. ■

Reduction 3.4. The following device allows us to assume that A is reduced, i.e., that $\Im$ is generated by square-free monomials. The nilradical $I = nil(A)$ of A is generated by square-free monomials, so $A_r = A/nil(A)$ is a reduced Hodge algebra. By [GwK] and [C], there are isomorphisms $\nu : K_n^{(i)}(A, nil(A)) \cong HC_n^{(i)}(A, nil(A))$. By chasing the following diagram we see that the theorem for A_r implies the theorem for A.

$$\begin{tikzcd} \widetilde{K}^{(i)}_{n+1}(A_r) \arrow[r] \arrow[d] & K^{(i)}_n(A,I) \arrow[r] \arrow[d, "\cong"] & \widetilde{K}^{(i)}_n(A) \arrow[r] \arrow[d, "\nu"] & \widetilde{K}^{(i)}_n(A_r) \arrow[r] \arrow[d] & K^{(i-1)}_{n-1}(A,I) \arrow[d, "\cong"] \\ \bullet \arrow[r] & HC^{(i)}_{n-1}(A,I) \arrow[r] & \widetilde{HC}^{(i-1)}_{n-1}(A) \arrow[r] & \widetilde{HC}^{(i-1)}_{n-1}(A_r) \arrow[r] & \bullet \end{tikzcd}$$

Proof of Theorem 3.1. We now assume that A is a reduced Hodge algebra. Reduced Hodge algebras are sometimes called *Stanley-Reisner rings* after the people who first investigated the combinatorial interpretation of such rings as "face rings" of simplicial complexes (see [St], [DEP, II.4]). Utilizing this interpretation, Vorst showed in [Vo, 3.4] that if A is a reduced Hodge algebra but not a polynomial ring then for one of the indeterminates x generating A the ideal $I = xA$ has a complement J generated by squarefree monomials in the sense that $I \cap J = 0$. Therefore A/xA and $D = A/(I + J)$ are reduced discrete Hodge algebras generated by fewer variables than A, $A/J \cong D[x]$, and there is a cartesian square:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/xA \arrow[d] \\ D[x] \arrow[r, "x=0"] & D. \end{tikzcd}$$

We proceed by induction on r, the number of variables generating A over ℓ. Note that $D[x]$ is a discrete Hodge algebra over $\ell[x]$ in fewer variables than A, so the inductive hypothesis and 3.3 apply to show that $K_n^{(i)}(D[x])/K_n^{(i)}(\ell[x]) \cong K_n^{(i)}(D)/K_n^{(i)}(\ell)$ injects into

$$HC_{n-1}^{(i)}(D[x])/HC_{n-1}^{(i)}(\ell) \cong HC_{n-1}^{(i)}(D)/HC_{n-1}^{(i)}(\ell) \oplus \begin{cases} \ell[x]/\ell \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, when $i = n$ the cokernel is effectively computeable and contains $(D[x]/\ell) \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}}$. Since $I \cong xD[x]$, it also follows from our inductive hypothesis that

$$K_n^{(i)}(D[x], I) \cong \widetilde{K}_n^{(i)}(D[x])/\widetilde{K}_n^{(i)}(D) \hookrightarrow HC_n^{(i)}(D[x], I) \cong \widetilde{HC}_n^{(i)}(D[x])/\widetilde{HC}_n^{(i)}(D)$$

is an isomorphism if $i \neq n$ and when $i = n$, the cokernel contains $I \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}} \cong D[x]/D \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}}$.

The main result of [OW] yields the isomorphisms in the following diagram.

$$\begin{tikzcd}
K_{n+1}^{(i)}(D[x], I) \arrow[r] \arrow[d, "inj"'] & K_n^{(i)}(A, I, J) \arrow[r] \arrow[d, "\cong"'] & K_n^{(i)}(A, I) \arrow[r] \arrow[d, "\nu"] & K_n^{(i)}(D[x], I) \arrow[r] \arrow[d, "inj"] & \bullet \arrow[d, "\cong"'] \\
\bullet \arrow[r] & HC_{n-1}^{(i-1)}(A, I, J) \arrow[r] & HC_{n-1}^{(i-1)}(A, I) \arrow[r] & HC_{n-1}^{(i-1)}(D[x], I) \arrow[r] & \bullet
\end{tikzcd}$$

The five-lemma shows that the map ν in the above diagram is an isomorphism for $i \neq n$, and if $i = n$ is an injection whose cokernel contains $I \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}}$.

Finally, we arrive at the diagram:

$$\begin{tikzcd}
\widetilde{K}_{n+1}^{(i)}(A/I) \arrow[r] \arrow[d] & K_n^{(i)}(A, I) \arrow[r] \arrow[d] & \widetilde{K}_n^{(i)}(A) \arrow[r] \arrow[d, "\nu"] & \widetilde{K}_n^{(i)}(A/I) \arrow[r] \arrow[d] & K_{n-1}^{(i)}(A, I) \arrow[d] \\
\bullet \arrow[r] & HC_{n-1}^{(i-1)}(A, I) \arrow[r] & \widetilde{HC}_{n-1}^{(i-1)}(A) \arrow[r] & \widetilde{HC}_{n-1}^{(i-1)}(A/I) \arrow[r] & \bullet.
\end{tikzcd}$$

Since the inductive hypothesis applies to A/I, we can do a case-by-case analysis. If $i > n$, then $K_n^{(i)}(A, I) = \widetilde{K}_n^{(i)}(A/I) = 0$, so $\widetilde{K}_n^{(i)}(A) = 0$. If $i < n$, the first, second and fourth vertical maps are isomorphisms, while the fifth map is an injection, so the third map (ν) is an isomorphism by the five-lemma. If $i = n$, the first map is an isomorphism, the second and fourth are injections, and the fifth map is the isomorphism $0 = 0$; the third map is therefore an injection by the five-lemma. The sequence of cokernels is therefore exact, so the cokernel of the third map (ν) is effectively computeable and contains $(A/\ell) \otimes_\ell \Omega^{n-1}_{\ell/\mathbf{Q}}$. ∎

§4. Low dimensional calculations

In this section we use the Eulerian idempotents $e_n^{(i)}$ for $n \leq 4$ (listed in 1.2) to study the Hodge decomposition of Loday symbols in low dimensions. The decomposition in K-theory is obtained by computing the Hodge index of the corresponding symbol in cyclic homology, and then appealing to either theorem 3.1 or the appendix. We shall write $\ll x, y \gg$ for both the K-theory Loday symbol and its image in cyclic homology.

4.1. The smallest Loday symbol which makes sense in a commutative ring A is the Dennis-Stein symbol $\ll x, y \gg = - < x, y >$. Since K_2 and HC_1 have trivial Hodge decompositions, we have:

$$\ll x, y \gg \in K_2^{(2)}(A) \text{ and } \ll x, y \gg \in HC_1^{(1)}(A).$$

4.2. Loday symbols in HC_2 and K_3. If $xy = yz = xz = 0$ then by 1.2 the cyclic homology Loday symbol $\ll x, y, z \gg$ has Hodge decomposition

$$\begin{aligned} e_2^{(1)} \ll x, y, z \gg &= \frac{1}{2}(\ll x, y, z \gg + \ll x, z, y \gg) \in HC_2^{(1)}(A) \\ e_2^{(2)} \ll x, y, z \gg &= \frac{1}{2}(\ll x, y, z \gg - \ll x, z, y \gg) \in HC_2^{(2)}(A) \\ &= -\frac{1}{2} < x, y, z > \end{aligned}$$

The corresponding Hodge decomposition in K-theory (the average in $K_3^{(2)}$, the difference in $K_3^{(3)}$) was described in the introduction in (0.1).

4.2.1. If we specialize $\ll x, y, z \gg$ by setting $y = z$, the universal ring is $R = \mathbf{Q}[x, y]/(xy, y^2)$ which is discussed further in §7 below. By inspection, the Loday symbol $\ll x, y, y \gg$ has pure Hodge index:

$$\ll x, y, y \gg \in HC_2^{(1)}(R), \text{ resp. } K_3^{(2)}.$$

4.2.2. Clearly, if we specialize further we still have pure Hodge index, so if $y^2 = 0$ then

$$\ll y, y, y \gg \in HC_2^{(1)}(R), \text{ resp. } K_3^{(2)}.$$

4.3. Loday symbols in HC_3 and K_4. We now consider the Hodge decomposition for the Loday symbol$\ll \mathbf{x} \gg = \ll x, y, z, w \gg$. Thus we assume that $xy = yz = zw = wz = 0$. In this generality the Loday symbol $\ll \bar{\mathbf{x}} \gg = \ll x, w, z, y \gg$ also makes sense. By 1.2 the Hodge decomposition of $\ll \mathbf{x} \gg$ in HC_3 is

$$\begin{aligned} e_3^{(1)} \ll \mathbf{x} \gg &= \frac{1}{3}(\ll \mathbf{x} \gg - \ll \bar{\mathbf{x}} \gg) + \frac{1}{6}\xi \in HC_3^{(1)}(A), \text{ where} \\ &\qquad \xi = x \otimes z \otimes y \otimes w \ + \ x \otimes y \otimes w \otimes z \ - \ x \otimes z \otimes w \otimes y \ - \ x \otimes w \otimes y \otimes z; \\ e_3^{(2)} \ll \mathbf{x} \gg &= \frac{1}{2}(\ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg) \in HC_3^{(2)}(A); \\ e_3^{(3)} \ll \mathbf{x} \gg &= -\frac{1}{6} < x, y, z, w > \in HC_3^{(3)}(A). \end{aligned}$$

Note that ξ has no expression as a sum of Loday symbols in general. Therefore, when we lift to $K_4(A)$ we have no simple expression for the component of Hodge index 2, but we can say what the components of $\ll \mathbf{x} \gg$ of Hodge index 3 and 4 are:

$$\begin{aligned} \frac{1}{2}(\ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg) \ &\in \ K_4^{(3)}(A); \\ -\frac{1}{6} < x, y, z, w > \ &\in \ K_4^{(4)}(A). \end{aligned}$$

4.3.1. If we add the equations $xz = yw = 0$, so that the universal case is the coordinate axes in 4-space, $R_3 = \mathbf{Q}[x_0, x_1, x_2, x_3]/(x_i x_j = 0, i \neq j)$, then we can lift ξ to the following sum of Loday symbols in $K_4(R_3)$:

$$\xi = \ll x, z, y, w \gg + \ll x, y, w, z \gg - \ll x, z, w, y \gg - \ll x, w, y, z \gg .$$

This allows us to not only write the Hodge index 2 component of $\ll \mathbf{x} \gg$ in $K_4^{(2)}(R_3)$, but also to write the Dennis-Stein symbol as a sum of Loday symbols in $K_4^{(4)}(R_3)$:

$$< x, y, z, w > = - \ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg + \xi.$$

4.3.2. If we specialize to $y = w$ so the universal case is $A = \mathbf{Q}[x, y, z]/(xy = yz = 0)$, then $\ll \mathbf{x} \gg = \ll \bar{\mathbf{x}} \gg$ and both ξ and $< x, y, z, y >$ vanish. Once more, the Loday symbol $\ll \mathbf{x} \gg$ has pure Hodge index:

$$\text{If } xy = yz = 0, \text{ then } \ll x, y, z, y \gg \in HC_3^{(2)}(A) \text{ resp. } K_4^{(3)}(A).$$

Clearly any further specialization will lead to a Loday symbol of pure Hodge index. In particular, this is true of the Loday symbols $\ll x, y, x, y \gg$ (coordinates axes in 2-space), $\ll x, y, y, y \gg$ and $\ll x, x, x, x \gg = 0$ (dual numbers).

4.3.3. If instead of setting $y = w$ we specialize to $z = w$ (with $z^2 = 0$), then we get slightly different phenomena. It is easy to see from 4.3.1 that $< x, y, z, z >= 0$. (This also follows from axiom D1 of [Lsym].) Therefore the Hodge decomposition of $\ll \mathbf{x} \gg = \ll x, y, z, z \gg$ has only 2 components:

$$\begin{aligned} \frac{1}{2}(\ll \mathbf{x} \gg - \ll \bar{\mathbf{x}} \gg) \ &\in HC_3^{(1)}(A) \text{ or } K_4^{(2)}(A) \\ \frac{1}{2}(\ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg) \ &\in HC_3^{(2)}(A) \text{ or } K_4^{(3)}(A). \end{aligned}$$

Both these components are nonzero in the general case, because $\ll \bar{\mathbf{x}} \gg = \ll x, z, z, y \gg = - \ll y, x, z, z \gg$ is linearly independent from $\ll \mathbf{x} \gg$.

4.3.4. There are two specializations of (4.3.3). They are both defined in $xy = y^2 = 0$, and both have pure Hodge index:

$$\begin{aligned} \ll x, x, y, y \gg \ &\in HC_3^{(1)}(A) \text{ or } K_4^{(2)}(A); \\ \ll x, y, y, y \gg \ &\in HC_3^{(2)}(A) \text{ or } K_4^{(3)}(A). \end{aligned}$$

Of course it follows from this (or the rotation axiom) that $\ll x, x, x, x \gg = 0$.

4.4. Loday symbols in HC_4 and K_5. Finally, we consider the Hodge decomposition of the Loday symbol $\ll \mathbf{x} \gg = \ll x_0, x_1, x_2, x_3, x_4 \gg$. Writing $\ll \bar{\mathbf{x}} \gg$ for $\ll x_0, x_4, x_3, x_2, x_1 \gg$, about the most we can easily say in this generality is that

$$\begin{aligned} (e_4^{(1)} + e_4^{(3)}) \ll \mathbf{x} \gg &= \frac{1}{2}(\ll \mathbf{x} \gg - \ll \bar{\mathbf{x}} \gg), \\ e_4^{(4)} \ll \mathbf{x} \gg &= -\frac{1}{24} < x_0, x_1, x_2, x_3, x_4 >, \\ e_4^{(2)} \ll \mathbf{x} \gg &= \frac{1}{2}(\ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg) + \frac{1}{24} < x_0, x_1, x_2, x_3, x_4 > . \end{aligned}$$

Thus we can describe the Hodge components of $\ll \mathbf{x} \gg$ in $K_5^{(3)}(A)$ and $K_5^{(5)}(A)$, but we have no general expression for the components of $\ll \mathbf{x} \gg$ in $K_5^{(2)}(A)$ and $K_5^{(4)}(A)$. All we can do is describe the images in $HC_4^{(1)}(A)$ and $HC_4^{(3)}(A)$ using the explicit description of $e_4^{(1)}$ and $e_4^{(3)}$ in $\mathbf{Q}[\Sigma_4]$ provided in (1.2) above.

4.4.1. If we further specialize to the ring $R_4 = \mathbf{Q}[x_0, ..., x_4]/(x_i x_j = 0, i \neq j)$ of the coordinate axes in 5-space, then we can describe the Hodge decomposition of $\ll \mathbf{x} \gg$ in $K_5(R_4)$. For this it is convenient to introduce the notation $\sigma \ll \mathbf{x} \gg$ as

$$\sigma \ll \mathbf{x} \gg = \ll x_0, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)} \gg \in K_5(R_4).$$

In this way the symmetric group Σ_4 acts on the homogeneous Loday symbols, and the component of $\ll \mathbf{x} \gg$ in $HC_4^{(i)}(R_4)$ or $K_5^{(i+1)}(R_4)$ is just $e_4^{(i)} \ll \mathbf{x} \gg$. For example, the Dennis-Stein symbol in $HC_4^{(4)}(R_4)$ or $K_5^{(5)}(R_4)$ is

$$< x_0, ..., x_4 > = -24 e_4^{(4)} \ll \mathbf{x} \gg = \sum_{\sigma \in \Sigma_4} (-1)^\sigma \sigma \ll \mathbf{x} \gg .$$

4.4.2. Because of axiom (D1), the generalized Dennis-Stein symbol $< x_0, ..., x_4 >$ vanishes whenever we identify any two of the x_i. By 1.3 we see that in such a case $e_4^{(2)} \ll \mathbf{x} \gg = \frac{1}{2}(\ll \mathbf{x} \gg + \ll \bar{\mathbf{x}} \gg)$ in $HC_4^{(2)}(R_4)$ or $K_5^{(3)}(R_4)$. This leads to the question of the Hodge decomposition of $\frac{1}{2}(\ll \mathbf{x} \gg - \ll \bar{\mathbf{x}} \gg)$. By inspection, $e_4^{(1)} \ll \mathbf{x} \gg$ and $e_4^{(3)} \ll \mathbf{x} \gg$ remain nonzero if we have one pair, or even three of the variables the same.

If we identify two pairs of variables, the results are more interesting. We leave it to the reader to use 1.2 and check the following assertions:

$$\begin{aligned} \ll t, x, x, y, y \gg - \ll t, y, y, x, x \gg &\text{ is in } HC_4^{(1)} \text{ or } K_5^{(2)}; \\ \ll t, x, y, x, y \gg - \ll t, y, x, y, x \gg &\text{ is in } HC_3^{(3)} \text{ or } K_5^{(4)}; \\ \ll t, x, y, y, x \gg &\text{ is in } HC_4^{(2)} \text{ or } K_5^{(3)}. \end{aligned}$$

People familiar with the card game of Poker will recognize that the further specializations are the two "full house" Loday symbols $\ll x, x, x, y, y \gg$ and $\ll x, x, y, x, y \gg$, the "4 of a kind" $\ll t, x, x, x, x \gg$ and the "5 of a kind" $\ll x, x, x, x, x \gg$. These are all specializations of $\ll t, x, y, y, x \gg$, so they have pure Hodge index, i.e., they all lie in $HC_4^{(2)}$ or $K_5^{(3)}$.

§5 Complete intersections

In this section k will always denote a field of characteristic 0. For nice k-algebras, the Hodge decomposition of Hochschild and cyclic homology partially vanishes. For example, if A is a smooth algebra over k, then

$$HH_n^{(i)}(A) = \begin{cases} 0 & i \neq n \\ \Omega_A^n & i = n; \end{cases}$$
$$HC_n^{(i)}(A) = \begin{cases} 0 & i < n/2 \\ H_{dR}^{(2i-n)}(A) & n/2 \leq i < n \\ \Omega_A^n / d\Omega_A^{n-1} & i = n. \end{cases}$$

(This decomposition follows easily from [LQ, 2.9]; see also [Lop, 4.6].) For example, if $i < n$ then $HC_n^{(i)}(k[x_1, x_1^{-1}, ..., x_m, x_m^{-1}])$ is the degree $2i - n$ part of the exterior algebra $k[\frac{dx_1}{x_1}, ..., \frac{dx_m}{x_m}]$, which is a vector space over k of dimension $\binom{m}{2i-n}$. This shows that all of the $HC_n^{(i)}(A)$ in the range $n/2 \leq i \leq n$ can indeed be nonzero.

Recall that a k-algebra A is a *complete intersection* if $A = k[x_0, ..., x_m]/(f_1, ..., f_r)$ for some regular sequence $f_1, ..., f_r$. The above vanishing result generalizes to complete intersections.

Theorem 5.1. *(Feigin-Tsygan) If A is a complete intersection (or even locally a complete intersection) then $HC_n^{(i)}(A) = 0$ for $i < n/2$.*

Proof. Since $HC_n^{(1)}(A) = HH_n^{(1)}(A)$ is Harrison/André-Quillen homology (see [Lop, 4.6]), the case $i = 1$ was proven by André and Quillen (see [Q, 5.4]). Feigin and Tsygan used this special case to deduce the general result in [FTC, Thm.5] and [FT, 6.5]. Note that the decomposition used by Feigin-Tsygan agrees with ours by [LP, §4]. ■

Corollary 5.2. *If a graded ring $A = k \oplus A_1 \oplus ...$ is locally a complete intersection over a field k of characteristic zero, then every Loday symbol in $K_n(A)$ belongs to the subspace*

$$\bigoplus_{n/2 < i \leq n} K_n^{(i)}(A).$$

That is, it projects to zero in $K_n^{(i)}(A)$ if either $i \leq n/2$ or $i > n$.

Proof. A is a direct limit of rings which are locally complete intersections over $\mathbf{Q}$, so we may assume $k = \mathbf{Q}$. The result then follows from 5.1 and 2.8. ∎

In the rest of this section, we shall strengthen a result proven by Burghelia and Vigué ([BV, 4.3]) for $k = \mathbf{C}$. Let $f = f(x_0, ..., x_m)$ be a homogeneous polynomial over k (a field of characteristic 0), and consider the ring $A = k[x_0, ..., x_m]/(f)$. The Jacobian Criterion implies that the following conditions are equivalent:

(i) $\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_m}$ form a regular sequence in $k[x_0, ..., x_m]$.

(ii) The equations $\frac{\partial f}{\partial x_0} = 0, ..., \frac{\partial f}{\partial x_m} = 0$ have no nonzero solutions in $\bar{k}^{m+1}$.

(iii) If I is the ideal of A which is generated by $\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_m}$, then $\mu = \dim_k(A/I)$ is finite.

(iv) The affine scheme $Spec(A)$ has an isolated singularity at zero.

(v) The equation $f = 0$ defines a smooth hypersurface in $\mathbf{P}^m_k$, namely $Proj(A)$.

We are going to show that, if any of these equivalent conditions hold, we can calculate $HH_*(A)$ and $HC_*(A)$ by following the method of [GRW, 1.10]. Write dx_i for $1 \otimes x \in k \otimes \bar{A}$, and let $t \in k \otimes \bar{A} \otimes \bar{A}$ be any element of Hodge index 1 such that, considering t as an element of $C_2(A)$:

$$b(t) = \nabla f = \sum \frac{\partial f_i}{\partial x_i} dx_i \text{ in } C_1(A).$$

(If $b(t) = \nabla f$ then $b(e_2^{(1)} t) = \nabla f$, so the assumption that t has Hodge index 1 is harmless.) The following lemma is proven exactly as in [GRW, 2.5], and we omit the proof.

Lemma 5.3. *$HH_*(A; k)$ is the graded algebra $k[dx_0, ..., dx_m, t]$. If f has degree d and we assign a "weight" of 2 to each x_i, then t has "weight" $2d$ and topological degree $2d - 2$.*

Proposition 5.4. *Suppose that f is a homogeneous polynomial defining a smooth hypersurface in $\mathbf{P}^m_k$, with homogeneous coordinate ring $A = k[x_0, ..., x_m]/(f)$. Let I denote the ideal of A generated by $\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_m}$ so that $\mu = \dim_k(A/I)$ is finite. Then each $HH_*(A)$ has pure Hodge index:*

$$HH_p(A) = \begin{cases} HH_p^{(p)}(A) \cong \Omega^p_A & 0 \le p \le m+1 \\ HH_p^{(m+j+1)}(A) \cong A/I & p = m+1+2j,\ j \ge 0 \\ HH_p^{(m+j)}(A) \cong A/I & p = m+2j,\ j \ge 1. \end{cases}$$

Moreover, $HH_{m+2j+1}(A)$ *is generated as an* A*-module by* $t^j dx_0 \wedge ... \wedge dx_m$*, and* $HH_{m+2j}(A)$ *is generated as an* A*-module by the cycle* $t^j \sum (-1)^i x_i dx_0 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_m$.

Using the well-known fact that $H^{(*)}_{dR}(A) = 0$ for $* \neq 0$, we immediately obtain

Corollary 5.5. *Let* $\widetilde{HC}_*(A)$ *denote* $HC_*(A)/HC_*(k)$*. Then*

$$\widetilde{HC}_p(A) = \begin{cases} \widetilde{HC}_p^{(p)}(A) \cong d\Omega^p_{A/k} \subseteq \Omega^{p+1}_{A/k} & \text{for } 0 \le p \le m \\ 0 & \text{for } p = m+2j+1,\ j \ge 0 \\ HC_p^{(m+j)}(A) \cong A/I & \text{for } p = m+2j,\ j \ge 1 \end{cases}$$

and the following maps are isomorphisms for $j \ge 1$*:*

$$HH^{(m+j)}_{m+2j}(A) \xrightarrow{I} \widetilde{HC}^{(m+j)}_{m+2j}(A) \xrightarrow{B} HH^{(m+j+1)}_{m+2j+1}(A).$$

Proof of 5.4. We use the spectral sequence $E_1^{pq} = A_p \otimes HH^q(A;k)$ of [GRW,1.3]. Writing Λ^*_A for the graded exterior A-algebra $A[dx_0, ..., dx_m]$, we see that the spectral sequence has $t \in E_1^{0,2d-2}$ and

$$E_1^{*q} = \bigoplus \{ t^j \Lambda^i_A :\ 2j(d-1) + i = q \}.$$

Note that $t^j \Lambda^i_A$ has pure Hodge index $j+i$.

By multiplicativity ([GRW, 1.9]), the first nontrivial differential occurs in E^{**}_{2d-2}, sending $t^j w$ to $j t^{j-1} (\nabla f \wedge w)$ for $w \in \Lambda^*_A$. Note that $\Lambda^i_A / (\nabla f \wedge \Lambda^{i-1}_A) \cong \Omega^i_{A/k}$, so for $j = 0$ the Kähler differentials $\Omega^i_{A/k}$ in $HH_i(A)$ appear in the row $q = i$ of E^{**}_{2d-1}. For $j \neq 0$, the E^{*q}_{2d-1} term coming from $t^j \Lambda^i_A$ is given by the homology of the chain complex

$$P_* :\ 0 \to A \xrightarrow{\nabla f} \Lambda^1_A \xrightarrow{\nabla f} \cdots \xrightarrow{\nabla f} \Lambda^{m+1}_A \to 0.$$

If we set $R = k[x_0, ..., x_m]$, then the complex P_* is the Koszul complex $K^R_\bullet(\frac{\partial f}{\partial x_\bullet}, A)$ for the sequence $\frac{\partial f}{\partial x_\bullet} :\ \frac{\partial f}{\partial x_0}, -\frac{\partial f}{\partial x_1}, ..., (-1)^m \frac{\partial f}{\partial x_m}$ ([SeM, IV-4]). Therefore the terms E^{**}_{2d-1} are given by the Koszul homology $H^R_*(\frac{\partial f}{\partial x_\bullet}, A)$. Since the $\frac{\partial f}{\partial x_i}$ form a regular sequence in R and generate I, we have ([SeM, IV-6])

$$H_*(P_*) = H^R_*(\frac{\partial f}{\partial x_\bullet}, A) \cong Tor^R_*(R/I, A).$$

Because $A = R/(f)$, the homology is zero for $* \neq 0, 1$. Moreover, $H_0(P_*) = \Omega^{m+1}_A \cong A/I$ and $H_1(P_*) \cong Tor^R_1(R/I, A) \cong A/I$ on generator $z = \sum(-1)^i x_i dx_0 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_m$. (The Tor_1 calculation is an elementary exercise left to the reader.) Therefore the only nonzero terms of Hodge index $j + m$ in E^{**}_{2d-1} are the $(A/I)t^j z$ in row $q = 2j(d-1) + m$ and the $(A/I)t^{j-1} dx_0 ... dx_m$ $2d - 3$ rows below it. Consequently, $E^{**}_{2d-1} = E^{**}_\infty$, and we may read off the Hochschild homology of A directly from the rows. ■

Application 5.6. (Truncated Polynomials) If k is a field of characteristic 0 and $A = k[x]/(x^{n+1})$, then $A/I = k[x]/(x^n)$. The calculations of HH_* and HC_* of A as a k-algebra given in 5.4 are well-known (see [GRW, 1.10], [C,1.2]): as an A-module, $HH^{(i)}_{2i}(A) = HH^{(i)}_{2i+1}(A) = A/I$. By base-change (see [GRW, §5]) and Goodwillie's Theorem ([GwK]), we obtain the following more precise version of [GRW, 9.6], [C, 1.2]: considering A as a $\mathbf{Q}$-algebra

$$\begin{aligned} HC^{(i)}_p(k[x]/(x^{n+1})) &= HC^{(i)}_p(k) \oplus (k[x]/(x^n)) \otimes_k \Omega^{2i-p}_{k/\mathbf{Q}} \qquad p/2 \le i \le p \\ K^{(i)}_p(k[x]/(x^{n+1})) &= K^{(i)}_p(k) \oplus (k[x]/(x^n)) \otimes_k \Omega^{2i-p-1}_{k/\mathbf{Q}} \qquad p/2 < i \le p. \end{aligned}$$

Summing over i yields

$$K_p(k[x]/(x^{n+1})) = K_p(k) \oplus (k[x]/(x^n)) \otimes_k \{\Omega^p_{k/\mathbf{Q}} \oplus \Omega^{p-2}_{k/\mathbf{Q}} \oplus \Omega^{p-4}_{k/\mathbf{Q}} \oplus ...\}.$$

Taking $k = \mathbf{Q}$, we see that there are no nonzero Loday symbols of the form $\ll x^{t_1}, ..., x^{t_p} \gg$ in $K_p(k[x]/(x^{n+1}))$ unless p is odd, in which case such Loday symbols (defined whenever $t_j + t_{j+1} > n$) have pure Hodge $(p+1)/2$, i.e., they lie in $K^{(i)}_{2i-1}(A)$.

Application 5.7. (axes in the plane) If $A = k[x, y]/(xy)$ then $A/I \cong k$, and the following is a more precise version of both [GRW, 5.7 and 7.1] and [V]. For the cyclic homology calculation, we consider A as a $\mathbf{Q}$-algebra.

$$HC^{(i)}_p(k[x,y]/(xy)) = HC^{(i)}_p(k) \oplus \begin{cases} \Omega^{2i-p-1}_{k/\mathbf{Q}} & p/2 < i < p \\ \Omega^{p-1}_{k/\mathbf{Q}} \oplus (A/k) \otimes_k \Omega^p_{k/\mathbf{Q}} & \text{for } i = p \\ 0 & \text{otherwise} \end{cases}$$

$$K^{(i)}_p(k[x,y]/(xy)) = K^{(i)}_p(k) \oplus \begin{cases} \Omega^{2i-p-2}_{k/\mathbf{Q}} & p/2 < i \le p \\ 0 & \text{otherwise} \end{cases}$$

The Loday symbols $\ll x, y, x, y, \cdots, x, y \gg$ lie in $HC^{(i)}_{2i-1}(A)$ and $K^{(i+1)}_{2i}(A)$, and they all have pure Hodge index.

§6 Hanlon's generating functions

To determine the Hodge indices of Loday symbols in graded algebras which are not complete intersections, we use the generating functions $\Pi H(A)$ and $\Pi C(A)$ developed by P. Hanlon in [H, §7]. An r-fold graded k-algebra A is an algebra graded by r-tuples $w = (\alpha_1, ..., \alpha_r)$ of nonnegative integers, i.e., $A = \bigoplus A_w$. We let ${}_wC_n(A)$ denote the subgroup of $C_n(A)$ generated by all homogeneous terms $a_0 \otimes a_1 \otimes \cdots \otimes a_n$, with $a_i \in A_{w_i}$ and $w = w_0 + ... + w_n$. The r-tuple w is called the *weight* of an element in ${}_wC_n(A)$; cf. [GRW, 1.1], [C], [H, 7.1]. As weight is preserved by the operators b and B, $C_*(A)$ is the direct sum of subcomplexes ${}_wC_*(A)$. As weight in $C_n(A)$ is preserved by the action of the symmetric group, ${}_wC_*(A)$ also has a Hodge decomposition when A is commutative. Setting ${}_wHH_*^{(i)}(A) = H_*({}_wC_*(A))$, etc., we have a double decomposition

$$HH_*(A) = \bigoplus_{i,w} {}_wHH_*^{(i)}(A); \quad HC_*(A) = \bigoplus_{i,w} {}_wHC_*^{(i)}(A), ...$$

Definition 6.1. (Hanlon [H, 7.3]) Suppose that k is a field and each A_w is finite-dimensional. Then for each w and i, only finitely many of the groups ${}_wHH_n^{(i)}(A)$ and ${}_wHC_n^{(i)}(A)$ are nonzero, and these are finite-dimensional over k. Thus we can define the Euler characteristics

$$\begin{aligned} {}_w\chi_H^{(i)}(A) &= \sum (-1)^n \dim\ {}_wHH_n^{(i)}(A) \\ {}_w\chi_C^{(i)}(A) &= \sum (-1)^n \dim\ {}_wHC_n^{(i)}(A). \end{aligned}$$

For $w = (\alpha_1, ..., \alpha_r)$, the symbol $\mathbf{z}^w$ represents the monomial $z_1^{\alpha_1} \cdots z_r^{\alpha_r}$. We can also define the generating functions

$$\begin{aligned} \Pi H(A) &= \sum_{i,w} {}_w\chi_H^{(i)}(A)\lambda^i \mathbf{z}^w \\ \Pi C(A) &= \sum_{i,w} {}_w\chi_C^{(i)}(A)\lambda^i \mathbf{z}^w \end{aligned}$$

which are power series in $\mathbf{Z}[\lambda, z_1, ..., z_r]$. In particular,

$$\Pi H(k) = 1 \text{ and } \Pi C(k) = \frac{1}{1-\lambda}.$$

Remark 6.1.1. Our notation differs slightly from Hanlon's in that our A has a unit. If $A_0 = k$ and A_+ is the augmentation ideal, in his notation we have $\Pi H(A) = \Pi(A_+, A)$ and $\Pi C(A) = \Pi C(k) + \Pi C(A_+)$.

Remark 6.1.2. Suppose ℓ is an extension field of k, and each A_w and ${}_wHH_n^{(i)}(A)$ is finite-dimensional over ℓ. Then we can still define Euler characteristics and $\Pi H(A)$ using dimension over ℓ. For example, if ℓ has transcendence degree d, then $\Pi H(\ell) = (1-\lambda)^d$ and $\Pi H(A \otimes_k \ell) = \Pi H(A)(1-\lambda)^d$ for every k-algebra A such that each $\dim_k(A_w)$ is finite. The definitions of ${}_w\chi_C^{(i)}(A)$ still make sense for $w \neq 0$ if we use the ℓ-module structure on ${}_wHC_*(A)$ given in [DW], but since $HC_*(\ell)$ is not an ℓ module, we need another definition when $w = 0$ to make sense out of $\Pi C(A)$, such as that suggested by the following lemma.

Lemma 6.2. *Suppose that k is a field, and each A_w is finite-dimensional over k. Then:*

$$\Pi C(A) = \frac{1}{1-\lambda}\Pi H(A).$$

Proof. For each w and i the SBI sequence

$$\cdots\ {}_wHC_{n+1}^{(i)}(A) \xrightarrow{S} {}_wHC_{n-1}^{(i-1)}(A) \xrightarrow{B} {}_wHH_n^{(i)}(A) \xrightarrow{I} {}_wHC_n^{(i)}(A) \cdots$$

is finite. Therefore

$${}_w\chi_H^{(i)}(A) - {}_w\chi_C^{(i)}(A) = \sum (-1)^n \dim\ {}_wHC_{n-1}^{(i-1)}(A) = -\ {}_w\chi_C^{(i-1)}(A).$$

Multiplying by $\lambda^i \mathbf{z}^w$ and adding yields $\Pi H(A) - \Pi C(A) = -\lambda \Pi C(A)$, whence the result. ∎

We now come to the main result of this section. Recall that the Poincaré series for A is the power series $P_A(\mathbf{z}) = \sum (\dim A_w)\mathbf{z}^w$. Let $P_A(\mathbf{z}^\ell)$ denote $P_A(z_1^\ell, ..., z_r^\ell)$, and let $\mu(d)$ denote the Möbius function of d.

Theorem 6.3. (Hanlon) *Let A be a commutative r-fold graded algebra over a field k such that $A_0 = k$ and each A_w is finite dimensional. Then*

$$\Pi H(A) = P_A(\mathbf{z}) \prod_{\ell=1}^{\infty} P_A(\mathbf{z}^\ell)^{-\frac{1}{\ell}\sum_{d|\ell}\mu(d)\lambda^{\frac{\ell}{d}}}$$

and $\Pi C(A)$ is given by $\Pi H(A)/(1-\lambda)$.

Proof. When $r = 1$ this is [H, 7.4 and 7.8] with a slight notational change (see 6.1.1). Phil Hanlon has pointed out to us (in [HL]) that the proof goes through verbatim when $r > 1$. ∎

Remark 6.3.1. The exponent for $\ell = 1$ is λ, so the first two terms may be written as $P_A(\mathbf{z})^{1-\lambda}$. The exponent for $\ell = 2$ is $-\frac{1}{2}(\lambda^2 - \lambda)$.

In order to perform calculations with 6.3, the following identity is very useful. Modulo a typographical error, it is identity (6.2a) of [H].

Identity 6.4.

$$\prod_{\ell=1}^{\infty}(1 - x^{\ell})^{-\frac{1}{\ell}\sum_{d|\ell}\mu(d)\lambda^{\frac{\ell}{d}}} = \frac{1}{1-\lambda x}$$

Lemma 6.5.

$$\prod_{\ell=1}^{\infty}(1 + x^{\ell})^{-\frac{1}{\ell}\sum_{d|\ell}\mu(d)\lambda^{\frac{\ell}{d}}} = \frac{1-\lambda x}{1-\lambda x^2} = 1 + \sum_{i=1}^{\infty}\lambda^i(x^{2i} - x^{2i-1})$$

Proof. Since $1 + x^{\ell} = (1 - x^{2\ell})/(1 - x^{\ell})$, the first equality follows from 6.4. The second equality is straight forward. ■

Application 6.6. Consider the truncated polynomial ring $A = k[x]/(x^2)$. Since $P_A(z) = 1 + z$, it follows from 6.5 that

$$\begin{aligned}
\Pi H(A) &= (1+z)(\frac{1-\lambda z}{1-\lambda z^2}) \\
&= (1+z)(1 + \sum_{i=1}^{\infty}\lambda^i(z^{2i} - z^{2i-1})) \\
&= 1 + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i z^{2i+1}; \\
\Pi C(A) &= \Pi C(k) + \sum_{i=0}^{\infty}\lambda^i z^{2i+1}.
\end{aligned}$$

There are only two terms of weight $2i + 1$ in $C_*(A)$, namely $\ll x, ..., x \gg \in C_{2i}(A)$ and $BI \ll x, ..., x \gg = 1 \otimes x \otimes \cdots \otimes x \in C_{2i+1}(A)$. As the coefficient of z^{2i+1} in $\Pi H(A)$ is $\lambda^i - \lambda^{i+1}$, it follows that

$$HH_{2i}^{(i)}(A) \cong HC_{2i}^{(i)}(A) \cong HH_{2i+1}^{(i+1)}(A) \cong k$$

and that $\ll x, ..., x \gg \in HC_{2i}^{(i)}(A)$. This provides an alternate proof of 5.6.

Application 6.7. Consider the "axes in the plane", $A = k[x, y]/(xy)$. Since

$$P_A(x, y) = 1 + (x + y) + \cdots + (x^n + y^n) + \cdots = \frac{1 - xy}{(1 - x)(1 - y)},$$

it follows from 6.4 that

$$\begin{aligned}
\Pi H(A) &= P_A(x, y)\frac{(1 - \lambda x)(1 - \lambda y)}{1 - \lambda xy} \\
&= P_A(x, y)\{(1 - \lambda) + \frac{\lambda(1 - x)(1 - y)}{1 - \lambda xy}\} \\
&= (1 - \lambda)P_A(x, y) + \frac{\lambda(1 - xy)}{1 - \lambda xy} \\
&= 1 + (1 - \lambda)(P_A(x, y) - 1) - (1 - \lambda)\frac{\lambda xy}{1 - \lambda xy}; \\
\Pi C(A) &= \Pi C(k) + (P_A(x, y) - 1) - \sum_{i=1}^{\infty} \lambda^i x^i y^i.
\end{aligned}$$

We know from 5.7 that $\lambda^i x^i y^i$ corresponds to the Loday symbol $\ll x, y, ..., x, y \gg$ in $HC^{(i)}_{2i-1}(A)$. In fact, knowing that $HC_n(A)$ is 0 for n even and generated by the Loday symbols for n odd, we can read off the Hodge indices of the Loday symbols from $\Pi C(A)$.

§7 Loday symbols in x and y.

In order to analyze the Hodge decomposition of Loday symbols in which the same variable appears many times, we need only compute some of the coefficients in the generating function $\Pi C(A)$. In the most trivial case, there is only one variable and we have already seen in 5.6 and 6.6 that $\ll x, ..., x \gg \in HC_{2i}^{(i)}(A)$. In the next case, there are many x's and only one y: $\ll y, x, ..., x \gg \in HC_n(A)$. In this case we can determine the Hodge decomposition using representation theory.

Theorem 7.1. *Suppose that $x^2 = xy = 0$. Then the Loday symbol $\ll y, x, ...x \gg$ has pure Hodge index. That is*

(i) In cyclic homology it lies in either $HC_{2i}^{(i)}(A)$ or $HC_{2i-1}^{(i)}(A)$;

(ii) In K-theory it lies in either $K_{2i}^{(i+1)}(A)$ or $K_{2i+1}^{(i+1)}(A)$.

Proof. The 1-dimensional subspace of $C_n(A)$ generated by the cycle $y \otimes x \otimes \cdots \otimes x$ is isomorphic to the trivial representation of Σ_n. By [H, 6.3] the multiplicity of this representation in $e_n^{(i)}\mathbf{Q}\Sigma_n$ is 1 if $i = [\frac{n+1}{2}]$ and 0 otherwise. Hence if $i \neq [\frac{n+1}{2}]$ we have $e_n^{(i)} \ll y, x, ..., x \gg = 0$ in $HH_n(A)$. This proves that the Hochschild Loday symbol $\ll y, x, ..., x \gg$ lies in $HH_n^{([\frac{n+1}{2}])}(A)$. Part (i) is immediate. Because the universal ring $A = \mathbf{Q}[x, y]/(xy)$ is a discrete Hodge algebra, part (ii) follows from Theorem 3.1 or from the appendix. ∎

Remark 7.1.1. The permutation representation of Σ_n is isomorphic to the subspace V of $C_n(A)$ with basis the other cycles $x \otimes \cdots \otimes x \otimes y \otimes x \otimes \cdots$ representing $\ll y, x, ..., x \gg$. By [H, 6.4], $e_n^{(i)}V \neq 0$ for $1 \leq i \leq \frac{n+1}{2}$. By 7.1, the nonzero cycles in $e_n^{(i)}V$ represent zero in $HH_n^{(i)}(A)$ for $i \neq [\frac{n+1}{2}]$. This shows that care must be taken using this approach.

Theorem 7.2. *Suppose that $x^2 = xy = 0$. Then in $HC_n(A)$:*

(1) The Loday symbols with two y's and $n-1$ x's lie in $HC_n^{(1)}(A) \oplus \cdots \oplus HC_n^{(\ell-2)}(A) \oplus HC_n^{(\ell)}(A)$, $\ell = [\frac{n+1}{2}]$, and form a basis for the weight $(n-1, 2)$ part of $HC_*(A)$.

(2) For $n \leq 5$, every Loday symbol has pure Hodge index: $\ll x, y, x, y \gg \in HC_3^{(2)}(A)$; $\ll y, x, y, x, x \gg \in HC_4^{(2)}(A)$ and $\ll y, x, y, x, x, x \gg \in HC_5^{(3)}(A)$.

(3) For $n \geq 6$, set $n = 4p + j$, $w = (n-1, 2)$ and $\ell = [\frac{n+1}{2}]$. Then $\dim\ {}_wHC_n^{(\ell)}(A) = 1$,

$$\dim\ {}_wHC_n^{(1)}(A) = \begin{cases} 1 & j = 2, 3 \\ 0 & j = 0, 1 \end{cases}$$

and for $2 \leq i \leq \ell - 2$

$$\dim\ {}_wHC_n^{(i)}(A) = \begin{cases} 1 & j = 1, 3 \\ 2 & j = 2 \text{ and } i \text{ odd} \\ 2 & j = 0 \text{ and } i \text{ even} \\ 0 & \text{otherwise .} \end{cases}$$

Corollary 7.3. *Consider the Loday symbol $\ll y, y, x, ..., x \gg$ in $HC_n(B)$, where $B = k[x, y]/(x^2, xy, y^2)$.*

(i) For $n \leq 5$, this symbol has pure Hodge index $[\frac{n}{2}]$. That is, it lies in $HC_2^{(1)}$, $HC_3^{(1)}$, $HC_4^{(2)}$, or $HC_5^{(2)}$.

(ii) If n is odd ($n \geq 3$), $\ll y, y, x, ..., x \gg$ has pure Hodge index $i = \frac{n-1}{2}$, i.e., it lies in $HC_{2i+1}^{(i)}(B)$.

(iii) If n is even ($n \geq 6$) the element $\ll y, y, x, x, x, ..., x \gg - \ll y, x, y, x, x, ..., x \gg$ has pure Hodge index $[\frac{n}{2}]$, i.e., it lies in $HC_{2i}^{(i)}(B)$.

Proof of 7.2. For simplicity, we shall write $x^i y^j$ for $\mathbf{z}^{(i,j)}$. For $A = k[x, y]/(x^2, xy)$ we have $P_A(x, y) = \frac{1}{1-y} + x$. Modulo y^3, we have $P_A(x, y) \equiv (1 + x + y + y^2)$ and

$$\Pi H(A) \equiv P_A(x, y) P_A(x, y)^{-\lambda} (1 + x^2 + y^2)^{-\frac{1}{2}(\lambda^2 - \lambda)} \prod_{\ell=3}^{\infty} (1 + x^\ell)^{-\frac{1}{\ell} \sum_{d|\ell} \mu(d) \lambda^{\frac{\ell}{d}}}.$$

To simplify this, we note that modulo y^3:

$$\begin{aligned} P_A(x, y)^{1-\lambda} &\equiv (1 + x)^{1-\lambda} + (1 - \lambda)(1 + x)^{-\lambda}(y + y^2) - \frac{\lambda(1-\lambda)}{2}(1 + x)^{-1-\lambda} y^2 \\ &= (1 + x)^{-\lambda}\{1 + x + (1 - \lambda)y + (1 - \lambda)[1 - \frac{\lambda}{2}(1 + x)^{-1}]y^2\}; \end{aligned}$$

$$(1+x^2+y^2)^{-\frac{1}{2}(\lambda^2-\lambda)} \equiv (1+x^2)^{-\frac{1}{2}(\lambda^2-\lambda)}\{1+(1+x^2)^{-1}\frac{\lambda(1-\lambda)}{2}y^2\}.$$

Again we compute modulo y^3 that

$$\begin{aligned}\{(1+x)+(1-\lambda)y+(1-\lambda)[1-\frac{\lambda}{2}(1+x)^{-1}]y^2\}\{1+(1+x^2)^{-1}\frac{\lambda(1-\lambda)}{2}y^2\}\\ \equiv (1+x)+(1-\lambda)y+(1-\lambda)y^2(1+\lambda T),\\ \text{where } T=\frac{1}{2}\{\frac{1+x}{1+x^2}-\frac{1}{1+x}\}=(1-x)\sum_{i=0}^{\infty}x^{4i+1}.\end{aligned}$$

Combining all this and using Identities 6.4 and 6.5, we see that modulo y^3

$$\begin{aligned}\Pi H(A) &\equiv (\frac{1-\lambda x}{1-\lambda x^2})\{(1+x)+(1-\lambda)y+(1-\lambda)y^2(1+\lambda T)\};\\ \Pi C(A) &\equiv \Pi C(k[x]/x^2)+(\frac{1-\lambda x}{1-\lambda x^2})y+(\frac{1-\lambda x}{1-\lambda x^2})y^2(1+\lambda T).\end{aligned}$$

The term in $\Pi C(A)$ involving just one y is given by 6.5:

$$(\frac{1-\lambda x}{1-\lambda x^2})y=\sum_{i=0}^{\infty}\lambda^i x^{2i}y-\sum_{i=1}^{\infty}\lambda^i x^{2i-1}y.$$

This provides an alternative proof of Theorem 7.1.

The term in $\Pi C(A)$ involving exactly two y's is y^2 times

$$\begin{aligned}(\frac{1-\lambda x}{1-\lambda x^2})(1+\lambda T) &= \{1-\sum_{i=1}^{\infty}\lambda^i x^{2i-1}(1-x)\}\{1+\lambda(1-x)\sum_{j=0}^{\infty}x^{4j+1}\}\\ &= 1+\lambda(1-x)\sum_{j=1}^{\infty}x^{4j+1}-\sum_{i=2}^{\infty}\lambda^i x^{2i-2}[1-x+\sum_{j=1}^{\infty}x^{4j}(1-2x+x^2)]\\ &= 1-\lambda^2x^2+\lambda^2x^3-\lambda^3x^4+(\lambda+\lambda^3)x^5-(\lambda+\lambda^2+\lambda^4)x^6+\cdots\end{aligned}$$

For $p\geq 1$, the coefficient of $(-1)^{j+1}x^{4p+j}y^2$ in $\Pi C(A)$ is:

$$\begin{aligned}(\lambda^2+\lambda^3+\cdots+\lambda^{2p-1})+\lambda^{2p+1} &\qquad j=0\\ \lambda+2(\lambda^3+\lambda^5+\cdots+\lambda^{2p-1})+\lambda^{2p+1} &\qquad j=1\\ (\lambda+\lambda^2+\lambda^3+\cdots+\lambda^{2p})+\lambda^{2p+2} &\qquad j=2\\ 2(\lambda^2+\lambda^4+\cdots+\lambda^{2p})+\lambda^{2p+2} &\qquad j=3.\end{aligned}$$

From our low-dimensional calculations, we see that the first few terms correspond to the fact that $\ll x,y,x,y\gg\in HC_3^{(2)}(A)$, and $\ll y,x,y,x,x\gg\in HC_4^{(2)}(A)$. Since the cyclic relation implies that $\ll y,x,x,y,x,x\gg=0$, we have $\ll y,x,y,x,x,x\gg\in HC_5^{(3)}(A)$. The rest of the theorem follows from the following Lemma.

Lemma 7.4. *For $A = k[x,y]/(x^2, xy)$, the only terms in $HC_*(A)$ of weight $w = (n-1, 2)$ lie in $HC_n(A)$. A basis for ${}_wHC_n(A)$, $n \geq 2$, is formed by the Loday symbols $\ll y, x, \ldots, x, y, x, \ldots, x \gg$ with the second y occurring in the i^{th} slot for $2 \leq i \leq n/2$ and for $i = (n+1)/2$* when $n = 4j - 1$.

Proof of Lemma 7.4. Set $N = \binom{n+1}{2} - (n+1)$. In the Hochschild complex, there are N terms in $C_n(A)$ and $C_{n+1}(A)$ of the form $\cdots \otimes y \otimes x \otimes \cdots \otimes x \otimes y \otimes \cdots$ (resp. $1 \otimes \cdots \otimes y \otimes x \otimes \cdots \otimes x \otimes y \otimes \cdots$) with the two y's not cyclically adjacent; call these of "Loday type". There are also $n+1$ terms in $C_n(A)$ and $C_{n+1}(A)$ of the form $\cdots \otimes y \otimes y \otimes \cdots$ (resp. $1 \otimes \cdots \otimes y \otimes y \otimes \cdots$ or $1 \otimes y \otimes x \cdots \otimes x \otimes y$); call these "bad". Finally, there are n terms in $C_{n-1}(A)$ and $C_n(A)$ of the form $\cdots \otimes y^2 \otimes \cdots$ (resp. $1 \otimes \cdots \otimes y^2 \otimes \cdots$); call these "induced". The above symbols form a basis for ${}_wC_*(A)$. By inspection, the subcomplex of symbols of Loday type is a summand of ${}_wC_*(A)$, and its n^{th} homology has the prescribed Loday symbols as a basis. The complementary summand is the complex

$$(*) \qquad 0 \to k^{n+1} \xrightarrow{b} k^{n+1} \oplus k^n \xrightarrow{b} k^n \to 0$$

generated by the bad symbols and induced symbols. Now the relation

$$bB(y \otimes y \otimes x \otimes \cdots \otimes x) = -B(y^2 \otimes x \otimes \cdots \otimes x)$$

and the argument given above for one y show that (*) is exact. Hence ${}_wHH_n(A)$ has the prescribed Loday symbols as a basis, ${}_wHH_{n-1}(A) = 0$, and we have isomorphisms

$${}_wHH_n(A) \underset{I}{\xrightarrow{\cong}} {}_wHC_n(A) \underset{B}{\xrightarrow{\cong}} {}_wHH_{n+1}(A). \quad \blacksquare$$

Proof of 7.3. For $n \leq 4$ the Hodge decompositions are given in (4.2.1), (4.3.2), and (4.4.2). For $n \geq 5$ we calculate using the generating functions. Set $w = (n-1, 2)$. Since $P_B(x,y) = 1 + x + y$, the calculation of $\Pi C(B)$ modulo y^3 is the same as for $\Pi C(A)$ except that the term $1 + \lambda T$ is replaced by λT. Thus modulo y^3

$$\Pi C(B) = \Pi C(A) - y^2 + \sum_{i=1}^{\infty} \lambda^i (x^{2i-1} y^2 - x^{2i} y^2).$$

Thus for $n \geq 2$ we have a short exact sequence

$$0 \to {}_wHC_n^{(i)}(A) \to {}_wHC_n^{(i)}(B) \to \begin{cases} k & i = [\frac{n}{2}] \\ 0 & \text{otherwise} \end{cases} \to 0.$$

Since ${}_wHC_n(B)$ is entirely generated by Loday symbols, and all but $\ll y, y, x, ..., x \gg$ come from Loday symbols in ${}_wHC_n(A)$, it follows that $\ll y, y, x, ...x \gg$ projects nontrivially into ${}_wHC_n^{([\frac{n}{2}])}(B)$. By theorem 7.2, ${}_wHC_n(A) = 0$ for all $i \geq 1$; it follows that $\ll y, y, x, ..., x \gg$ lies in $HC_{2i+1}^{(i)}(B)$. For n even we need another argument because ${}_wHC_{2i}^{(i)}(A) \cong k$. We use the shuffle product $HH_1^{(1)}(B) \otimes HH_{2i-1}^{(i-1)}(B) \to HH_{2i}^{(i)}(B)$ described in (1.6). We have just shown that $\ll y, y, x, ..., x \gg \in HH_{2i-1}^{(i-1)}(B)$. Since

$$(1 \otimes x)\#(y \otimes y \otimes x \otimes \cdots \otimes x) = (y \otimes x \otimes y \otimes x \otimes x \otimes \cdots \otimes x) - (y \otimes y \otimes x \otimes \cdots \otimes x),$$

it follows that $\ll y, x, y, x, x, ..., x \gg - \ll y, y, x, ..., x \gg$ lies in $HH_{2i}^{(i)}(B)$. ■

§8 Loday symbols in x, y and z

In this section we consider Loday symbols in three variables. For simplicity, we first consider Loday symbols with exactly one y, exactly one z and many x's.

Theorem 8.1. *Let $A = k[x,y,x]/(x,y,z)^2$, and let V denote the subspace of $HC_n(A)$ generated by the n Loday symbols $\ll z,y,x,..,x \gg, ..., \ll z,x,...,x,y \gg$ with exactly one y and exactly one z. Then V is isomorphic to the permutation representation of Σ_n in such a way that the projection of V into $HC_n^{(i)}(A)$ is $e_n^{(i)}V$.*

For $n=1$, $\ll z,y \gg \in HC_1^{(1)}(A)$; for $n=2$ the Hodge decomposition $e_2^{(1)}V \cong e_2^{(2)}V \cong k$ is described in (4.2). For $n \geq 3$

$$\dim(e_n^{(i)}V) = \begin{cases} 1 & i=1 \\ 2 & 2 \leq i \leq \frac{n+1}{2} \\ 1 & i = \frac{n}{2}+1 \text{ and } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

For $2 \leq i \leq \frac{n+2}{2}$, each of these n Loday symbols projects nontrivially into $HC_n^{(i)}(A)$.

Corollary 8.2. *When $A = k[x,y,z]/(x,y,z)^2$, every single one of the n Loday symbols $\ll z,y,x,...,x \gg, ..., \ll z,x,...,x,y \gg$ projects nontrivially into $K_n^{(i)}(A)$, $2 \leq i \leq \frac{n+1}{2}$ and projects to zero in $K_n^{(i)}(A)$ if $i > \frac{n+1}{2}$.*

Proof of 8.1. Let V' denote the subspace of the Hochschild chain complex $C_n(A)$ generated by the cycles $z \otimes y \otimes x \otimes \cdots \otimes x, ..., z \otimes x \otimes \cdots \otimes x \otimes y$. By (1.1), Σ_n acts on V', and V' is isomorphic to the permutation representation. Since V' maps isomorphically onto $V \subseteq HC_n(A)$, this proves the first part of the theorem.

Now V' is the direct sum of the trivial representation V_0 and the irreducible representation V_1 corresponding to the partition $\mu = (n-1,1)$. On V_j, $e_n^{(i)}$ is multiplication by the multiplicity m_{ij} of V_j in $e_n^{(i)}\mathbf{Q}\Sigma_n$, so dim $e_n^{(i)}V$ is $m_{i0}+m_{i1}$. Reading off the multiplicities from [H,6.3 and 6.4] yields the final assertions of the theorem. ∎

Alternate proof of 8.1. We are interested in the coefficient of $x^n yz$ in $\Pi C(A)$, so we may compute modulo y^2 and z^2. Setting $B = k[x,y]/(x^2,xy,y^2)$, and referring to the proof of Theorem 7.2, we compute mod y^2 that

$$\begin{aligned}\Pi C(A) &\equiv \Big\{\frac{P_B+z}{P_B}\Big\}^{1-\lambda}\Pi C(B)\\ &\equiv (1+(1-\lambda)zP_B^{-1})(\frac{1-\lambda x}{1-\lambda x^2})(\frac{1+x}{1-\lambda}+y)\\ &\equiv \Pi C(B)+z(1+x)^{-1}(1-\frac{y}{1+x})(\frac{1-\lambda x}{1-\lambda x^2})(1+x+y-\lambda y).\end{aligned}$$

Using Lemma 6.5, we see that the coefficient of yz in $\Pi C(A)$ is

$$\frac{-\lambda}{1+x}(\frac{1-\lambda x}{1-\lambda x^2})=\frac{-\lambda}{1+x}+\sum_{i=2}^{\infty}\lambda^i x^{2i-3}(1-2x+2x^2-3x^3+\cdots).$$

Therefore for $n\geq 3$ the coefficient of $x^{n-1}yz$ is $(-1)^{n+1}$ times

$$\sum_{i=0}^{\infty}\lambda^{1+[\frac{i+1}{2}]}=\begin{cases}\lambda+2\lambda+...+2\lambda^p+\lambda^{p+1} & \text{if } n=2p\geq 4\\ \lambda+2\lambda+...+2\lambda^p & \text{if } n=2p-1\geq 3.\ \blacksquare\end{cases}$$

We can use the shuffle product in order to find linear combinations of Loday symbols that have pure Hodge index. For example:

Proposition 8.3. *Let $A=k[x,y,z]/(x,y,z)^2$. Then using only one z and one y, we have:*

$$\ll z,y,x,...,x\gg-\ll z,x,y,x,...,x\gg+...-\ll z,x,...,x,y\gg\in HC_{2n}^{(n+1)}(A);$$

$$\ll z,x,y,x,...,x\gg+\ll z,x,x,x,y,x,...,x\gg+...+\ll z,x,...,x,y,x\gg\in HC_{2n-1}^{(n)}(A).$$

The corresponding K-theory symbols live in $K_{2n+1}^{(n+2)}(A)$and $K_{2n}^{(n+1)}(A)$, respectively.

Proof. In order to use the shuffle product we will work in Hochschild homology. By 7.1, $y\otimes x\otimes...\otimes x\in HH_{2n-1}^{(n)}(A)$. Specializing to $y=1$ yields $1\otimes x\otimes...\otimes x\in HH_{2n-1}^{(n)}(A)$. By 4.1, $z\otimes y\in HH_1^{(1)}(A)$. Using (1.6), this yields $z\otimes y\ \#\ 1\otimes x\otimes...\otimes x=z\otimes y\otimes x\otimes...\otimes x-z\otimes x\otimes y\otimes x\otimes...\otimes x+...-z\otimes x\otimes...\otimes x\otimes y\in HH_{2n}^{(n+1)}(A)$. The second calculation is similar, using the element $\frac{1}{2}(z\otimes y\otimes x+z\otimes x\otimes y)$, which belongs to $HH_2^{(1)}(A)$ by (4.2). ∎

Remark 8.3.1. We saw in (4.3.2) that $\ll z,x,y,x\gg$ is of pure Hodge index 2; this is a special case of the second part of (8.3). For $n\geq 4$ no single Loday symbol has pure weight. We leave it to the reader to compute such other smash products as suits the reader's fancy.

However, we note that if there are $2n$ x's, then $1 \otimes x \otimes ... \otimes x$ is not a cycle, and hence is not in $HH_{2n+1}(A)$. Thus we cannot use the same argument as in 8.3 with an even number of x's.

Theorem 8.4. *Set* $A = k[x,y,z]/(xy,xz,yz)$.

(i) For every n *there are only two independent Loday symbols in* $HC_n(A)$ *with one* z*, and the following symbols have pure Hodge index:*

$$\begin{aligned}
&\ll z,x,y,x,y,...,x \gg \text{ and } \ll z,y,x,y,x,...,y \gg \text{ in } HC^{(i)}_{2i-1}(A);\\
&\ll z,x,y,x,y,...,x,y \gg + \ll z,y,x,y,x,...,y,x \gg \text{ in } HC^{(i)}_{2i}(A);\\
&\ll z,x,y,x,y,...,x,y \gg - \ll z,y,x,y,x,...,y,x \gg \text{ in } HC^{(i+1)}_{2i}(A);
\end{aligned}$$

(ii) In $HC_5(A)$*, the Loday symbols* $\ll z,x,z,x,y,x \gg$ *and* $\ll z,y,z,y,x,y \gg$ *have pure Hodge index 3. The three Loday symbols with weight* $(2,2,2)$ *have mixed Hodge indices 1 and 3.*

(iii) For $n \geq 6$*, the Loday symbols with exactly two* z*'s lie in the sum of the* $HC^{(i)}_n(A)$ *with* $1 \leq i \leq \frac{n+3}{2}$*, and with the exception of* $HC^{(1)}_n(A)$*when* $n \equiv 3(\text{mod } 4)$*, all the* $HC^{(i)}_n(A)$ *contain a nonzero linear combination of these Loday symbols.*

Proof. For $A = k[x,y,z]/(xy,xz,yz)$ we have $P_A = 1 + \sum(x^i + y^i + z^i)$. Setting $B = k[x,y]/(xy)$ and computing modulo z^2 yields

$$\begin{aligned}
\Pi H(A) &\equiv (P_B + z)^{1-\lambda}(P_B)^{\lambda-1}\Pi H(B)\\
&\equiv \Pi H(B) + (1-\lambda)z(P_B)^{-1}\Pi H(B)\\
\Pi C(A) &\equiv \Pi C(B) + z(P_B)^{-1}\Pi H(B).
\end{aligned}$$

Referring to (6.7), we see that the coefficient of z in $\Pi C(A)$ is

$$(1-\lambda) + \frac{\lambda(1-x)(1-y)}{1-\lambda xy} = 1 + \lambda(-x-y+xy) + \sum_{i=2}^{\infty} \lambda^i (xy)^{i-1}(1-x-y+xy).$$

The coefficient of λ corresponds to the Loday symbols $\ll x,z \gg$, $\ll y,z \gg \in HC^{(1)}_1(A)$ and $\frac{1}{2}(\ll z,x,y \gg + \ll z,y,x \gg) \in HC^{(1)}_2(A)$ described in (4.1) and (4.2). The elements of Hodge index 2 lying in HC_2, HC_3 and HC_4 were described in (4.2), (4.3.1) and (4.4.2); they are $\frac{1}{2}(\ll z,x,y \gg - \ll z,y,x \gg)$, $\ll z,x,y,x \gg$, $\ll z,y,x,y \gg$, and

$$\frac{1}{2}(\ll z,x,y,x,y \gg + \ll z,y,x,y,x \gg).$$

We know by [GRW, 3.12] that $HC_*(A)$ is spanned by Loday symbols, and the only Loday symbols with one z are $\ll z,x,y,x,... \gg$ and $\ll z,y,x,y,x,... \gg$. Thus the assertion (i) follows from considering the symmetrizing idempotent (see 1.3).

In order to study the coefficient of z^2 in $\Pi C(A)$ it is useful to start with some low-dimensional remarks. The only term of weight $(0,0,2)$ is $z^2 \in HH_0^{(0)}(A)$. After that, the smallest Loday symbols with 2 z's are

$$\ll z,x,z,x \gg,\ \ll z,x,z,y \gg \text{ and} \ll z,y,z,y \gg$$

which lie in $HC_3^{(2)}(A)$ by (3.3.2). By (3.4.2), next come symbols in $HC_4^{(2)}(A)$:

$$\ll z,x,z,x,y \gg + \ll z,x,z,y,x \gg \text{ and } \ll z,y,z,y,x \gg + \ll z,y,z,y,x \gg .$$

The differences $\ll z,x,z,x,y \gg - \ll z,x,z,y,x \gg$ and $\ll z,y,z,y,x \gg -$ $\ll z,y,z,x,y \gg$ lie in $HC_4^{(3)}(A)$. Therefore the first few terms of the coefficient of z^2 in $\Pi C(A)$ form the polynomial

$$t = 1 - \lambda^2(x^2+xy+y^2) + \lambda^2(x^2y+xy^2) + \lambda^3(x^2y+xy^2).$$

Computing $\Pi H(A)$ modulo z^3 yields

$$\begin{aligned}\frac{\Pi H(A)}{\Pi H(B)} &\equiv \left\{\frac{P_B+z+z^2}{P_B}\right\}^{1-\lambda}\left\{\frac{P_B(x^2,y^2)+z^2}{P_B(x^2,y^2)}\right\}^{\frac{1}{2}\lambda(1-\lambda)} \\ &\equiv \{1+(1-\lambda)(z+z^2)P_B^{-1} - \frac{\lambda(1-\lambda)}{2}z^2P_B^{-2}\}\{1+\frac{\lambda(1-\lambda)}{2}z^2P_B^{-1}(x^2,y^2)\}.\end{aligned}$$

Referring to (6.7), we see that $P_B(x^2,y^2) = \frac{1+xy}{(1+x)(1+y)}P_B(x,y)$. Therefore the coefficient of z^2 is $(1-\lambda)\Pi H(B)P_B^{-1}$ times

$$1 - \frac{\lambda}{2}\frac{(1-x)(1-y)}{1-xy} + \frac{\lambda}{2}\frac{(1+x)(1+y)}{1+xy} = 1+\lambda(\frac{x+y-xy-x^2y^2}{1-x^2y^2}).$$

From here a straightforward but tedious calculation shows that the coefficient of λz^2 in $\Pi C(A)$ is

$$-x^2y^2 + (x+y-xy-x^2y^2)\sum_{i=1}^{\infty}(xy)^{2i};$$

the coefficient of $\lambda^2 z^2$ in $\Pi C(A) - t$ is

$$x^2y^2 + \{-1 + x + y - (x^2 + 3xy + y^2) + (2x^2y + 2xy^2)\} \sum_{j=1}^{\infty} (xy)^{2j},$$

and for $i \geq 3$ the coefficient of $\lambda^i z^2$ in $\Pi C(A)$ is

$$\begin{gathered}(xy)^{i-2}\big\{(x + y) - (x^2 + 2xy + y^2) + (x^2y + xy^2)\big\} + (xy)^i \\ +\big\{(-2 + 2(x + y) - (x^2 + 4xy + y^2))\big\} \sum_{j=1}^{\infty} (xy)^{i+2j}.\end{gathered}$$

Since the Loday symbols of weight $w = (p, q, 2)$ can only occur in $HC_{p+q+1}(A)$, we can read off the dimension of the ${}_wHC_n^{(i)}(A)$ from this. ■

Appendix. A map from relative HC to K-theory

The purpose of this appendix is to prove the following result, which has Proposition 0.6 as a special case.

Theorem A.0. *Let $I_1, \dots, I_m$ be ideals in a $\mathbb{Q}$-algebra A such that $\cap I_i = 0$. Then there is a natural map*

$$HC_{*-1}(A; I_1, \dots, I_m) \xrightarrow{E} K_*(A; I_1, \dots, I_m).$$

If $x_i \in I_i$ are such that $x_1 x_2 = \dots = x_n x_1 = 0$, then E sends the cyclic homology Loday symbol $x_1 \otimes \dots \otimes x_n$ to the K-theory Loday symbol $\ll x_1 \dots, x_n \gg$. Moreover, if A is commutative then E commutes with the operations λ^k and ψ^k.

Our proof closely follows Cathelineau's proof for a nilpotent ideal in [C], except that we have used ideas from Ogle's thesis [Ogle T] to finesse the gap in [OW]. (If the gap were fixed, the result would be that the map E is an isomorphism.) We remark that we have chosen to work with combinatorial models rather than $GL(\mathbb{Q})$-invariant models in order to simplify the argument. This means that we must replace the invariant theory in [OW] by a more elementary combinatorial result due to R. Aboughazi (this part of her thesis appears in [AO] as Thm. 1.1.12).

Let I be an ideal of a ring A. For each n, let $T_n(A, I)$ denote the subgroup of $GL_n(A)$ consisting of all $n \times n$ invertible matrices g with are strictly upper triangular modulo I, i.e., such that $g_{ij} \in I$ if $i > j$ and $g_{ii} - 1 \in I$ for all i. The corresponding Lie subalgebra $\mathfrak{t}_n(A, I)$ of $\mathfrak{gl}_n(A)$ consists of all matrices $a \in M_n(A)$ such that $a_{ij} \in I$ if $i \geq j$.

The above groups depend upon the choice of an ordered basis of A^n. To remove this dependence, note that for any $\sigma \in GL_n(A)$ we can form the conjugate subgroup $T_n(A, I)^\sigma = \{g | \ \sigma g \sigma^{-1} \in T_n(A, I)\}$ and the Lie subalgebra $\mathfrak{t}_n(A, I)^\sigma$. Via the regular representation $\Sigma_n \to GL_n(A)$ of the symmetric group Σ_n this allows us to write T_n^σ and $\mathfrak{t}_n^\sigma$ for any permutation σ of the canonical basis of A^n.

Now suppose we are given an m-tuple $\sigma = (\sigma_1, \dots, \sigma_m)$ and m ideals $I_1, \dots, I_m$ of A such that $\cap I_i = 0$. Write $T_n^\sigma(A; I_1, \dots, I_m)$ and $\mathfrak{t}_n^\sigma(A; I_1, \dots, I_m)$ for the intersection of the $T_n(A, I_i)^{\sigma_i}$ and of the $\mathfrak{t}_n(A, I_i)^{\sigma_i}$, respectively. The groups T_n^σ are nilpotent, because any commutator of length n is congruent to the identity matrix modulo I_i for all i. Similarly, the $\mathfrak{t}_n^\sigma$ are nilpotent Lie groups. If A is a $\mathbb{Q}$-algebra, this implies that the exponential map

$$exp : \mathfrak{t}_n^\sigma(A; I_1, \dots, I_m) \to T_n^\sigma(A; I_1, \dots, I_m)$$

is well-defined and bijective. One of the main theorems of rational homotopy (see [GwL, pp.391-3] or [OW, 2.3]) is that *exp* induces a natural isomorphism on rational homology:

$$\Phi : H_*^{Lie}(\mathfrak{t}_n^\sigma;\mathbb{Q}) \overset{\cong}{\to} H_*(T_n^\sigma;\mathbb{Q}) \overset{\cong}{\to} H_*(BT_n^\sigma;\mathbb{Q}).$$

Here the left-hand term is Lie algebra homology, the middle is group homology, and the right-hand term is the topological homology of the classifying space BT_n^σ of the group T_n^σ.

In order to construct λ-operations, we need to define exterior products on the T_n^σ and $\mathfrak{t}_n^\sigma$ based on the canonical identification of $\Lambda^k A^n$ with $A^{\binom{n}{k}}$, which is obtained by putting the usual basis of $\Lambda^k A^n$ in lexicographical order. This exterior product construction requires that A be commutative. Since Λ^k is a functor, it induces maps $\Lambda_+^k : \mathfrak{gl}_n(A) \to \mathfrak{gl}_{\binom{n}{k}}(A)$ and $\Lambda_\times^k : GL_n(A) \to GL_{\binom{n}{k}}(A)$. As observed in [C, 2.2], these are given by the explicit formulas:

$$\left(\Lambda_+^k g\right)(v_1 \wedge \ldots \wedge v_k) = \sum v_1 \wedge \ldots \wedge g(v_i) \wedge \ldots \wedge v_k;$$
$$\left(\Lambda_\times^k g\right)(v_1 \wedge \ldots \wedge v_k) = g(v_1) \wedge \ldots \wedge g(v_k).$$

From these formulas, it is easy to make the following two observations: (1) if g is upper triangular then so is $\Lambda^k g$; (2) if g is a nilpotent matrix then $\Lambda_\times^k \exp(g) = \exp(\Lambda_+^k g)$. From this (and naturality in A), we obtain the following commutative diagram for all $\sigma \in GL_n(A)$:

$$(A.1)\qquad \begin{tikzcd} \mathfrak{t}_n^\sigma(A;I_1,\ldots,I_m) \arrow[r, "\Lambda_+^k"] \arrow[d, "exp"'] & \mathfrak{t}_{\binom{n}{k}}^{\Lambda^k\sigma}(A;I_1,\ldots,I_m) \arrow[d, "\exp"'] \\ T_n^\sigma(A;I_1,\ldots,I_m) \arrow[r, "\Lambda_\times^k"] & T_{\binom{n}{k}}^{\Lambda^k\sigma}(A;I_1,\ldots,I_m) \end{tikzcd}$$

This is the analogue of diagram (1) in [C]. We remark that if $\sigma \in \Sigma_n$ then $\Lambda^k\sigma$ is a monomial matrix which is not in general an element of the canonical subgroup $\Sigma_{\binom{n}{k}}$ of $GL_{\binom{n}{k}}(A)$. However, there is a $\tau \in \Sigma_{\binom{n}{k}}$ such that $(\Lambda^k\sigma)\tau^{-1} = \delta$ is a diagonal matrix. Since δ fixes $\mathfrak{t}_{\binom{n}{k}}$ and $T_{\binom{n}{k}}$, we have $\mathfrak{t}_{\binom{n}{k}}^{\Lambda^k\sigma} = \mathfrak{t}_{\binom{n}{k}}^\tau$ and $T_{\binom{n}{k}}^{\Lambda^k\sigma} = T_{\binom{n}{k}}^\tau$ in (A.1).

Lemma A.2. *The following diagram is commutative for every commutative $\mathbb{Q}$-algebra A, $\sigma \in GL_n(A)$ and ideals $I_1,\ldots,I_m$ such that $\cap I_i = 0$:*

$$\begin{tikzcd} H_*\left(\mathfrak{t}_n^\sigma(A;I_1,\ldots,I_m);\mathbb{Q}\right) \arrow[r, "\Lambda_+^k"] \arrow[d, "\Phi"'] & H_*\left(\mathfrak{t}_{\binom{n}{k}}^{\Lambda^k\sigma}(A;I_1,\ldots,I_m);\mathbb{Q}\right) \arrow[d, "\Phi"] \\ H_*\left(T_n^\sigma(A;I_1,\ldots,I_m);\mathbb{Q}\right) \arrow[r, "\Lambda_+^k"] & H_*\left(T_{\binom{n}{k}}^{\Lambda^k\sigma}(A;I_1,\ldots,I_m);\mathbb{Q}\right). \end{tikzcd}$$

Proof: Cathelineau's proof of Lemma 1 in [C] applies verbatim. We remark that this proof is implicit in Goodwillie's argument on p. 392 of [GwL].

Following [OW], we let $X_n = X_n^C(A; I_1, \dots, I_m)$ denote the union of topological subspaces $BT_n^\sigma(A; I_1, \dots, I_m)$ of $BGL_n(A)$, the union being taken as σ runs over all m-tuples of permutations in Σ_n. The superscript C stands for "combinatorial", in order to distinguish it from the "linear" model $X_n^L = X_n^L(A; I_1, \dots, I_m)$ of [OW], which is the union of the $BT_n^\sigma(A; I_1, \dots, I_m)$ as σ runs over all m-tuples of elements in $GL_n(\mathbb{Q})$. X_n has an infinitesimal "Lie" analogue represented by the subcomplex

$$x_n = x_n^C(A; I_1, \dots, I_n) = \sum_{\sigma \in \Sigma_n} \Lambda^* \mathfrak{t}_n^\sigma(A; I_1, \dots, I_m)$$

of the Chevalley–Eilenberg complex $\Lambda^* \mathfrak{gl}_n(A)$ used to compute the Lie algebra homology $H_*(\mathfrak{gl}_n(A); \mathbb{Q})$.

Proposition A.3. *For every $\mathbb{Q}$-algebra A and ideals I_i such that $\cap I_i = 0$, and all n, the exponential map induces isomorphisms*

$$\Phi_n : H_*\left(x_n^C\left(A; I_1, \dots, I_m\right)\right) \xrightarrow{\cong} H_*\left(X_n^C\left(A; I_1, \dots, I_m\right); \mathbb{Q}\right)$$

fitting into a commutative diagram for each n and k:

$$\begin{tikzcd}
H_*\left(x_n\right) \arrow[r, "\Lambda_+^k"] \arrow[d, "\Phi"', "\cong"] & H_*\left(x_{\binom{n}{k}}\right) \arrow[d, "\Phi"', "\cong"] \\
H_*\left(X_n; \mathbb{Q}\right) \arrow[r, "\Lambda_\times^k"] & H_*\left(X_{\binom{n}{k}}; \mathbb{Q}\right).
\end{tikzcd}$$

Proof: This is Cor. 2.4 of [OW]; cf. III.5 of [GwL]. The subcomplexes $\Lambda^* \mathfrak{t}_n^\sigma$ form an "atomic" functor in the sense of [OW], so that $H_*(x_n)$ is the homology of the chain complex *hocolim*$(\Lambda^* \mathfrak{t}_n^\sigma)$. Since $H_*(x_n)$ is the homology of the complex *hocolim*$(C_*(BT_n^\sigma))$, the result follows from the hocolim spectral sequence 1.4 of [OW].

Now set $x = \cup x_n$ and $X = \cup X_n$. Following [LP] and [C], we define operations

$$\begin{aligned}
\lambda_{+,n}^k &= \sum_{i=0}^k (-1)^i \binom{n-1+i}{i} \Lambda_+^{k-i} : H_*(x_n) \to H_*(x) \\
\lambda_{\times,n}^k &= \sum_{i=0}^k (-1)^i \binom{n-1+i}{i} \Lambda_\times^{k-i} : H_*(X_n; \mathbb{Q}) \to H_*(X; \mathbb{Q})
\end{aligned} \tag{A.3.1}$$

Now Σ_∞ acts trivially on $H_*(x)$ and $H_*(X; \mathbb{Q})$ by an argument of Suslin, detailed in [OW, 3.2]. Thus the maps $\lambda_{+,n}^k$ and $\lambda_{\times,n}^k$ are compatible with stabilization, and we can define endomorphisms $\lambda^k_+ = \varinjlim \lambda^k_{+,n}$ and $\lambda^k_\times = \varinjlim \lambda^k_{\times,n}$ on $H_*(x)$ and $H_*(X;\mathbb{Q})$.

As pointed out in [C], the direct sum of matrices provides the stable homology $H_*(x)$ and $H_*(X;\mathbb{Q})$ with a product. Since the diagonal map gives them a coproduct, they are in fact Hopf algebras. The map $\Phi = \varinjlim \Phi_n$ is in fact an isomophism of Hopf algebras, from $H_*(x)$ to $H_*(X;\mathbb{Q})$. Under this isomorphism, the operations λ^k_+ and $\lambda^k_\times$ correspond. Writing P_* for the primitive part of $H_*(x)$, we conclude as in [C] that the operations λ^k restrict to make P_* into a graded λ-ring. It remains to relate P_* to cyclic homology and to K-theory.

Let $C_p(A;I_1,\dots,I_m)$ denote the intersection of the kernels of the maps $\pi_i, i = 1,...,m$. Since all the π_i are surjections, the homology of $C_*(A;I_1,\dots,I_m)$ is the m-fold relative cyclic homology group $HC_*(A;I_1,\dots,I_m)$.

Now consider the Loday–Quillen map $Tr\circ\lambda : \Lambda^*(\mathfrak{gl}(A) \to C_{*-1}(A)$, restricted to the subcomplex $x(A;I_1,\dots,I_m)$ of $\Lambda^* gl(A)$. Since composing $Tr\circ\lambda$ with any projection $\pi_i : C_p(A) \to C_p(A/I_i)$ sends each $\Lambda^{p+1}\mathfrak{t}^\sigma_n(A/I_i)$ to zero, it vanishes on all of x_{p+1}. Therefore the image of $x_{p+1}(A;I_1,\dots,I_m)$ lands in $C_p(A;I_1,\dots,I_m)$, and the restriction $x_* \to C_{*-1}(A;I_1,\dots,I_m)$ therefore induces a map on homology

$$P_* \subset H_*(x) \longrightarrow HC_{*-1}(A;I_1,\dots,I_m).$$

As pointed out in [C, 2.4], this Loday–Quillen map commutes with the λ-operations by the definition in [LP] of the λ-operations on cyclic homology. This proves the second part of

Theorem A.4. *The map $Tr\circ\lambda : P_* \to HC_{*-1}(A;I_1,\dots,I_m)$ is an isomorphism. If A is commutative, this isomorphism commutes with all the operations λ^k.*

Proof: (Cf. [OW, 4.5]). $C_{p-1}(A;I_1,\dots,I_m)$ is generated by terms $a_1\otimes\dots\otimes a_p$ such that, for every i, some a_j is in I_i. For an appropriate σ, the p-form

$$w = e_{12}(a_1)\wedge\dots\wedge e_{p,1}(a_p)$$

is in $\Lambda^p\mathfrak{t}^\sigma_p(A;I_1,\dots,I_m)$ and $Tr(\lambda(w)) = a_1\otimes\dots\otimes a_p$. Hence the linear span W_p of forms of the type w forms a subspace of x_p mapping onto $C_{p-1}(A;I_1,\dots,I_m)$. By inspection, each w is primitive. By Theorem 1.1.12 of [AO], P_* is isomorphic under the Loday–Quillen map with the homology of $Tr\circ\lambda(W_*) = C_{*-1}(A;I_1,\dots,I_m)$, which is what the first part of our theorem asserts.

Remark A.4.1. One could also consider the $GL(\mathbb{Q})$-invariant subcomplex $x^L = \cup x^L_n$ of $\Lambda^*\mathfrak{gl}(A)$ generated by the $\mathfrak{t}_n(A;I_1,\dots,I_m)$. Theorems 3.3 and 4.5 of [OW] state that $H_*(x^L)$ is a Hopf algebra whose primitive part is $HC_{*-1}(A;I_1,\dots,I_m)$.

Therefore $H_*(x^C) \cong H_*(x^L)$. Using the natural map $H_*(x^L) \to H_*(X^L;\mathbb{Q}) \to H_*(K(A;I_1,\dots,I_m);\mathbb{Q})$, this would yield a completely parallel proof of the main result A.0 of this appendix.

We now turn to K-theory. Given ideals $I_1,\dots,I_m$ in a ring A, the m-fold relative K-groups $K_*(A;I_1,\dots,I_m)$ are the homotopy groups of a topological space $K(A;I_1,\dots,I_m)$. One way to construct this space is to take the interated homotopy fiber of the m-cube of spaces which at the vertex $(\epsilon_1,\dots,\epsilon_m)$ of the m-cube has the space

$$K\left(A/\bigcap\{I_i : \epsilon_i = 1\}\right).$$

By this construction, the natural maps $\lambda^k : K_*(A) \to K_*(A)$ induce endomorphisms λ^k of each group $K_*(A;I_1,\dots,I_m)$ when A is commutative.

Theorem A.5. *Let $I_1,\dots,I_m$ be any family of ideals in a ring A. There is a natural topological map $X^C(A;I_1,\dots,I_m) \to K(A;I_1,\dots,I_m)$. If A is commutative, the induced map of homotopy groups commutes with the operations λ^k.*

Proof: Suslin proved in [Sus] that $X(A) = X^C(A;0) \subset BGL(A)$ is naturally the homotopy fiber of the plus construction $BGL(A) \to BGL(A)^+$. For every ideal I of A, the map $X^C(A;I) \to BGL(A/I)^+$ factors through $X(A/I)$ and so is naturally contractible. Therefore the map

$$X^C(A;I_1,\dots,I_m) \to BGL(A) \to K(A)$$

lifts to a natural map from $X = X^C(A;I_1,\dots,I_m)$ to $K(A;I_1,\dots,I_m)$. When A is commutative, the argument used by Cathelineau in [C, 2.4] shows that this map commutes with the λ^k. Indeed, the maps $\lambda^k_n : BGL_n(A) \to BGL(A)^+$ used to define λ^k on $K_*(A)$ are given by formula (A.3.1), using the H-space structure of $BGL(A)^+$. These maps restrict to maps $\lambda^k_n : X \to X^+$, again given by (A.3.1).

Remark A.5.1. In [OW, 7.3] it is asserted, but not proven, that the map of Theorem A.5 is a homology isomorphism. This is the crucial gap in the proof of the Main Theorem 7.5 of [OW].

Definition A.6. For every $\mathbb{Q}$-algebra A and ideals I_i such that $\cap I_i = 0$, let E denote the composition:

$$\begin{aligned} HC_{n-1}(A;I_1,\dots,I_m) &\cong \text{ Prim } H_n x^C(A;I_1,\dots,I_m) && \text{by A.4} \\ &\cong \text{ Prim } H_n X^C(A;I_1,\dots,I_m) && \text{by A.3} \\ &\to \text{ Prim } H_n K(A;I_1,\dots,I_m) && \text{by A.5} \\ &\cong K_n(A;I_1,\dots,I_m). \end{aligned}$$

If A is commutative, it follows from A.4, A.3 and A.5 that E commutes with the operations λ^k. Therefore Theorem A.0 follows from the following calculation, which is due to Ogle.

Proposition A.6.1. *Suppose given elements* $a_i \in I_i$ *such that* $a_1a_2 = \dots a_na_1 = 0$. *Then* E *sends the cyclic homology Loday symbol* $\ll a_1, \dots , a_n \gg$ *to the* K*-theory Loday symbol* $\ll a_1, \dots , a_n \gg$.

Proof: Given $\{a_i\}$ with $a_1a_2 = \dots = a_na_1 = 0$, there is a map from the group $\mathbb{Z}^n$ to $T_n^\sigma(A; I_1, \dots , I_m)$ sending the i^{th} generator to $e_{i,i+1}(a_i)$. This induces a map from the torus $T^n = B\mathbb{Z}^n$ to BT_n^σ, and therefore a map of the associated Lie algebras from $\mathbb{Q}^n$ to $\mathfrak{t}_n^\sigma$. By naturality, rational homotopy theory gives a commutative square

$$\begin{tikzcd}
\mathbb{Q} \cong H_n^{\mathrm{Lie}}(\mathbb{Q}^n;\mathbb{Q}) \arrow[r] \arrow[d, "\Phi"', "\cong"] & H_n^{\mathrm{Lie}}(\mathfrak{t}_n^\sigma;\mathbb{Q}) \arrow[r] \arrow[d, "\cong"', "\Phi"] & H_n(x^C(A; I_1, \dots , I_m)) \arrow[d, "\cong"', "\Phi"] \\
\mathbb{Q} \cong H_n(T^n;\mathbb{Q}) \arrow[r] & H_n(BT_n^\sigma;\mathbb{Q}) \arrow[r] & H_n(X^C(A; I_1, \dots , I_m);\mathbb{Q}).
\end{tikzcd}$$

From the proof of A.4 we see that the generator of the top left side maps to $\ll a_1, \dots , a_n \gg = [a_1 \otimes \dots \otimes a_n] \in H_n(x^C(A; I_1, \dots , I_m))$. The proof of [Ogle I, 3.1] now applies to show that the image of $a_1 \otimes \dots \otimes a_n$ under the map E to $K_n(A; I_1 \dots , I_n)$ is the K-theory Loday symbol $\ll a_1, \dots , a_n \gg$.

References

[AO] R. Aboughazi and C. Ogle, On Chevalley-Eilenberg homology, preprint, 1991.

[Barr] M. Barr, Harrison homology, Hochschild homology and triples, J. Alg., **8**(1968), 314-323.

[B] A. Beilinson, Higher regulators and the values of L-functions, J. Soviet Math., **30**(1985), 2036-2070.

[BV] D. Burghelea and M. Vigué-Poirrier, Cyclic homology of comutative algebras I, Springer Lecture Notes in Math. #1388, New York-Heidelberg-Berlin, 1988, 51-72.

[C] J.-L. Cathelineau, λ-structures in algebraic K-theory and cyclic homology, J. K-theory, **4**(1991), 591-606.

[Co] A. Connes, *Noncommutative Differential Geometry*, Publ. IHES, **62**(1985), 41-144.

[D] K. Dennis, Differentials in algebraic K-theory, unpublished manuscript, circa 1975.

[DW] B. Dayton and C. Weibel, Module structures on the Hochschild and cyclic homology of graded rings, preprint, 1991.

[FT] B. Feigin and B. Tsygan, Additive K-theory, Springer Lecture Notes in Math. #1289, New York-Heidelberg-Berlin, 1987, 67-209.

[TC] B. Feigin and B. Tsygan, Additive K-theory and crystalline cohomology, Funct. Anal. Appl., **19**(1985), 124-132.

[GRW] S. Geller, L. Reid, and C. Weibel, The cyclic homology and K-theory of curves, J. reine und angew. Math., **393** (1989), 39–90.

[GW] S. Geller and C. Weibel, $K(A, B, I) : II$, K-Theory, **2**(1989), 753-760.

[GS] M. Gerstenhaber and S. D. Schack, A Hodge-type decomposition for commutative algebra homology, J. Pure and Appl. Alg., **48**(1987), 229-247.

[GS2] M. Gerstenhaber and S. D. Schack, The shuffle bialgebra and the cohomology of commutative algebras, J. Pure and Appl. Alg., **70**(1991), 263-272.

[Gw] T. G. Goodwillie, Cyclic homology, derivations and the free loop space, Topology **24**(1985), 187-215.

[GwK] T. Goodwillie, Algebraic K-theory and cyclic homology, Annals Math., **124**(1986), 344-399.

[GwL] T. Goodwillie, On the general linear group and Hochschild homology, Annals Math., **121**(1985), 383-407.

[Hain] R. M. Hain, On the indecomposable elements of the bar construction, Proc. AMS, **98**(1986), 312-316.

[H] P. Hanlon, The action of S_n on the components of the Hodge decomposition of Hochschild homology, Mich. Math. J., **37**(1990), 105-124.

[HL] P. Hanlon, Letter to C. Weibel, Sept. 5, 1991.

[Hil] H. Hiller, λ-rings and algebraic K-theory, J. Pure and Appl. Alg., **20**(1981), 241-266.

[I] K. Igusa, What happens to Hatcher and Wagoner's formula for $\pi_0(CM)$ when the first Postnikov invariant of M is trivial? Springer Lecture Notes in Math. #1046, New York-Heidelberg-Berlin, 1984, 104-172.

[Kas] C. Kassel, Une formule de Künneth pour la décomposition de l'homologie cyclique des algèbres commutatives, Math. Scand., to appear.

[K] C. Kratzer, λ-structure en K-théorie algébrique, Comm. Math. Helv., **55**(1980), 233-254.

[Lsym] J.-L. Loday, Symbols en K-théorie algébrique supérieure, CR Acad. Sci.(Paris), **292**(1981), 863-866.

[Lop] J.-L. Loday, Opérations sur l'homologie cyclique des algébres commutatives, Inv. Math., **96**(1989), 205-230.

[LP] J.L. Loday and C. Procesi, Cyclic homology and lambda operations, Algebraic K-theory: Connections with Geometry and Topology, NATO ASI Series C, vol. 279, Klewer, 1989.

[LQ] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comm. Math Helv., **59** (1984), 565–591.

[NS] T. Natsume and S.D. Schack, A decomposition for the cyclic cohomology of a commutative algebra, J. Pure and Appl. Alg., **61**(1989), 273-282.

[Nuss] P. Nuss, Décomposition del la cohomologie cyclique bivariante des algèbres commutatives, Math. Scand., to appear.

[Ogle I] C. Ogle, On the K-theory and cyclic homology of a square-free ideal I, J. Pure Appl. Alg., **46**(1987), 233-247.

[Ogle T] C. Ogle, A map from cyclic homology into K-theory, Ph.D. thesis, Brandeis, 1984.

[OW] C. Ogle and C. Weibel, Relative K-theory and cyclic homology, preprint, 1986.

[Q] D. Quillen, On the (co)homology of commutative rings, Proc. Symp. Pure Math. XVII (1970), 65-87.

[Sch] P. Schneider, Introduction to the Beilinson conjectures, *Beilinson's conjectures on special values of L-functions*, Perspectives in Math., vol. 4, Academic Press, 1988.

[SeM] J.-P. Serre, Algèbre Locale Multiplicités, Springer Lecture Notes in Math. #11, New York-Heidelberg-Berlin, 1975.

[S] C. Soulé, Opérations en K-théorie algébrique, Can. J. Math., **37**(1985), 488-550.

[St] R. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, 1983.

[Sus] A. Suslin, On the equivalence of K-theories, Comm. Algebra, **9**(1981), 1559-1565.

[V] M. Vigué-Poirrier, Cyclic homology of algebraic hypersurfaces, J. Pure Appl. Alg., **72**(1991), 95-108.

[Vo] T. Vorst, The Serre problem for discrete Hodge algebras, Math. Zeit. **184**(1983), 425-433.

[Wms] C. Weibel, Module structures on the K-theory of graded rings, J. Alg., **105**(1987), 465-483.

[Wnil] C. Weibel, Nil K-theory maps to cyclic homology, Trans. Amer. Math. Soc., **303**(1987), 541-557.