

# Extension Categories and their Homotopy

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## 0 Introduction

In [9], Robinson proved that, given a ring  $R$  and two  $R$ -modules  $A$  and  $B$ , there is a spectrum whose homotopy groups are the torsion groups  $\text{Tor}_i^R(A, B)$ . More precisely, he proves

**Theorem 1 of [9]** *Let  $R$  be a ring. Let  $A$  be a right  $R$ -module, let  $B$  be a left  $R$ -module. Consider the category  $\text{Tor}^R(A, B)$  whose objects are pairs  $\epsilon : P \rightarrow B$  and  $\eta : P^* \rightarrow A$ , where  $P$  is a projective left  $R$ -module, and  $P^* = \text{Hom}(P, R)$  is its dual. Then if we geometrically realise the category  $\text{Tor}^R(A, B)$  we obtain a space with the natural structure of an infinite loop space, and its homotopy is given by the formula*

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$$\pi_i [Tor^R(A, B)] = Tor_i^R(A, B).$$

Inspired by Robinson's work, the second author generalised this to obtain a spectrum whose  $-i^{th}$  homotopy group is  $Ext^i(A, B)$ . More precisely, in [7], the second author announced the following theorem:

**Theorem 2 of [7]** *Let  $\mathcal{E}$  be an exact category. Let  $A$  and  $B$  be objects of  $\mathcal{E}$ . Consider the categories  $Ext^n(A, B)$  whose objects are exact sequences*

$$0 \longrightarrow B \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow A \longrightarrow 0$$

*and whose morphisms are morphisms of such exact sequences. Then the loop space  $\Omega Ext^n(A, B)$  of the category  $Ext^n(A, B)$  is naturally identified with  $Ext^{n-1}(A, B)$ . If one considers the  $\Omega$ -spectrum defined by the homotopy equivalences*

$$Ext^{n-1}(A, B) \longrightarrow \Omega Ext^n(A, B)$$

*then this defines a spectrum we will denote  $Ext(A, B)$ , and clearly*

$$\pi_{-i} [Ext(A, B)] = Ext_{\mathcal{E}}^i(A, B)$$

*where  $Ext_{\mathcal{E}}^i(A, B)$  means the usual extension groups, as opposed to extension categories.*

In this article we will generalise both results to an arbitrary homotopy category associated to a Waldhausen category. Let us begin by reminding the reader about the formalism of a Waldhausen category. Note that our presentation here will be very sketchy and incomplete. Throughout this article we will assume familiarity with Waldhausen's foundational article [11].

A *Waldhausen category* will mean a category with cofibrations and weak equivalences, satisfying the Gluing Lemma, Extension Axiom, Saturation Axiom and Cylinder Axiom, as in Waldhausen's article [11]. Starting with such a category, one can construct an associated stable homotopy category, by inverting the suspension functor and the weak equivalences. Inverting the suspension functor is harmless; the resulting category is still Waldhausen. For a discussion, see Section 1. Inverting the weak equivalences is drastic. One obtains something which is decidedly *not* a Waldhausen category, but only a triangulated category.

Throughout the article we will assume that the Waldhausen categories we are dealing with have an invertible suspension functor. As we have already said, this is harmless once we replace a Waldhausen category by its stabilisation.

For the uninitiated, let us briefly recall what a Waldhausen category is. It is a category  $C$ , with two subcategories  $w(C)$  and  $c(C)$ . The objects in the three categories are the same. The morphisms in  $w(C)$  are called *weak equivalences* and denoted by the letter  $w$ , the morphisms in  $c(C)$  are called *cofibrations* and denoted by the letter  $c$ .

These must satisfy a long list of axioms, which we do not want to repeat here. Perhaps the most important is that pushouts of cofibrations exist. That is, given a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \\ Y' & & \end{array}$$

then it is possible to complete it to a pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z \end{array}$$

as long as either  $X \longrightarrow Y$  or  $X \longrightarrow Y'$  is a cofibration. The other piece of structure a Waldhausen category comes equipped with is a *cylinder functor*. Given any map  $f : X \longrightarrow Y$ , there is an object

$$\text{Cyl} \left( X \xrightarrow{f} Y \right)$$

which is functorial in  $f : X \longrightarrow Y$ , comes with several natural transformations, and also satisfies a long list of axioms we do not want to repeat. It might be more enlightening if we gave a simple example.

**Example 0.1** Let  $\mathcal{A}$  be an abelian category. Define  $C = C(\mathcal{A})$  to be the category of all chain complexes of objects in  $\mathcal{A}$ . A morphism  $f$  in  $C = C(\mathcal{A})$  is a chain map of chain complexes. The morphism  $f$  is called a *cofibration* if it is a monomorphism in each degree. The morphism  $f$  is called a *weak equivalence* if it induces a homology isomorphism. Given a morphism  $f : X \longrightarrow Y$  in  $C$ , the *mapping cylinder*  $\text{Cyl} \left( X \xrightarrow{f} Y \right)$  is by definition the mapping cone on the map of chain complexes

$$\begin{pmatrix} 1 \\ -f \end{pmatrix} : X \longrightarrow X \oplus Y$$

where by *mapping cone* we understand the usual mapping cone of homological algebra. In this case, the suspension functor, which is always given by the pushout diagram

$$\begin{array}{ccc} X \vee X & \longrightarrow & \text{Cyl} \left( X \xrightarrow{1} X \right) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

is nothing other than the functor shifting the complex to the left, and is therefore invertible in  $C = C(\mathcal{A})$ . Thus the category  $C$  is a Waldhausen category with an invertible suspension functor.

It is possible to formally invert the weak equivalences. One then gets a triangulated category, which in the above case is nothing other than  $D(\mathcal{A})$ , the derived category of the abelian category  $\mathcal{A}$ .

**Notation 0.2** Let  $C$  be a Waldhausen category with an invertible suspension functor. Then the associated homotopy category, obtained by formally inverting the weak equivalences, will be denoted  $w^{-1}C$ .

Before we state our theorems, we need some definitions. Let  $C$  be a Waldhausen category. Let

$$X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n+1}$$

be a chain complex in  $C$ ; that is, the composites  $\partial \circ \partial$  are all zero. It is then possible to define a “totalization” of the complex, denoted  $T(X.)$ . This is the precise analogue of taking the total complex of a double complex in the category  $C = C(\mathcal{A})$ , and for a definition which makes sense in an arbitrary Waldhausen category, see Definition 1.2. A complex  $X.$  is called *acyclic* if the natural map  $0 \rightarrow T(X.)$  is a weak equivalence.

**Definition 0.3** Let  $n$  be an integer  $\geq 1$ . Then define  $Ext^n(A, B)$  to be the category whose objects are diagrams

$$\begin{array}{ccccccc} B & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ & & & & & & & & \uparrow \\ & & & & & & & & A \end{array}$$

where the chain complex

$$B \xrightarrow{\partial} X_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n+1}$$

is acyclic, and the map  $A \rightarrow X_{n+1}$  is both a weak equivalence and a cofibration. It is a map of type  $\bar{w}$ , in the standard notation. The morphisms in the category  $Ext^n(A, B)$  are morphisms of diagrams, which are the identity on  $A$  and  $B$ .

**Definition 0.4** Define  $Ext^0(A, B)$  to be the category whose objects are the diagrams

$$\begin{array}{ccc} A & & B \\ & \searrow & \swarrow \\ & X & \end{array}$$

where  $B \rightarrow X$  is of type  $\bar{w}$  (that is a cofibration and a weak equivalence), and  $A \rightarrow X$  is any morphism. A morphism in the category  $Ext^0(A, B)$  is a map of such diagrams that is the identity on  $A$  and  $B$ .

Now we are ready to state our theorems.

**Theorem 5.2** Let  $n \geq 1$  be an integer. The loop space of the category  $Ext^n(A, B)$  is naturally homotopy equivalent to  $Ext^{n-1}(A, B)$ .

A useful addendum is

**Proposition 7.4** *There is a natural homotopy equivalence  $Ext^n(\Sigma A, B) \simeq Ext^{n-1}(A, B)$ .*

In particular, we obtain a spectrum, which is perhaps simplest to describe in terms of the  $Ext^0$ . There is a sequence of homotopy equivalences

$$Ext^0(\Sigma^i A, B) \longrightarrow \Omega Ext^0(\Sigma^{i+1} A, B)$$

which defines an  $\Omega$ -spectrum we will call  $Ext(A, B)$ . Clearly,

$$\pi_{-i}[Ext(A, B)] = \pi_0[Ext^0(\Sigma^i A, B)].$$

To turn this into a computation of the homotopy of the spectrum  $Ext(A, B)$ , one appeals to

**Proposition 6.1** *The group  $\pi_0[Ext^0(A, B)]$  is naturally isomorphic to  $\text{Hom}_{w^{-1}C}(A, B)$ . In other words, it is the homomorphisms, in the associated triangulated category where the weak equivalences have been inverted.*

Combining these results, we deduce that the spectrum  $Ext(A, B)$  satisfies

$$\pi_{-i}[Ext(A, B)] = \text{Hom}_{w^{-1}C}(\Sigma^i A, B).$$

This isomorphism is also compatible with the composition in the category  $w^{-1}C$ . Precisely, let  $A, A'$  and  $A''$  be three objects of the category  $C$ . Then there is a product

$$Ext(A, A') \times Ext(A', A'') \longrightarrow Ext(A, A'').$$

It is even defined on the level of categories. There is a functor

$$\phi_{AA'A''} : Ext^0(A, A') \times Ext^0(A', A'') \longrightarrow Ext^0(A, A'').$$

which is very simple to describe. It takes the pair of diagrams

$$\begin{array}{ccc} A & & A' \\ & \searrow & \swarrow \\ & X & \end{array} \quad \text{and} \quad \begin{array}{ccc} A' & & A'' \\ & \searrow & \swarrow \\ & X' & \end{array}$$

to the diagram

$$\begin{array}{ccc} A & & A'' \\ & \searrow & \swarrow \\ & Y & \end{array}$$

where  $Y$ , and the maps to it, are defined up to canonical isomorphism by the pushout square

$$\begin{array}{ccc} A' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y \end{array}$$

and the pushout exists because  $A' \longrightarrow X$  is a cofibration. Now if we declare

$$\begin{array}{ccc} A & & A' \\ & \searrow & \swarrow \\ & A' & \end{array}$$

to be the base point of  $Ext^0(A, A')$ , where  $A' \longrightarrow A'$  is the identity and  $A \longrightarrow A'$  is the zero map, then there is a natural transformation from the zero map to the maps

$$Ext^0(A, A') \times * \quad \subset \quad Ext^0(A, A') \times Ext^0(A', A'') \quad \xrightarrow{\phi_{AA'A''}} \quad Ext^0(A, A'')$$

and

$$* \times Ext^0(A', A'') \quad \subset \quad Ext^0(A, A') \times Ext^0(A', A'') \quad \xrightarrow{\phi_{AA'A''}} \quad Ext^0(A, A'').$$

These natural transformations combine to define multiplication as a map

$$\phi_{AA'A''} : Ext^0(A, A') \wedge Ext^0(A', A'') \quad \longrightarrow \quad Ext^0(A, A'').$$

It can be checked that this is compatible with the infinite loop structure, and defines a map of spectra

$$\phi_{AA'A''} : Ext(A, A') \wedge Ext(A', A'') \quad \longrightarrow \quad Ext(A, A'').$$

and this map satisfies the associative law, up to coherent homotopies. It follows that the category  $w^{-1}C$  can be enriched over the homotopy category of spectra.

The results of Robinson [9] and the second author [7] are now immediate consequences. Let  $\mathcal{A}$  be the category of left modules over a ring  $R$ , and let  $C = C(\mathcal{A})$  be the Waldhausen category of chain complexes in  $\mathcal{A}$ . Given a left  $R$ -module  $B$  and a right  $R$ -module  $A$ ,

$$Tor_i^R(A, B) = \text{Hom}_{D(\mathcal{A})}(A^*, \Sigma^i B) = \text{Hom}_{w^{-1}C}(A^*, \Sigma^i B) = \pi_i [Ext(A^*, B)]$$

where  $A^* = R\text{Hom}(A, R)$  is the dual of  $A$ . In other words, the spectrum  $Ext(A^*, B)$  is in fact homotopy equivalent to Robinson's. The theorem of the second author is an even more direct consequence; one just looks at the special case where the category  $C = C(\mathcal{E})$  is the Waldhausen category of chain complexes in an exact category  $\mathcal{E}$ .

It follows directly from the above, and from the work of Porter in [5], that one can define higher products in any Waldhausen category, and they take their values in

the associated homotopy category. Porter’s machinery applies immediately. Another approach, which avoids using extensions, can be found in [8].

Perhaps one should make a comment regarding the proof. As is well-known, the dual of a Waldhausen category need not be Waldhausen, and Waldhausen categories are not in general additive. If one assumes at the start that we will only deal with Waldhausen categories  $C$  whose duals are Waldhausen in a compatible way, and such that  $C$  is additive, then the proof becomes substantially simpler. In particular, the cases where  $C = C(\mathcal{A})$  or  $C = C(\mathcal{E})$ , where  $\mathcal{A}$  is abelian and  $\mathcal{E}$  is exact are such very nice Waldhausen categories, and the theorem can be proved with substantially simpler arguments. This covers both the theorem of Robinson and of the second author.

However, it is not clear whether one learns anything new from this simple case. Let us spend a couple of paragraphs discussing the simple case, before returning to the more general situation of an arbitrary Waldhausen category. What makes the theory “special” in the additive case is

**Proposition 8.1** *Suppose the Waldhausen category  $C$  is additive. Then the spectra  $Ext(A, B)$  are all wedges of suspensions of Eilenberg–MacLane spectra.*

**Remark 0.5** Both Robinson [9] and the second author [7] prove versions of Proposition 8.1 appropriate to each of their theories.

As we have said, the authors are not entirely sure whether this makes the result completely trivial in the case of an additive Waldhausen category. Let us explain the difficulty by posing it as a problem to the reader. Let  $\mathcal{T}$  be the homotopy category of spectra,  $\mathcal{T}[\frac{1}{2}]$  the homotopy category of prime-to-2 spectra. Observe first that from the article [4] we know that there is a triangulated functor  $\Pi : \mathcal{T}[\frac{1}{2}] \rightarrow D(\mathbb{Z}[\frac{1}{2}])$  which takes a spectrum to a complex computing its stable homotopy. Note that when we restrict  $\Pi$  to spectra that are wedges of suspensions of Eilenberg–MacLane spectra, the result becomes trivial and there is no need to invert 2. There is also a functor  $K : D(\mathbb{Z}) \rightarrow \mathcal{T}$ , which takes a chain complex of abelian groups to the associated chain complex of Eilenberg–MacLane spectra. The existence of the functor  $K$  is essentially trivial, and can be seen in many ways. One way to see that the functor  $K$  exists is the following. There is a  $t$ -structure on the category of spectra whose heart is just the category of abelian groups, embedded in  $\mathcal{T}$  as the Eilenberg–MacLane spectra (see [4], Section 1). The functor  $D(\mathbb{Z}) \rightarrow \mathcal{T}$  is just the map from the derived category of the heart to the category with  $t$ -structure.

For any  $X$  which is a wedge of suspensions of Eilenberg–MacLane spectra, there is an isomorphism of  $X$  with  $K\Pi(X)$ . The isomorphism is decidedly *not* natural in  $X$ . What is not clear to the authors is

**Problem 0.6** Let  $C$  be an additive Waldhausen category. Is there a *natural* isomorphism

$$Ext(A, B) = K\Pi[Ext(A, B)]$$

where naturality means naturality in  $A$  and  $B$ ?

If the answer to Problem 0.6 is “yes”, then the spectrum  $Ext(A, B)$  carries no more information than the object  $\Pi[Ext(A, B)]$  of  $D(\mathbb{Z})$ . But in most cases of interest, it has been known for a long time that the derived category can be enriched over  $D(\mathbb{Z})$ ; this is the  $RHom$  construction. If the answer to Problem 0.6 is “no”, then this means that one cannot rectify the functor  $Ext(A, B)$  so that, in the wedges of suspensions of Eilenberg–MacLane spectra that come up, all the maps are in the image of the functor  $K$ . Recall that maps between wedges of suspensions of Eilenberg–MacLane spectra are generalised Steenrod operations. The ones in the image of the functor  $K$  are Bocksteins. To say that the answer to Problem 0.6 is “no” would imply that the higher Steenrod operations come naturally into the structure. That would be interesting.

While it is not clear that this article has proved anything the least bit profound about additive Waldhausen categories, the statement about arbitrary Waldhausen categories is interesting. Consider for example the category  $C$  of simplicial schemes. Variants of this category can be given the structure of Waldhausen categories, in a number of different ways. This is a subject with some subtle technicalities, which would be quite inappropriate for this article. Nevertheless, it seems to the authors that for a suitable choice of Waldhausen category  $C$ , the category  $w^{-1}C$  is Voevodsky’s triangulated category of mixed motives. There are many details we have not checked carefully, but this seems very probable.

Before this article, it does not seem to have been known that  $w^{-1}C$  can be enriched over  $D(\mathbb{Z})$ , let alone over the category  $\mathcal{T}$  of spectra. But the work of this article has an even more interesting bearing on Voevodsky’s work.

It is conjectured that the triangulated category of mixed motives is  $D(\mathcal{A})$ , the derived category of an *abelian* category  $\mathcal{A}$  of mixed motives. As a consequence of the results here, we now know that both  $D(\mathcal{A})$  and  $w^{-1}C$  can be enriched over the category of spectra. We also know from Proposition 8.1 that for  $D(\mathcal{A})$  this enriching structure is trivial; the spectra one gets are wedges of Eilenberg–MacLane spectra. It becomes natural to ask whether in Voevodsky’s category  $w^{-1}C$ , the spectra  $Ext(A, B)$  are all wedges of Eilenberg–MacLane spectra—equivalently, whether their Postnikov invariants all vanish.

Thus the really interesting consequence of the work presented here is that we have attached to every category  $w^{-1}C$  a set of invariants, namely the spectra  $Ext(A, B)$ . The Postnikov invariants of these spectra will vanish when  $C$  is additive, but not in general; for the homotopy category of finite spectra, they do not vanish. Thus we have an invariant that can distinguish “algebraic” triangulated categories from more general “topological” ones.

The only other such invariant that the authors are aware of comes from Franke’s work; see [3]. Franke shows that if one suitably enriches the categories in question,

then the only categories possessing Adams' spectral sequences with certain properties are derived categories.

The enriching structure of  $w^{-1}C$  over the category of spectra depends on the Waldhausen model we start with, but not much. A map  $C \rightarrow C'$  of Waldhausen models inducing an equivalence  $w^{-1}C \rightarrow w^{-1}C'$  will induce homotopy equivalences  $Ext_C(A, B) \rightarrow Ext_{C'}(A, B)$ . Thus if it so happens that in Voevodsky's category  $w^{-1}C$ , the spectra  $Ext(A, B)$  in general have non-vanishing Postnikov invariants, then we have not disproved the conjecture that there is an equivalence  $w^{-1}C \simeq D(\mathcal{A})$ . All we have shown is that such an equivalence cannot possibly be induced by a map of Waldhausen models.

It is known that there are triangulated functors of triangulated categories with no Waldhausen models. For two examples, see [4], Remarks 4.2 and 4.8. There is, however, no known example of an *equivalence* of triangulated categories with no model. Finding such an example would be very interesting. If nothing else, it would allow one to check a problem raised by Thomason in [10]. Thomason asks the following. Suppose  $C$  and  $C'$  are Waldhausen categories, and suppose  $w^{-1}C$  and  $w^{-1}C'$  are equivalent. Is it true that the  $K$ -theory spectra  $K(C)$  and  $K(C')$  are homotopy equivalent? It is a theorem of Waldhausen's (the approximation theorem) that if the equivalence is induced by an exact functor of the Waldhausen categories  $F : C \rightarrow C'$ , then in fact  $K(F) : K(C) \rightarrow K(C')$  is a homotopy equivalence.

Until now, it has been impossible to check Thomason's conjecture, since we have had no examples of Waldhausen categories  $C$  and  $C'$ , such that  $w^{-1}C$  and  $w^{-1}C'$  are equivalent, but the equivalence cannot be expressed as a sequence of exact functors of Waldhausen categories

$$\begin{array}{ccccccc}
 C = C_0 & & C_1 & & \dots & & C_{n-1} & & C_n = C' \\
 & \searrow & & \searrow & & \searrow & & \searrow & \\
 & & C'_0 & & & & C'_{n-1} & & 
 \end{array}$$

The above suggests a way to construct examples.

## 1 The total complex

A *Waldhausen category* will mean a category with cofibrations and weak equivalences, satisfying the Gluing Lemma, Saturation Axiom, Extension Axiom and Cylinder Axiom as in Waldhausen's foundational article [11]. There is one notation we will adopt that will differ from [11]; it is appropriate to alert the reader to it immediately. The cylinder functor will be denoted by  $Cyl(-)$ . This is of course quite different from Waldhausen, who denotes it  $T(-)$ . Thus, given a map  $f : X \rightarrow Y$ , then the object

$$Cyl\left(X \xrightarrow{f} Y\right)$$

is nothing other than the mapping cylinder of the map  $f$ . There is a cofibration

$$X \vee Y \longrightarrow \text{Cyl} \left( X \xrightarrow{f} Y \right)$$

which is the inclusion of the first and last face of the cylinder object, and there is a weak equivalence

$$\text{Cyl} \left( X \xrightarrow{f} Y \right) \longrightarrow Y$$

so that the composite

$$Y \hookrightarrow X \vee Y \longrightarrow \text{Cyl} \left( X \xrightarrow{f} Y \right) \longrightarrow Y$$

is the identity. The mapping cone  $\text{Cone} \left( X \xrightarrow{f} Y \right)$  is defined by the pushout square

$$\begin{array}{ccc} X & \longrightarrow & \text{Cyl} \left( X \xrightarrow{f} Y \right) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Cone} \left( X \xrightarrow{f} Y \right) \end{array}$$

In any Waldhausen category  $C$ , there is a suspension endomorphisms  $\Sigma : C \longrightarrow C$ . It is defined by sending an object  $X$  to the object  $\Sigma(X)$ , given by the pushout diagram

$$\begin{array}{ccc} X \vee X & \longrightarrow & \text{Cyl} \left( X \xrightarrow{1} X \right) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(X) \end{array}$$

and since  $X \vee X$  and  $\text{Cyl} \left( X \xrightarrow{1} X \right)$  are functorial in  $X$  and the inclusion of the front and back faces in the cylinder is a natural transformation, it is clear that the pushout diagram defining  $\Sigma(X)$  makes it a functor in  $X$ . For a general Waldhausen category, the functor  $\Sigma$  need not be invertible. However, it is an endomorphism of the category  $C$  preserving the Waldhausen structure. One can therefore form a Waldhausen category  $\Sigma^{-1}C$ , which is simply the direct limit category for the sequence

$$C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \dots$$

In the Waldhausen category  $\Sigma^{-1}C$ , the suspension functor is an invertible endomorphism of the category. From now on we will assume that all our Waldhausen categories have invertible suspension functors. All our theorems are therefore theorems about  $\Sigma^{-1}C$ .

Now we come to our first definition.

**Definition 1.1** Let  $C$  be a Waldhausen category. Let  $\mathbf{n}$  be the category

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow (n-1) \longrightarrow n$$

Let  $Cpx_n(C)$  be the category of complexes in  $C$  of length  $n$ ; that it is the full subcategory of the functor category  $\text{Hom}(\mathbf{n}, C)$  whose objects are the chain complexes. They are the sequences

$$X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_n$$

such that  $\partial \circ \partial = 0$ .

Now we are ready for the main definition of this section:

**Definition 1.2** Let  $C$  be a Waldhausen category, and let  $Cpx_n(C)$  be as in Definition 1.1. There is a functor  $T : Cpx_n(C) \longrightarrow C$  which totalises the complex. We define it inductively.

If  $n = 1$ , define  $T(X_0 \longrightarrow X_1)$  to be  $\text{Cone}(X_0 \longrightarrow X_1)$ .

Suppose  $T$  has been defined on complexes of length  $\leq (n-1)$ . There is a natural functor

$$Cpx_n(C) \longrightarrow \text{Hom}(\mathbf{1}, Cpx_{n-1}(C))$$

which takes the complex

$$X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_n$$

to the map of complexes

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & X_{n-2} & \xrightarrow{\partial} & X_{n-1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & X_n \end{array}$$

By induction, we already have the functor  $T$  defined on  $Cpx_{n-1}(C)$ ; thus we have an induced map

$$\begin{array}{c} T(X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_{n-2} \xrightarrow{\partial} X_{n-1}) \\ \alpha \downarrow \\ T(0 \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow X_n) = X_n \end{array}$$

and  $T(X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_n)$  is defined to be the mapping cone of  $\alpha$ .

**Remark 1.3** As has already been mentioned, we varied Waldhausen's notation to let  $\text{Cyl}(-)$  stand for the cylinder functor. The reason was that we wanted to reserve the letter  $T$  for the total complex of Definition 1.2.

In this section, we will recall briefly the elementary properties of the functor  $T$ . We begin with another definition:

**Definition 1.4** *Let  $C$  be a Waldhausen category. Let  $Cpx_n(C)$  be the category of length  $n$  complexes in  $C$ , as in Definition 1.1. Then a morphism*

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Y_0 & \xrightarrow{\partial} & Y_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n \end{array}$$

*in  $Cpx_n(C)$  is called a weak equivalence if all the vertical morphism in  $C$  are weak equivalences. The morphism is called a cofibration if all the vertical maps are cofibrations in  $C$ .*

The following lemmas are trivial and the proof is left to the reader.

**Lemma 1.5** *Let  $f : X. \rightarrow Y.$  be a morphism in  $Cpx_n(C)$ . If  $f$  is a cofibration (resp. weak equivalence), then so is  $T(f) : T(X.) \rightarrow T(Y.)$   $\square$*

**Lemma 1.6** *Let  $f : X. \rightarrow Y.$  be a cofibration in  $Cpx_n(C)$ , and let  $g : X. \rightarrow Z.$  be an arbitrary morphism. Then one can form the pushout  $Y. \vee_X Z.$  in  $Cpx_n(C)$ , which is the complex*

$$Y_0 \vee_{X_0} Z_0 \xrightarrow{\partial} Y_1 \vee_{X_1} Z_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} Y_{n-1} \vee_{X_{n-1}} Z_{n-1} \xrightarrow{\partial} Y_n \vee_{X_n} Z_n$$

*Pushouts of cofibrations are cofibrations, and the category  $Cpx_n(C)$  satisfies the Gluing Lemma and the Extension Axiom. To remind the reader: the gluing lemma says the following. Suppose we are given the two commutative squares*

$$\begin{array}{ccc} X. & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y. & \longrightarrow & Y' \end{array} \quad \begin{array}{ccc} X. & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z. & \longrightarrow & Z' \end{array}$$

*Suppose further that  $X. \rightarrow Y.$  and  $X' \rightarrow Y'$  are cofibrations, and that  $X. \rightarrow X'$ ,  $Y. \rightarrow Y'$  and  $Z. \rightarrow Z'$  are weak equivalences. Then the natural map*

$$Y. \vee_X Z. \longrightarrow Y' \vee_{X'} Z'$$

*is also a weak equivalence. The extension axiom says that suppose we are given a commutative square*

$$\begin{array}{ccc} X. & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y. & \longrightarrow & Y' \end{array}$$

*where the vertical maps are cofibrations, and suppose that  $X. \rightarrow X'$  and*

$$Y. \vee_X 0 \longrightarrow Y' \vee_{X'} 0$$

*are both weak equivalences. Then the map  $Y. \rightarrow Y'$  is also a weak equivalence.  $\square$*

## 2 The categories of acyclics

Let  $C$  be a Waldhausen category, and as always we assume it to have an invertible suspension functor. The category  $Acy_n(C)$  is the category of acyclic objects in  $Cpx_n(C)$ . Precisely

**Definition 2.1** *The category  $Acy_n(C)$  is the full subcategory of  $Cpx_n(C)$  whose objects are given by*

$$Acy_n(C) = \{X. \in Cpx_n(C) \mid \text{the zero map } 0 \longrightarrow T(X.) \text{ is a weak equivalence.}\}$$

**Lemma 2.2** *It is immediate from the Gluing Lemma for  $Cpx_n(C)$  that if  $X. \longrightarrow Y.$  is a cofibration,  $X. \longrightarrow Z.$  any morphism, and  $X., Y.$  and  $Z.$  are in  $Acy_n(C)$ , then so is  $Y. \vee_{X.} Z.$ . It is immediate from the Extension Axiom that if  $X. \longrightarrow Y.$  is a cofibration in  $Cpx_n(C)$  and  $X.$  as well as  $Y. \vee_{X.} 0$  are objects of  $Acy_n(C)$ , then  $Y.$  must also be an object of  $Acy_n(C)$ .  $\square$*

The important technical lemma of this section is:

**Lemma 2.3** *Let  $f : X. \longrightarrow Y.$  be a morphism in  $Acy_n(C)$ . Then it is possible to construct a diagram in  $Acy_n(C)$*

$$\begin{array}{ccc} X. & & Y. \\ & \searrow & \swarrow \\ & \bar{Y}. & \\ & \downarrow & \\ & Y. & \end{array}$$

where  $X. \longrightarrow \bar{Y}.$  and  $Y. \longrightarrow \bar{Y}.$  are cofibrations, and the composites

$$X. \longrightarrow \bar{Y}. \longrightarrow Y. \quad \text{and} \quad Y. \longrightarrow \bar{Y}. \longrightarrow Y.$$

are, respectively

$$f : X. \longrightarrow Y. \quad \text{and} \quad 1 : Y. \longrightarrow Y.$$

Furthermore, the map  $f$  is after all a map of complexes

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Y_0 & \xrightarrow{\partial} & Y_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n \end{array}$$

Suppose now that for some  $i$ , with  $0 \leq i \leq n$ , the map  $f_i : X_i \longrightarrow Y_i$  is a weak equivalence. Then so are  $X_i \longrightarrow \bar{Y}_i$ ,  $X_i \longrightarrow \bar{Y}_i$  and  $\bar{Y}_i \longrightarrow Y_i$ .

**Proof** Let us consider the diagram that comes from the Cylinder Axiom

$$\begin{array}{ccc}
 X. & & Y. \\
 & \searrow & \swarrow \\
 & \text{Cyl} \left( X. \xrightarrow{f} Y. \right) & \\
 & \downarrow & \\
 & Y. & 
 \end{array}$$

where  $\text{Cyl} \left( X. \xrightarrow{f} Y. \right)$  is the complex

$$\text{Cyl} \left( X_0 \xrightarrow{1} Y_0 \right) \xrightarrow{\partial} \text{Cyl} \left( X_1 \xrightarrow{f_0} Y_1 \right) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \text{Cyl} \left( X_n \xrightarrow{f_n} Y_n \right)$$

and the maps  $X. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right)$ ,  $Y. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right)$  and  $\text{Cyl} \left( X. \xrightarrow{f} Y. \right) \longrightarrow Y.$  are all the natural maps of chain complexes. It is immediate that  $X. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right)$  and  $Y. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right)$  are both cofibrations, and that the composites

$$X. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right) \longrightarrow Y. \quad \text{and} \quad Y. \longrightarrow \text{Cyl} \left( X. \xrightarrow{f} Y. \right) \longrightarrow Y.$$

are, respectively

$$f : X. \longrightarrow Y. \quad \text{and} \quad 1 : Y. \longrightarrow Y.$$

So we let  $\bar{Y}.$  be  $\text{Cyl} \left( X. \xrightarrow{f} Y. \right)$ , and we consider the diagram above. Two things remain to prove about the diagram. The first is that  $\text{Cyl} \left( X. \xrightarrow{f} Y. \right)$  is acyclic; that is that it lies in the category  $\text{Acy}_n(C)$ . The second is that if  $f_i : X_i \longrightarrow Y_i$  is a weak equivalence, then so are  $X_i \longrightarrow \bar{Y}_i$ ,  $X_i \longrightarrow \bar{Y}_i$  and  $\bar{Y}_i \longrightarrow Y_i$ .

We prove first the acyclicity. By the Cylinder Axiom for the Waldhausen category  $C$ , the map  $\text{Cyl} \left( X. \xrightarrow{f} Y. \right) \longrightarrow Y.$  is a weak equivalence. Hence the map

$$T \left( \text{Cyl} \left( X. \xrightarrow{f} Y. \right) \right) \longrightarrow T(Y.)$$

is also a weak equivalence. We know that the composite

$$0 \longrightarrow T \left( \text{Cyl} \left( X. \xrightarrow{f} Y. \right) \right) \longrightarrow T(Y.)$$

is a weak equivalence, because  $Y.$  is acyclic. It follows from the Saturation Axiom that

$$0 \longrightarrow T \left( \text{Cyl} \left( X. \xrightarrow{f} Y. \right) \right)$$

is a weak equivalence, and hence  $\bar{Y} = \text{Cyl}\left(X \xrightarrow{f} Y\right)$  is acyclic.

Now we already observed above that the map  $\bar{Y} = \text{Cyl}\left(X \xrightarrow{f} Y\right) \rightarrow Y$  is a weak equivalence by the Cylinder Axiom. The composite

$$Y \rightarrow \bar{Y} \rightarrow Y$$

is the identity, hence definitely a weak equivalence. It follows again from the Saturation Axiom that  $Y \rightarrow \bar{Y}$  is a weak equivalence. Thus, with no hypothesis at all on the map  $f : X \rightarrow Y$ , we always have that  $Y_i \rightarrow \bar{Y}_i$  and  $\bar{Y}_i \rightarrow Y_i$  are weak equivalences.

Suppose now that  $f_i : X_i \rightarrow Y_i$  is a weak equivalence. The map  $f_i$  can be expressed as the composite

$$X_i \rightarrow \bar{Y}_i \rightarrow Y_i;$$

the composite is a weak equivalence, as is the second map. It follows by the Saturation Axiom that  $X_i \rightarrow \bar{Y}_i$  is a weak equivalence.  $\square$

### 3 The categories of extensions

Now we come to the categories whose homotopy we will compute. For technical reasons, we will need several models for the same topological space. Let us therefore give a flexible definition.

**Definition 3.1** *Let  $C$  be a Waldhausen category. Let  $A$  and  $B$  be two fixed objects of  $C$ . Let  $n \geq 1$  be an integer. We would like to consider a number of categories, which we will denote by symbols like  $\text{Ext}^n(aa'A, bb'B)$ . Thus the symbol for the category is has 7 inputs which help specify which of the wide assortment of possible categories we want to consider. An object in one of these extension categories is a diagram*

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & & & \uparrow \\ X'_0 & & & & & & & & X'_{n+1} \\ \downarrow & & & & & & & & \downarrow \\ B & & & & & & & & A \end{array}$$

where the first row is an object of  $\text{Acy}_{n+1}(C)$ . That is, the row

$$X_0 \xrightarrow{\partial} X_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n+1}$$

is an acyclic complex in  $C$ . The objects  $A$  and  $B$  are our fixed objects in  $C$ , and in the symbol for the category  $\text{Ext}^n(aa'\mathbf{A}, bb'\mathbf{B})$  their position has been highlighted. This explains the role of the  $n$ , the  $A$  and the  $B$  in the symbol for the category. It remains to explain the significance of the  $a$ ,  $a'$ ,  $b$  and  $b'$ .

The symbols  $a$ ,  $a'$ ,  $b$  and  $b'$  tell us, respectively, the restrictions on the morphisms  $a : X'_{n+1} \longrightarrow A$ ,  $a' : X'_{n+1} \longrightarrow X_{n+1}$ ,  $b : X'_0 \longrightarrow B$  and  $b' : X'_0 \longrightarrow X_0$ . Thus, in the category

$$\text{Ext}^n(wcA, wwB)$$

the morphism  $a' : X'_{n+1} \longrightarrow X_{n+1}$  is assumed a cofibration, while the others, that is  $a : X'_{n+1} \longrightarrow A$ ,  $b : X'_0 \longrightarrow B$  and  $b' : X'_0 \longrightarrow X_0$ , are assumed weak equivalences. Thus  $w$  stands for weak equivalence,  $c$  for cofibration. If the morphism is unrestricted, we indicate this with the letter  $f$  for free. If it is restricted to be the identity, the symbol for that is  $=$ . If we assume the morphism to be simultaneously a cofibration and a weak equivalence, we will indicate this with an  $\bar{w}$ . Thus each of the letters  $a$ ,  $a'$ ,  $b$  and  $b'$  in the symbol  $\text{Ext}^n(aa'A, bb'B)$  is allowed to be any of  $f$ ,  $w$ ,  $c$ ,  $\bar{w}$  or  $=$ , and this gives  $5^4 = 625$  possibilities.

The morphisms in any of the above categories are just maps of diagrams, restricted to be the identity on  $A$  and  $B$ .

**Example 3.2** Consider for example the category  $\text{Ext}^n(==A, ==B)$ . An object is an acyclic complex

$$B \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_n \xrightarrow{\partial} A$$

and a morphism is a map of such complexes. Suppose now that  $C = C(\mathcal{A})$  is the Waldhausen category of chain complexes in an abelian category  $\mathcal{A}$  as in Example 0.1. Suppose furthermore that  $A$ ,  $B$  and all the  $X_i$  lie in the subcategory  $\mathcal{A} \subset C(\mathcal{A})$ , that is they are complexes supported in degree 0. Then to say that the complex totalises to an acyclic object is to say that it has no cohomology; the sequence

$$B \xrightarrow{\partial} X_1 \xrightarrow{\partial} \dots \xrightarrow{\partial} X_n \xrightarrow{\partial} A$$

had better be exact. Thus such an object of  $\text{Ext}^n(==A, ==B)$  is nothing more than an extension of length  $n$  of  $A$  by  $B$ . The morphisms are just morphisms of extensions.

From now on, we will be studying the homotopy of the categories of the form  $\text{Ext}^n(aa'A, bb'B)$ , and to make sense of this we better assume henceforth that  $C$  is a *small* category. Thus  $C$  has only a set of objects and a set of morphisms. This makes the categories  $\text{Ext}^n(aa'A, bb'B)$  also small. Their nerves are simplicial sets, which we can realise to get topological spaces. The topological space given by the realisation of the nerve of  $\text{Ext}^n(aa'A, bb'B)$  will be freely confused with the category  $\text{Ext}^n(aa'A, bb'B)$ . We will allow ourselves to say that two categories are homotopy equivalent, meaning that their realisations are.

**Lemma 3.3** *Let  $a$ ,  $a'$  and  $b$  be any of the possibilities  $f$ ,  $w$ ,  $c$ ,  $\bar{w}$  or  $=$ . The inclusions*

$$\text{Ext}^n(aa'A, b = B) \subset \text{Ext}^n(aa'A, b\bar{w}B) \subset \text{Ext}^n(aa'A, bwB)$$

*induce homotopy equivalences.*

**Proof** The inclusion functors

$$\begin{array}{ccc} \text{Ext}^n(aa'A, b = B) & \subset & \text{Ext}^n(aa'A, b\bar{w}B) \\ & \text{and} & \\ \text{Ext}^n(aa'A, b = B) & \subset & \text{Ext}^n(aa'A, bwB) \end{array}$$

each has a right adjoint, sending the diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & & & \uparrow \\ X'_0 & & & & & & & & X'_{n+1} \\ \downarrow & & & & & & & & \downarrow \\ B & & & & & & & & A \end{array}$$

to the diagram

$$\begin{array}{ccccccc} X'_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \parallel & & & & & & & & \uparrow \\ X'_0 & & & & & & & & X'_{n+1} \\ \downarrow & & & & & & & & \downarrow \\ B & & & & & & & & A \end{array}$$

□

**Lemma 3.4** *Let  $a, a'$  and  $b$  be any of the possibilities  $f, w, c, \bar{w}$  or  $=$ . The inclusion*

$$\text{Ext}^n(aa'A, = \bar{w}B) \subset \text{Ext}^n(aa'A, b\bar{w}B)$$

*induces a homotopy equivalence.*

**Proof** We will produce two natural transformations on  $\text{Ext}^n(aa'A, b\bar{w}B)$  which, taken together, connect the identity on  $\text{Ext}^n(aa'A, b\bar{w}B)$  to a map factoring through to the subset  $\text{Ext}^n(aa'A, = \bar{w}B)$ . The homotopy will be relative to the subset, proving that the inclusion of the subset is a homotopy equivalence.

The first natural transformation is a map  $F_1 \implies 1$ , and is given by the diagram

$$\begin{array}{cccccccccccc} X_0 & & \xrightarrow{\partial} & & X_1 & & \xrightarrow{\partial} & & X_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & & X_n & \xrightarrow{\partial} & & X_{n+1} \\ \uparrow & & & & \uparrow & & & & \uparrow & & & & & \uparrow & & & \uparrow \\ \text{Cyl}(X'_0 \longrightarrow X_0) & \xrightarrow{\partial} & \text{Cyl}(X'_0 \longrightarrow X_1) & \xrightarrow{\partial} & X_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & & & & X_n & \xrightarrow{\partial} & & X_{n+1} \\ \uparrow & & & & & & & & & & & & & & & & \uparrow \\ X'_0 & & & & & & & & & & & & & & & & X'_{n+1} \\ \downarrow & & & & & & & & & & & & & & & & \downarrow \\ B & & & & & & & & & & & & & & & & A \end{array}$$

The second natural transformation is a map  $F_1 \implies F_2$ , and is given by the diagram

$$\begin{array}{ccccccc}
B \vee_{X'_0} \text{Cyl}(X'_0 \longrightarrow X_0) & \xrightarrow{\partial} & B \vee_{X'_0} \text{Cyl}(X'_0 \longrightarrow X_1) & \xrightarrow{\partial} & X_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\text{Cyl}(X'_0 \longrightarrow X_0) & \xrightarrow{\partial} & \text{Cyl}(X'_0 \longrightarrow X_1) & \xrightarrow{\partial} & X_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
X'_0 & & & & & & & & X'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
B & & & & & & & & A
\end{array}$$

□

**Remark 3.5** So far, we have proved that the homotopy type of

$$\text{Ext}^n(aa'A, \{=, \bar{w}, w, c, f\} \{=, \bar{w}, w\} B)$$

depend only on  $a$  and  $a'$ . That means that if for  $b$  we take any of  $\{=, \bar{w}, w, c, f\}$  and for  $b'$  any of  $\{=, \bar{w}, w\}$ , the resulting simplicial set has a homotopy type depending only on  $a$  and  $a'$ .

**Lemma 3.6** *Let  $b$  and  $b'$  be any of the possibilities  $f, w, c, \bar{w}$  or  $=$ . The inclusions*

$$\text{Ext}^n(=\bar{w}A, bb'B) \subset \text{Ext}^n(\bar{w}\bar{w}A, bb'B) \subset \text{Ext}^n(w\bar{w}A, bb'B)$$

and

$$\text{Ext}^n(=cA, bb'B) \subset \text{Ext}^n(\bar{w}cA, bb'B) \subset \text{Ext}^n(wcA, bb'B)$$

induce homotopy equivalences.

**Proof** The point is that the four inclusion functors

$$\begin{array}{ll}
\text{Ext}^n(=\bar{w}A, bb'B) & \subset \text{Ext}^n(\bar{w}\bar{w}A, bb'B) \\
\text{Ext}^n(=\bar{w}A, bb'B) & \subset \text{Ext}^n(w\bar{w}A, bb'B) \\
\text{Ext}^n(=cA, bb'B) & \subset \text{Ext}^n(\bar{w}cA, bb'B) \\
\text{Ext}^n(=cA, bb'B) & \subset \text{Ext}^n(wcA, bb'B)
\end{array}$$

all have left adjoints, which send the object

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
X'_0 & & & & & & & & X'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
B & & & & & & & & A
\end{array}$$

to the object

$$\begin{array}{ccccccccccc}
 X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} & \vee_{X'_{n+1}} & A \\
 \uparrow & & & & & & & & & \uparrow & \\
 X'_0 & & & & & & & & & A & \\
 \downarrow & & & & & & & & & \parallel & \\
 B & & & & & & & & & A & 
 \end{array}$$

□

**Lemma 3.7** For  $a, b$  and  $b'$  arbitrary, the inclusions

$$Ext^n(a\bar{w}A, bb'B) \subset Ext^n(awA, bb'B)$$

and

$$Ext^n(acA, bb'B) \subset Ext^n(afA, bb'B)$$

induce homotopy equivalences.

**Proof** We will, in each case, produce a homotopy which connects the identity to a map factoring through the subset. In each case, the homotopy will be relative to the subset; hence the Lemma.

The homotopies are all given by a single natural transformation  $\eta : F \implies 1$ . On an object  $s$  of the category, that is on a diagram

$$\begin{array}{ccccccccccc}
 X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} & & \\
 \uparrow & & & & & & & & \uparrow & & \\
 X'_0 & & & & & & & & X'_{n+1} & & \\
 \downarrow & & & & & & & & \downarrow & & \\
 B & & & & & & & & A & & 
 \end{array}$$

the natural transformation  $\eta$  is determined by giving the morphism  $\eta(s)$ , which is the diagram

$$\begin{array}{ccccccccccc}
 X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} & & \\
 \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
 X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & Cyl(X'_{n+1} \longrightarrow X_{n+1}) & & \\
 \uparrow & & & & & & & & \uparrow & & \\
 X'_0 & & & & & & & & X'_{n+1} & & \\
 \downarrow & & & & & & & & \downarrow & & \\
 B & & & & & & & & A & & 
 \end{array}$$

□

**Remark 3.8** From what we have proved so far, it certainly follows that for the following two classes of categories

$$Ext^n (\{=, \bar{w}, w\} \{\bar{w}, w\} A, \{=, \bar{w}, w, c, f\} \{=, \bar{w}, w\} B)$$

and

$$Ext^n (\{=, \bar{w}, w\} \{c, f\} A, \{=, \bar{w}, w, c, f\} \{=, \bar{w}, w\} B)$$

the homotopy of the category depends only on its class. That means that the natural inclusions among the categories in either class induce homotopy equivalences. We would like to prove that these two classes are the same. We need to show that some inclusion of a category in one class to a category in the other class induces a homotopy equivalence.

**Notation 3.9** From now on, we will frequently consider cylinders on identity maps. We will write  $Cyl(X)$  for  $Cyl(X \xrightarrow{1} X)$ . We will also need a notation for wedges of such cylinders. Thus, given a pair of maps

$$\begin{array}{ccc} S & & X \\ & \swarrow & \nearrow \\ & R & \end{array}$$

with  $R \longrightarrow S$  a cofibration, then

$$S \vee_R Cyl(X)$$

will stand for the pushout where we attach  $S$  to  $Cyl(X)$  via the inclusion of  $R$  in the first face of  $Cyl(X)$ , while

$$Cyl(X) \vee_R S$$

will stand for attaching via the inclusion of  $R$  in the second face. We can even string these together, forming objects like

$$S' \vee_{R'} Cyl(S) \vee_R Cyl(X)$$

which means that  $S'$  is attached to the first face of  $Cyl(S)$ , then the second face of  $Cyl(S)$  is attached to the first face of  $Cyl(X)$  over  $R$ . The reader can amuse himself with other possibilities.

If the object we are attaching is the 0 object, we will leave it out. Thus, as a matter of notation

$${}_R Cyl(X) = 0 \vee_R Cyl(X)$$

In particular,  $\Sigma X$ , which is defined to be the quotient of  $Cyl(X)$  by the inclusion of the front and back face, can be denoted

$$\Sigma X = {}_X Cyl(X)_X = 0 \vee_X Cyl(X) \vee_X 0$$

**Lemma 3.10** *The inclusion*

$$\text{Ext}^n(\text{ww}A, \text{ww}B) \subset \text{Ext}^n(\text{wf}A, \text{ww}B)$$

*induces a homotopy equivalence.*

**Remark 3.11** We will actually need a refinement of Lemma 3.10. Consider the category  $\text{Ext}^n(C \rightarrow A, a', B)$  whose objects are diagrams

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & \uparrow & & \\ X'_0 & & & & & & X'_{n+1} & \longrightarrow & A \\ \downarrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & & & & Y' & \longrightarrow & Y' & \longrightarrow & C \end{array}$$

where the object  $B$  and the map  $C \rightarrow A$  is given and fixed, and the restrictions on the diagram are that the top row is acyclic, the maps  $b : X'_0 \rightarrow B$ ,  $b' : X'_0 \rightarrow X_0$ ,  $Y' \rightarrow C$  and  $a : X'_{n+1} \rightarrow A$  are all weak equivalences, while  $a' : X'_{n+1} \rightarrow X_{n+1}$  is of type  $a'$ , as indicated in the symbol of the category. If we forget  $Y'$  and  $C$ , what we have is exactly  $\text{Ext}^n(\text{wa}'A, \text{ww}B)$ . The refinement we need is that the natural inclusion

$$\text{Ext}^n(C \rightarrow A, w, B) \subset \text{Ext}^n(C \rightarrow A, f, B)$$

induces a homotopy equivalence.

**Proof** Once again, we will produce a homotopy connecting the identity on  $\text{Ext}^n(\text{wf}A, \text{ww}B)$  to a map factoring through  $\text{Ext}^n(\text{ww}A, \text{ww}B)$ . The homotopy will be relative to the inclusion of the subset  $\text{Ext}^n(\text{ww}A, \text{ww}B)$ , and hence the Lemma. We will give the homotopy on the more complicated pair of simplicial sets  $\text{Ext}^n(C \rightarrow A, w, B) \subset \text{Ext}^n(C \rightarrow A, f, B)$ ; to obtain a proof of Lemma 3.10, the reader should simply delete  $Y$  and  $C$  from our diagrams.

The difficult thing about this homotopy is that +we will give a suspension of it. Recall that in the Waldhausen category  $C$ , the suspension functor has been inverted. We are therefore free to give the suspension of a natural transformation, and the reader should everywhere supply desuspensions for the diagrams we write.

Other than this difficulty, the homotopy could hardly be simpler. It is given by exactly one natural transformation  $F \Rightarrow \Sigma$ . The functor  $F$  takes the object

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & \uparrow & & \\ X'_0 & & & & & & X'_{n+1} & \longrightarrow & A \\ \downarrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & & & & Y' & \longrightarrow & Y' & \longrightarrow & C \end{array}$$

to the object

$$\begin{array}{ccccccc}
\bar{X}_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \bar{X}_{n-1} & \xrightarrow{\partial} & \bar{X}_n & \xrightarrow{\partial} & \bar{X}_{n+1} \\
\uparrow & & & & & & \uparrow & & \uparrow \\
\bar{X}'_0 & & & & & & \bar{X}'_{n+1} & \longrightarrow & \Sigma A \\
\downarrow & & & & & & \uparrow & & \uparrow \\
\Sigma B & & & & \bar{Y}' & \longrightarrow & \bar{Y}' & \longrightarrow & \Sigma C
\end{array}$$

where for all but two terms  $Z$  in the diagram,  $\bar{Z}$  means  ${}_Z Cyl(Z) \vee_Z Cyl(Z)_Z$ . The two exceptions are  $Z = X_n$  and  $Z = X_{n+1}$ . For these two,  $\bar{Z}$  is defined by the formula

$$\bar{X}_n = {}_{X'_{n+1}} Cyl(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \vee_{X_n} Cyl(X_n)_{X_n}$$

and

$$\bar{X}_{n+1} = {}_{X'_{n+1}} Cyl(X'_{n+1}) \vee_{X'_{n+1}} Cyl(X_{n+1})_{X_{n+1}}$$

In every case, the natural transformation  $F \implies \Sigma$  is given by collapsing the first cylinder. Thus for  $Z$  not  $X_n$  or  $X_{n+1}$ , the map  $\bar{Z} \longrightarrow \Sigma Z$  is

$${}_Z Cyl(Z) \vee_Z Cyl(Z)_Z \longrightarrow {}_0 Cyl(0) \vee_Z Cyl(Z)_Z = {}_Z Cyl(Z)_Z = \Sigma Z.$$

For the exceptional cases  $Z = X_n$  or  $X_{n+1}$ , the map is still the collapse;  $\bar{X}_n \longrightarrow \Sigma X_n$  is

$$\bar{X}_n = {}_{X'_{n+1}} Cyl(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \vee_{X_n} Cyl(X_n)_{X_n} \longrightarrow {}_0 Cyl(0) \vee_0 Cyl(X_n)_{X_n} = \Sigma X_n$$

while for  $\bar{X}_{n+1} \longrightarrow \Sigma X_{n+1}$  the map is

$$\bar{X}_{n+1} = {}_{X'_{n+1}} Cyl(X'_{n+1}) \vee_{X'_{n+1}} Cyl(X_{n+1})_{X_{n+1}} = {}_0 Cyl(0) \vee_{X_{n+1}} Cyl(X_{n+1})_{X_{n+1}} = \Sigma X_{n+1}$$

Of course, it need hardly be added that the structure maps  $\bar{Y}' \longrightarrow \Sigma C$ ,  $\bar{X}'_{n+1} \longrightarrow \Sigma A$  and  $\bar{X}'_0 \longrightarrow \Sigma B$  are also given by the collapse of the first cylinder.

We leave to the reader most of the checking that this is a well-defined natural transformation. Let us just verify here that the sequence

$$\bar{X}_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bar{X}_{n-1} \xrightarrow{\partial} \bar{X}_n \xrightarrow{\partial} \bar{X}_{n+1}$$

is an acyclic complex in the category  $C$ . In any case, we have a diagram

$$\begin{array}{cccccccc}
\Sigma X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Sigma X_{n-1} & \xrightarrow{\partial} & \Sigma X_n & \xrightarrow{\partial} & \Sigma X_{n+1} \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\overline{X}_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \overline{X}_{n-1} & \xrightarrow{\partial} & \overline{X}_n & \xrightarrow{\partial} & \overline{X}_{n+1} \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\overline{\overline{X}}_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \overline{\overline{X}}_{n-1} & \xrightarrow{\partial} & \overline{\overline{X}}_n & \xrightarrow{\partial} & \overline{\overline{X}}_{n+1}
\end{array}$$

where  $\Sigma X_i$  is the quotient of  $\overline{X}_i$  by  $\overline{\overline{X}}_i$ . In fact, for  $i \neq n, (n+1)$ , the cofibration  $\overline{\overline{X}}_i \rightarrow \overline{X}_i$  is given by the inclusion into the first cylinder

$${}_{X_i} \text{Cyl}(X_i) \longrightarrow {}_{X_i} \text{Cyl}(X_i) \vee_{X_i} \text{Cyl}(X_i)_{X_i}$$

The inclusion  $\overline{\overline{X}}_n \rightarrow \overline{X}_n$  is

$${}_{X'_{n+1}} \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \longrightarrow {}_{X'_{n+1}} \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \vee_{X_n} \text{Cyl}(X_n)_{X_n}$$

whereas the inclusion  $\overline{\overline{X}}_{n+1} \rightarrow \overline{X}_{n+1}$  is

$${}_{X'_{n+1}} \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \longrightarrow {}_{X'_{n+1}} \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} \text{Cyl}(X_{n+1})_{X_{n+1}}$$

From the above it follows that, for  $i < n$ ,  $\overline{\overline{X}}_i$  is weakly equivalent to 0, while  $\overline{\overline{X}}_n \rightarrow \overline{\overline{X}}_{n+1}$  is an isomorphism. Hence the sequence

$$\overline{\overline{X}}_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \overline{\overline{X}}_{n-1} \xrightarrow{\partial} \overline{\overline{X}}_n \xrightarrow{\partial} \overline{\overline{X}}_{n+1}$$

is acyclic, as is

$$\Sigma X_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Sigma X_{n-1} \xrightarrow{\partial} \Sigma X_n \xrightarrow{\partial} \Sigma X_{n+1}$$

By the Extension Axiom (see Lemma 2.2), it follows that the sequence

$$\overline{X}_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \overline{X}_{n-1} \xrightarrow{\partial} \overline{X}_n \xrightarrow{\partial} \overline{X}_{n+1}$$

is also acyclic. □

**Remark 3.12** All the results of the section put together certainly imply that the homotopy type of

$$\text{Ext}^n(\{=, \overline{w}, w\} \{ \overline{w}, w, c, f\} A, \{=, \overline{w}, w, c, f\} \{=, \overline{w}, w\} B)$$

is well-defined; this means, for any category  $\text{Ext}^n(aa'A, bb'B)$  where  $a, a', b$  and  $b'$  satisfy the above conditions, the homotopy type of the category is the same. In fact, the natural inclusions of these categories all induce homotopy equivalences.

## 4 A contractible category

Consider now the category  $EXT^n(A, B)$ , whose objects are diagrams

$$\begin{array}{cccccccc}
 Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & 0 \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
 X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
 \uparrow & & & & & & & & \uparrow \\
 X'_0 & & & & & & & & X'_{n+1} \\
 \downarrow & & & & & & & & \downarrow \\
 B & & & & & & & & A
 \end{array}$$

where we are assuming that both the row

$$X_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n+1}$$

and the row

$$Y_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} Y_{n-1} \xrightarrow{\partial} Y_n \xrightarrow{\partial} 0$$

are acyclic, i.e. are objects of  $Acy_{n+1}(C)$ . The maps  $a : X'_{n+1} \rightarrow A$ ,  $a' : X'_{n+1} \rightarrow X_{n+1}$ ,  $b : X'_0 \rightarrow B$  and  $b' : X'_0 \rightarrow X_0$  and  $b'' : X'_0 \rightarrow Y_0$  are all weak equivalences, but not necessarily anything more. There is a natural forgetful functor

$$EXT^n(A, B) \longrightarrow Ext^n(wwA, wwB)$$

which forgets the  $Y$ 's; it sends the object above to the diagram

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
 \uparrow & & & & & & & & \uparrow \\
 X'_0 & & & & & & & & X'_{n+1} \\
 \downarrow & & & & & & & & \downarrow \\
 B & & & & & & & & A
 \end{array}$$

In this section, we will prove that the category  $EXT^n(A, B)$  is contractible. In the next section, we will show that the functor above, namely

$$F : EXT^n(A, B) \longrightarrow Ext^n(wwA, wwB)$$

is a quasifibration, and the fiber can easily be computed to be  $Ext^{n-1}(=fA, =wB)$ . Modulo the results of the last section, we know that  $Ext^{n-1}(=fA, =wB)$  can be naturally identified with  $Ext^{n-1}(wwA, wwB)$  (among many other models). Thus the loop space of  $Ext^n(wwA, wwB)$  is  $Ext^{n-1}(wwA, wwB)$ .

It turns out to be surprisingly hard to prove the contractibility of  $EXT^n(A, B)$ , and much easier to prove that the functor  $F : EXT^n(A, B) \longrightarrow Ext^n(wwA, wwB)$  is a quasifibration. The difficulty with the contractibility proof is that, once again, our homotopies are defined on suspensions.

**Proposition 4.1** *The category  $EXT^n(A, B)$  above is contractible.*

**Proof** We will give a series of functors and natural transformations, starting with the suspension on  $EXT^n(A, B)$  and ending with a contraction. We begin with a functor  $F_1$  and a natural transformation  $F_1 \implies \Sigma$ . Let  $s$  be the object

$$\begin{array}{ccccccc}
Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & 0 \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
X'_0 & & & & & & & & X'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
B & & & & & & & & A
\end{array}$$

of the category  $EXT^n(A, B)$ . Then  $F_1(s)$  is the diagram

$$\begin{array}{ccccccc}
\Sigma Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Sigma Y_{n-1} & \xrightarrow{\partial} & \Sigma Y_n & \xrightarrow{\partial} & 0 \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\overline{X}_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \overline{X}_{n-1} & \xrightarrow{\partial} & \overline{X}_n & \xrightarrow{\partial} & \overline{X}_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
\overline{X}'_0 & & & & & & & & \overline{X}'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
\Sigma B & & & & & & & & \Sigma A
\end{array}$$

where  $\overline{Z}$  is  ${}_Z Cyl(Z) \vee_Z Cyl(Z)_Z$ , and any map  $\overline{Z} \longrightarrow \Sigma W$  in the diagram is the collapse of the first cylinder. The natural transformation  $F_1 \implies \Sigma$  is also the collapse of the first cylinder.

Next we give a natural transformation  $F_1 \implies F_2$ . The object  $F_2(s)$  is given by the diagram

$$\begin{array}{ccccccc}
\Sigma Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Sigma Y_{n-1} & \xrightarrow{\partial} & \Sigma Y_n & \xrightarrow{\partial} & 0 \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\Sigma Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Sigma Y_{n-1} & \xrightarrow{\partial} & \Sigma X_{n+1} \vee \Sigma Y_n & \xrightarrow{\partial} & \Sigma X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
\overline{X}'_0 & & & & & & & & \overline{X}'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
\Sigma B & & & & & & & & \Sigma A
\end{array}$$

The natural map  $F_1 \implies F_2$  is only interesting on the  $X_i$ 's. For  $i \leq (n-1)$ , The map  $\overline{X}_i \longrightarrow \Sigma X_i$  is just the collapse of the first cylinder. The map  $\overline{X}_{n+1} \longrightarrow \Sigma X_{n+1}$  is the collapse of the *second* cylinder. The map  $\overline{X}_n \longrightarrow \Sigma X_{n+1} \vee \Sigma Y_n$  is the map which takes the first cylinder to  $\Sigma X_{n+1}$ , the second to  $\Sigma Y_n$ . That is, it is the map

$${}_{X_n} \text{Cyl}(X_n) \vee_{X_n} \text{Cyl}(X_n)_{X_n} \longrightarrow {}_{X_{n+1}} \text{Cyl}(X_{n+1}) \vee_{X_{n+1} \vee Y_n} \text{Cyl}(Y_n)_{Y_n} = \Sigma X_{n+1} \vee \Sigma Y_n$$

Next, we follow with a natural transformation  $F_3 \implies F_2$ . The object  $F_3(s)$  is the diagram

$$\begin{array}{cccccccccccc} \overline{X}'_0 & \xrightarrow{\partial} & \overline{X}'_0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ \overline{X}'_0 & \xrightarrow{\partial} & \overline{X}'_0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \overline{X}'_{n+1} & \longrightarrow & \overline{X}'_{n+1} \\ \uparrow & & & & & & & & & & \uparrow & & \uparrow \\ \overline{X}'_0 & & & & & & & & & & \overline{X}'_{n+1} & & \overline{X}'_{n+1} \\ \downarrow & & & & & & & & & & \downarrow & & \downarrow \\ \Sigma B & & & & & & & & & & \Sigma A & & \Sigma A \end{array}$$

and the natural transformation  $F_3 \implies F_2$  is obvious. To finish matters off, consider the natural transformation  $F_3 \implies F_4$ , where  $F_4(s)$  is the diagram

$$\begin{array}{cccccccccccc} \Sigma B & \xrightarrow{\partial} & \Sigma B & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ \Sigma B & \xrightarrow{\partial} & \Sigma B & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \Sigma A & \longrightarrow & \Sigma A \\ \uparrow & & & & & & & & & & \uparrow & & \uparrow \\ \Sigma B & & & & & & & & & & \Sigma A & & \Sigma A \\ \downarrow & & & & & & & & & & \downarrow & & \downarrow \\ \Sigma B & & & & & & & & & & \Sigma A & & \Sigma A \end{array}$$

and since  $F_4$  is a contraction, we are done.  $\square$

## 5 A quasi-fibration

The key result of this section will be:

**Lemma 5.1** *The natural projection*

$$EXT^n(A, B) \longrightarrow Ext^n(wwA, wwB)$$

*satisfies the hypotheses of Quillen's Theorem B. That is, given a morphism  $s \longrightarrow s'$  in the category  $Ext^n(wwA, wwB)$ , the induced map of comma categories  $s' \setminus F \longrightarrow s \setminus F$  is a homotopy equivalence.*

**Proof** Recall that an object in the category  $s' \setminus F$  is an object  $x$  of  $EXT^n(A, B)$  together with a map  $s' \longrightarrow F(x)$ . The map  $s' \setminus F \longrightarrow s \setminus F$  is induced by composing  $s' \longrightarrow F(x)$  with the given map  $s \longrightarrow s'$ .

Let  $s$  be given by the diagram

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
X'_0 & & & & & & & & X'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
B & & & & & & & & A
\end{array}$$

and  $s'$  by the diagram

$$\begin{array}{ccccccc}
Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & Y_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
Y'_0 & & & & & & & & Y'_{n+1} \\
\downarrow & & & & & & & & \downarrow \\
B & & & & & & & & A
\end{array}$$

We are given a morphism  $s \longrightarrow s'$ . I assert that if  $s \longrightarrow s'$  is such that the induced map

$$\begin{array}{ccccccc}
Y_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & Y_{n+1} \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1}
\end{array}$$

is a *cofibration*, then the natural map  $s' \setminus F \longrightarrow s \setminus F$  has a left adjoint. That is, if for each  $i$ ,  $f_i : X_i \longrightarrow Y_i$  is a cofibration, then there is a left adjoint. The adjoint is given simply by pushing out along the cofibrations  $X_i \longrightarrow Y_i$ .

The Lemma will therefore follow immediately if we can reduce to this case. Now recall that by Lemma 2.3, any map  $f : X. \longrightarrow Y.$  in  $Acy_{n+1}(C)$  can be extended to a diagram

$$\begin{array}{ccc}
X. & & Y. \\
& \searrow & \swarrow \\
& \bar{Y}. & \\
& \downarrow & \\
& Y. &
\end{array}$$

where  $X. \longrightarrow \bar{Y}.$  and  $Y. \longrightarrow \bar{Y}.$  are cofibrations, and the composites

$$X. \longrightarrow \bar{Y}. \longrightarrow Y. \quad \text{and} \quad Y. \longrightarrow \bar{Y}. \longrightarrow Y.$$

are, respectively

$$f : X. \longrightarrow Y. \quad \text{and} \quad 1 : Y. \longrightarrow Y.$$

and where furthermore if  $f_i : X_i \longrightarrow Y_i$  is a weak equivalence, then so are all the  $i^{th}$  maps in the new diagram; that is

$$\begin{array}{ccc}
X_i & & Y_i \\
& \searrow & \swarrow \\
& \bar{Y}_i & \\
& \downarrow & \\
& Y_i & 
\end{array}$$

are all weak equivalences. In our case, we have a map  $X \rightarrow Y$  such that  $f_0 : X_0 \rightarrow Y_0$  and  $f_{n+1} : X_{n+1} \rightarrow Y_{n+1}$  are weak equivalences. Applying Lemma 2.3, we deduce a diagram in the category  $Ext^n(wwA, wwB)$

$$\begin{array}{ccc}
s & & s' \\
& \searrow & \swarrow \\
& \bar{s} & \\
& \downarrow & \\
& s' & 
\end{array}$$

where  $s \rightarrow \bar{s}$  and  $s' \rightarrow \bar{s}$  yield cofibrations in  $Acy_{n+1}(C)$ , and the composites

$$s \rightarrow \bar{s} \rightarrow s' \quad \text{and} \quad s' \rightarrow \bar{s} \rightarrow s'$$

are, respectively

$$f : s \rightarrow s' \quad \text{and} \quad 1 : s' \rightarrow s'.$$

We already know that  $s \rightarrow \bar{s}$  and  $s' \rightarrow \bar{s}$  induce homotopy equivalences  $\bar{s} \setminus F \rightarrow s \setminus F$  and  $\bar{s} \setminus F \rightarrow s' \setminus F$ . On the other hand, the composite  $s' \rightarrow \bar{s} \rightarrow s'$  is the identity, and hence definitely induces a homotopy equivalence. It follows that  $\bar{s} \rightarrow s'$  induces a homotopy equivalence  $s' \setminus F \rightarrow \bar{s} \setminus F$ . Therefore, composition with  $f : s \rightarrow s'$ , which can be factored as composition with  $\bar{s} \rightarrow s'$  followed by composition with  $s \rightarrow \bar{s}$ , induces a homotopy equivalence  $s' \setminus F \rightarrow s \setminus F$ .  $\square$

It follows that the homotopy type of the fiber  $s \setminus F$  is independent of  $s$ , and is homotopy equivalent to the homotopy fiber. Since  $EXT^n(A, B)$  is contractible, this is a model for the loop space of  $Ext^n(wwA, wwB)$ . We will now prove:

**Theorem 5.2** *The loop space of  $Ext^n(wwA, wwB)$  is naturally identified with  $Ext^{n-1}(= fA, = wB)$ .*

**Proof** By Lemma 5.1 it remains only to identify the space  $s \setminus F$  with  $Ext^{n-1}(= fA, = wB)$  for some suitable choice of  $s$ . We choose  $s$  to be the object

$$\begin{array}{cccccccc}
B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A \\
\uparrow & & & & & & & & & & & & \uparrow \\
B & & & & & & & & & & & & A \\
\downarrow & & & & & & & & & & & & \downarrow \\
B & & & & & & & & & & & & A
\end{array}$$

Observe now that an object in  $s \setminus F$  is a morphism  $s \longrightarrow F(t)$ , where the object  $t$  is given by the diagram

$$\begin{array}{ccccccccccccccc}
 Y_0 & \xrightarrow{\partial} & Y_1 & \xrightarrow{\partial} & Y_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & 0 \\
 \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
 X_0 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & X_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
 \uparrow & & & & & & & & & & & & \uparrow \\
 X'_0 & & & & & & & & & & & & X'_{n+1} \\
 \downarrow & & & & & & & & & & & & \downarrow \\
 B & & & & & & & & & & & & A
 \end{array}$$

Inside the category  $s \setminus F$  is a subcategory  $s = F$ , the full subcategory of objects so that  $s \longrightarrow F(t)$  is the identity. I assert first that the inclusion  $\{s = F\} \subset \{s \setminus F\}$  has a right adjoint. It is the functor sending  $t$  above to

$$\begin{array}{ccccccccccccccc}
 Y_0 & \xrightarrow{\partial} & Y_1 & \xrightarrow{\partial} & Y_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n & \xrightarrow{\partial} & 0 \\
 \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
 B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A \\
 \uparrow & & & & & & & & & & & & \uparrow \\
 B & & & & & & & & & & & & A \\
 \downarrow & & & & & & & & & & & & \downarrow \\
 B & & & & & & & & & & & & A
 \end{array}$$

It therefore suffices to determine the homotopy type of the subcategory  $s = F$ . But as a category, this is visibly nothing other than  $Ext^{n-1}(= fA, = wB)$ . After all, most of the diagram above is redundant. The part that matters is

$$\begin{array}{ccccccccccc}
 Y_0 & \xrightarrow{\partial} & Y_1 & \xrightarrow{\partial} & Y_2 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & Y_{n-1} & \xrightarrow{\partial} & Y_n \\
 \uparrow & & & & & & & & & & \uparrow \\
 B & & & & & & & & & & A
 \end{array}$$

and the restrictions on the morphisms are as stated;  $B \longrightarrow Y_0$  is a weak equivalence, while  $A \longrightarrow Y_n$  is free.  $\square$

**Remark 5.3** The case where  $n = 1$  deserves special mention, since the loop space of  $Ext^1(wA, wB)$  has been identified with  $Ext^0(= fA, = wB)$ , and until now we have explicitly excluded  $Ext^0$ . Of course, the proof of Theorem 5.2 is valid in the case  $n = 1$ , and the loop space, which we will call  $Ext^0(= fA, = wB)$ , can be defined to be what we computed it to be; it is the category of diagrams

$$\begin{array}{ccc}
 Y_0 & \longrightarrow & Y_1 \\
 \uparrow & & \uparrow \\
 B & & A
 \end{array}$$

where  $B \rightarrow Y_0$  is a weak equivalence,  $A \rightarrow Y_1$  is free, and  $Y_0 \rightarrow Y_1$  is acyclic. It follows that  $\Sigma Y_0 \rightarrow \Sigma Y_1$  is a weak equivalence, and since the functor  $\Sigma$  is invertible, that  $Y_0 \rightarrow Y_1$  is a weak equivalence.

What is not true is that the results of Section 3 apply. We have not proved, for the case  $n = 0$ , that the category  $Ext^n(aa'A, bb'B)$  has a homotopy type which, for a large class of possible  $a, a', b$  and  $b'$  is independent of  $a, a', b$  and  $b'$ . Not only did we not prove it, but it turns out to be false. The reason we excluded the case  $n = 0$  before is that it is something of a special case.

However, let us include the following two lemmas, which deal with the homotopy of  $Ext^0$ .

**Lemma 5.4** *The natural inclusion*

$$Ext^0(= fA, == B) \subset Ext^0(= fA, = wB)$$

*induces a homotopy equivalence.*

**Proof** The inclusion has a right adjoint which sends the object

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y_1 \\ \uparrow & & \uparrow \\ B & & A \end{array}$$

to the object

$$\begin{array}{ccc} B & \longrightarrow & Y_1 \\ \parallel & & \uparrow \\ B & & A \end{array}$$

□

**Lemma 5.5** *Consider the subcategory of  $Hom^h(A, B) \subset Ext^0(= fA, == B)$  consisting of diagrams*

$$\begin{array}{ccc} B & \longrightarrow & Y_1 \\ \parallel & & \uparrow \\ B & & A \end{array}$$

*where the morphism  $B \rightarrow Y_1$  is not just a weak equivalence, but also a cofibration. In our notation, it is of type  $\bar{w}$ . Then the inclusion  $Hom^h(A, B) \subset Ext^0(= fA, == B)$  induces a homotopy equivalence.*

**Proof** We give a natural transformation  $F \Rightarrow 1$ , where  $F$  factors through the inclusion of the subcategory  $Hom^h(A, B) \subset Ext^0(= fA, == B)$ . The natural transformation takes the object

$$\begin{array}{ccc} B & \longrightarrow & Y_1 \\ \parallel & & \uparrow \\ B & & A \end{array}$$

to the morphism

$$\begin{array}{ccc} B & \longrightarrow & Y_1 \\ \parallel & & \uparrow \\ B & \longrightarrow & \text{Cyl}(B \longrightarrow Y_1) \\ \parallel & & \uparrow \\ B & & A \end{array}$$

□

The category  $\text{Hom}^h(A, B)$  will be our preferred model for  $\Omega^n |\text{Ext}^n(wwA, wwB)|$ , the  $n$ -fold loop space on  $\text{Ext}^n(wwA, wwB)$ .

## 6 A computation of $\pi_n(\text{Ext}^n(wwA, wwB))$

The  $n^{\text{th}}$  homotopy group of  $\text{Ext}^n(wwA, wwB)$  is naturally identified with the group of components of its  $n$ -fold loop space. We know by Section 5 that  $\text{Hom}^h(A, B)$  is a model for this  $n$ -fold loop space. What is its group of components?

Define a category, denoted  $w^{-1}C$ , whose objects are the objects of  $C$  and whose morphisms are elements of  $\pi_0(\text{Hom}^h(A, B))$ . Composition in this category is defined as follows. There is a continuous map

$$\phi_{ABC} : \text{Hom}^h(A, B) \times \text{Hom}^h(B, C) \longrightarrow \text{Hom}^h(A, C)$$

It is given by a functor of the corresponding categories; given an object of  $\text{Hom}^h(A, B)$

$$\begin{array}{ccc} B & \longrightarrow & Y_1 \\ \parallel & & \uparrow \\ B & & A \end{array}$$

and an object of  $\text{Hom}^h(B, C)$

$$\begin{array}{ccc} C & \longrightarrow & Y_2 \\ \parallel & & \uparrow \\ C & & B \end{array}$$

one obtains an object of  $\text{Hom}^h(A, C)$

$$\begin{array}{ccc} C & \longrightarrow & Z \\ \parallel & & \uparrow \\ C & & A \end{array}$$

from the diagram

$$\begin{array}{ccccc}
C & \longrightarrow & Y_2 & \longrightarrow & Z \\
\parallel & & \uparrow & & \uparrow \\
C & & B & \longrightarrow & Y_1 \\
\parallel & & \parallel & & \uparrow \\
C & & B & & A
\end{array}$$

where the square

$$\begin{array}{ccc}
Y_2 & \longrightarrow & Z \\
\uparrow & & \uparrow \\
B & \longrightarrow & Y_1
\end{array}$$

is defined by pushing out along the cofibration  $B \longrightarrow Y_1$ . Since it is obvious that the two composites

$$\begin{array}{ccc}
Hom^h(A, B) \times Hom^h(B, C) \times Hom^h(C, D) & \longrightarrow & Hom^h(A, C) \times Hom^h(C, D) \\
\downarrow & & \downarrow \\
Hom^h(A, B) \times Hom^h(B, D) & \longrightarrow & Hom^h(A, D)
\end{array}$$

are equal, even as maps of categories, it follows that the maps on the level of  $\pi_0$  are equal, and hence our composition law for the category  $w^{-1}C$  is associative. Thus we have a category.

Given any functor  $F : C \longrightarrow Z$  where  $F(w)$  is invertible for each weak equivalence  $w$ , the map factors uniquely through the category  $w^{-1}C$ ; one sends the connected component of

$$\begin{array}{ccc}
B & \longrightarrow & Y_1 \\
\parallel & & \uparrow \\
B & & A
\end{array}$$

to the morphism  $F(w)^{-1}F(f)$ , where  $w : B \longrightarrow Y_1$  and  $f : A \longrightarrow Y_1$  are the maps defining the object. It is trivial to check that this is independent of the representative

$$\begin{array}{ccc}
B & \longrightarrow & Y_1 \\
\parallel & & \uparrow \\
B & & A
\end{array}$$

that we choose in the path component. Now recall that, in the proof that the map  $F : EXT^n(A, B) \longrightarrow Ext^n(wA, wB)$  satisfies the conditions of Quillen's Theorem B we established, among other things, that the homotopy of  $s \setminus F$  is independent of  $s$ . To identify the fiber with  $Ext^{n-1}(= fA, = wB)$ , we chose  $s$  to be

$$\begin{array}{ccccccccccc}
B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A \\
\uparrow & & & & & & & & & & & & \uparrow \\
B & & & & & & & & & & & & A \\
\downarrow & & & & & & & & & & & & \downarrow \\
B & & & & & & & & & & & & A
\end{array}$$

Given weak equivalences  $A' \longrightarrow A$  and  $B \longrightarrow B'$ , we get another object  $s'$  given by the diagram

$$\begin{array}{ccccccccccc}
B' & \longrightarrow & B' & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A' & \longrightarrow & A' \\
\uparrow & & & & & & & & & & & & \uparrow \\
B & & & & & & & & & & & & A' \\
\downarrow & & & & & & & & & & & & \downarrow \\
B & & & & & & & & & & & & A
\end{array}$$

and morphism  $s \longrightarrow \bar{s}$  and  $s' \longrightarrow \bar{s}$ . The fiber  $s' \setminus F$  is identified with  $\text{Hom}^h(A', B')$  by precisely the computation that identified  $s \setminus F$  with  $\text{Hom}^h(A, B)$ . The reader can easily check that the map inducing the homotopy equivalence  $s' \setminus F \longrightarrow s \setminus F$  is nothing other than the map

$$\text{Hom}^h(A, B) \longrightarrow \text{Hom}^h(A', B')$$

given by precomposing with  $A' \longrightarrow A$  and postcomposing with  $B \longrightarrow B'$ . It follows that this map is an isomorphism in the category  $w^{-1}C$ . In other words, given any weak equivalence  $w : A' \longrightarrow A$  in  $C$ , we have proved that precomposing with  $w$  and postcomposing with  $w$  yields isomorphisms

$$\begin{array}{ccc}
\text{Hom}_{w^{-1}C}(A, B) & \longrightarrow & \text{Hom}_{w^{-1}C}(A', B) \\
\text{Hom}_{w^{-1}C}(X, A') & \longrightarrow & \text{Hom}_{w^{-1}C}(X, A)
\end{array}$$

In other words, for the functor  $F : C \longrightarrow w^{-1}C$ , every weak equivalence  $w$  in  $C$  satisfies  $F(w)$  invertible. Thus  $w^{-1}C$  is universal with this property. The category  $w^{-1}C$  is known as the homotopy category associated to the Waldhausen category  $C$ . What we have proved in this section is

**Proposition 6.1** *The  $n^{\text{th}}$  homotopy group  $\pi_n(\text{Ext}_w^n(wA, wB))$  is naturally isomorphic to  $\text{Hom}_{w^{-1}C}(A, B)$ , where  $w^{-1}C$  is the homotopy category associated to the Waldhausen category  $C$ .*

**Exercise 6.2** The reader can amuse himself with the following exercise. Let  $C$  be a Waldhausen category. Let  $f_1 : X \longrightarrow Y$ ,  $f_2 : X \longrightarrow Y$  be two morphisms in  $C$ . A homotopy between  $f_1$  and  $f_2$  is a map

$$F : \text{Cyl}(X) \longrightarrow Y$$

so that if  $i_1 : X \rightarrow \text{Cyl}(X)$  is the inclusion of the first face,  $i_2 : X \rightarrow \text{Cyl}(X)$  the inclusion of the second face, then  $f_1 = F \circ i_1$ ,  $f_2 = F \circ i_2$ .

Suppose there exists a homotopy between two maps  $f_1 : X \rightarrow Y$  and  $f_2 : X \rightarrow Y$ . That is, there is an  $F : \text{Cyl}(X) \rightarrow Y$  as above. Let  $p : \text{Cyl}(X) \rightarrow X$  be the natural projection. Then  $p \circ i_1 = 1_X = p \circ i_2$ . But by the Cylinder Axiom,  $p : \text{Cyl}(X) \rightarrow X$  is a weak equivalence. Thus, in any category in which weak equivalences become invertible,  $p$  becomes invertible, and in such a category  $i_1 = i_2$ . In particular, it follows that in the universal example  $w^{-1}C$ ,

$$f_1 = F \circ i_1 = F \circ i_2 = f_2.$$

The exercise for the reader is to show directly that  $f_1 : X \rightarrow Y$  and  $f_2 : X \rightarrow Y$  lie in the same path component of  $\text{Hom}^h(X, Y)$ .

## 7 Another model for $\text{Ext}^n(wA, wB)$

We already have so many models for the space  $\text{Ext}^n(wA, wB)$ , that it may seem excessive to find yet another one. However, we would like to compute the homotopy groups of  $\text{Ext}^n(wA, wB)$ , and so far we have only been able to get a clean description of  $\pi_n(\text{Ext}^n(wA, wB))$ . To get further, we will need a new model.

Let us recall that in Remark 3.11 we considered a category  $\text{Ext}^n(C \rightarrow A, a', B)$ , whose objects are diagrams

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & \uparrow & & \uparrow \\ X'_0 & & & & & & X'_{n+1} & \longrightarrow & A \\ \downarrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & & & & Y' & \longrightarrow & Y' & \longrightarrow & C \end{array}$$

where the object  $B$  and the map  $C \rightarrow A$  is given and fixed, and the restrictions on the diagram are that the top row is acyclic, the maps  $b : X'_0 \rightarrow B$ ,  $b' : X'_0 \rightarrow X_0$ ,  $Y' \rightarrow C$  and  $a : X'_{n+1} \rightarrow A$  are all weak equivalences, while  $a' : X'_{n+1} \rightarrow X_{n+1}$  is of type  $a'$ , as indicated in the symbol for the category. Let  $A$  be a given object of the category  $C$ , and  $A \rightarrow \text{Cyl}(A)_A$  be the inclusion of the first face. We can consider the category  $\text{Ext}^n(A \rightarrow \text{Cyl}(A)_A, w, B)$ . There is a natural inclusion map

$$\text{Ext}^{n-1}(wA, wB) \subset \text{Ext}^n(A \rightarrow \text{Cyl}(A)_A, w, B)$$

which sends the object

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n \\ \uparrow & & & & & & \uparrow \\ X'_0 & & & & & & Y' \\ \downarrow & & & & & & \downarrow \\ B & & & & & & A \end{array}$$

to the object

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & 0 \\
\uparrow & & & & & & \uparrow & & \uparrow \\
X'_0 & & & & & & Y' & \longrightarrow & Cyl(A)_A & \longrightarrow & Cyl(A)_A \\
\downarrow & & & & & & \downarrow & & \downarrow & & \downarrow \\
B & & & & & & Y' & \longrightarrow & Y' & \longrightarrow & A
\end{array}$$

**Lemma 7.1** *The natural inclusion map above*

$$Ext^{n-1}(wwA, wwB) \subset Ext^n(A \longrightarrow Cyl(A)_A, w, B)$$

*induces a homotopy equivalence.*

**Proof** The point is that the inclusion functor has a left adjoint, sending the object

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\
\uparrow & & & & & & \uparrow & & \uparrow \\
X'_0 & & & & & & Y' & \longrightarrow & X'_{n+1} & \longrightarrow & Cyl(A)_A \\
\downarrow & & & & & & \downarrow & & \downarrow & & \downarrow \\
B & & & & & & Y' & \longrightarrow & Y' & \longrightarrow & A
\end{array}$$

to the object

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n \\
\uparrow & & & & & & \uparrow \\
X'_0 & & & & & & Y' \\
\downarrow & & & & & & \downarrow \\
B & & & & & & A
\end{array}$$

The reader should note that since  $X'_{n+1} \longrightarrow X_{n+1}$  and  $X'_{n+1} \longrightarrow Cyl(A)_A$  are assumed weak equivalences,  $X_{n+1}$  is weakly equivalent to 0. It follows that

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & 0 \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1}
\end{array}$$

is a weak equivalence of objects of  $Cpx_{n+1}(C)$ . Since the bottom row is acyclic, so is the top row.  $\square$

**Corollary 7.2** *We know from Remark 3.11 that the inclusion*

$$Ext^n(A \longrightarrow Cyl(A)_A, w, B) \subset Ext^n(A \longrightarrow Cyl(A)_A, f, B)$$

*induces a homotopy equivalence. It therefore follows that  $Ext^n(A \longrightarrow Cyl(A)_A, f, B)$  is yet another model for  $Ext^{n-1}(wwA, wwB)$ .*

Now there is a map

$$Ext^n(= f \{ \Sigma A \}, wwB) \longrightarrow Ext^n(A \longrightarrow Cyl(A)_A, f, B)$$

given by sending the object

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & & & \uparrow \\ X'_0 & & & & & & & & {}_A Cyl(A)_A \\ \downarrow & & & & & & & & \parallel \\ B & & & & & & & & {}_A Cyl(A)_A \end{array}$$

to the object

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & \uparrow & & \uparrow \\ X'_0 & & & & & & Cyl(A)_A & \longrightarrow & Cyl(A)_A \\ \downarrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & & & & A & \longrightarrow & A & \longrightarrow & A \end{array}$$

where the map  $A \longrightarrow X_n$  is, of course, the zero map.

**Lemma 7.3** *The inclusion above*

$$F : Ext^n(= f \{ \Sigma A \}, wwB) \longrightarrow Ext^n(A \longrightarrow Cyl(A)_A, f, B)$$

*induces a homotopy equivalence.*

**Proof** Although this time there is no adjoint, there is a functor

$$G : Ext^n(A \longrightarrow Cyl(A)_A, f, B) \longrightarrow Ext^n(= f \{ \Sigma A \}, wwB)$$

which is relatively straight-forward to describe, and is a homotopy inverse to  $F$ . It takes the object  $s$  given by the diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & X_n & \xrightarrow{\partial} & X_{n+1} \\ \uparrow & & & & & & \uparrow & & \uparrow \\ X'_0 & & & & & & X'_{n+1} & \longrightarrow & Cyl(A)_A \\ \downarrow & & & & \uparrow & & \uparrow & & \uparrow \\ B & & & & Y' & \longrightarrow & Y' & \longrightarrow & A \end{array}$$

to the object  $G(s)$ , which is

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & {}_{Y'}\text{Cyl}(Y') \vee_{Y'} X_n & \xrightarrow{\partial} & [{}_A\text{Cyl}(A)_A] \vee_{X'_{n+1}} \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} \\
\uparrow & & & & & & & & \uparrow \\
X'_0 & & & & & & & & {}_A\text{Cyl}(A)_A \\
\downarrow & & & & & & & & \parallel \\
B & & & & & & & & {}_A\text{Cyl}(A)_A
\end{array}$$

There is clearly a natural transformation  $G \circ F \implies 1$ . As for  $F \circ G$ , there is a natural transformation  $H \implies F \circ G$ , and another natural transformation  $H \implies 1$ . We will write down  $H$ , and allow the reader to supply the (obvious) natural transformations. Thus,  $H(s)$  is the diagram

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} & \text{Cyl}(Y') \vee_{Y'} X_n & \xrightarrow{\partial} & \text{Cyl}(X'_{n+1}) \vee_{X'_{n+1}} X_{n+1} & & \\
\uparrow & & & & & & \uparrow & & \uparrow & & \\
X'_0 & & & & & & Y' & & X'_{n+1} & \longrightarrow & \text{Cyl}(A)_A \\
\downarrow & & & & & & \longrightarrow & & \uparrow & & \\
B & & & & & & Y' & \longrightarrow & A & & 
\end{array}$$

□

Summarising the results of this section so far, we obtain:

**Proposition 7.4** *There is a natural homotopy equivalence of the categories*

$$\text{Ext}^n(= f \{ \Sigma A \}, wwB) \quad \text{and} \quad \text{Ext}^{n-1}(wwA, wwB)$$

*Since we already know that there is a homotopy equivalence of*

$$\text{Ext}^n(= f \{ \Sigma A \}, wwB) \quad \text{and} \quad \text{Ext}^n(ww \{ \Sigma A \}, wwB)$$

*we deduce a homotopy equivalence of*

$$\text{Ext}^n(ww \{ \Sigma A \}, wwB) \quad \text{and} \quad \text{Ext}^{n-1}(wwA, wwB)$$

We use this to compute the homotopy groups.

**Theorem 7.5** *For any  $r \geq -n$ , the homotopy groups are given by*

$$\pi_{n+r} \text{Ext}^n(wwA, wwB) = \pi_{n+r} \text{Ext}^{n+r}(ww \{ \Sigma^r A \}, wwB) = \text{Hom}_{w^{-1}C}(\Sigma^r A, B)$$

**Proof** The first equality is the result of this section, while the second was proved in Section 6. □

## 8 The case of an additive Waldhausen category

Now we need to prove the last unproved claim made in the Introduction, namely

**Proposition 8.1** *Let  $C$  be an additive Waldhausen category. Then the spectra  $Ext(A, B)$  that one obtains from the sequence of homotopy equivalences*

$$Ext^0(\Sigma^i A, B) \longrightarrow \Omega Ext^0(\Sigma^{i+1} A, B)$$

is a wedge of Eilenberg–MacLane spectra.

**Proof** Observe that if we are given three additive Waldhausen categories  $C$  and  $C'$  and  $D$ , and a tensor product map

$$C \times C' \longrightarrow D$$

which is a map of Waldhausen categories, then there is an induced map

$$Ext_C^0(A, A') \times Ext_{C'}^0(B, B') \longrightarrow Ext_D^0(A \otimes B, A' \otimes B')$$

where the map is given by a functor. Given objects

$$\begin{array}{ccc} A & & A' \\ & \searrow & \swarrow \\ & X & \\ & \swarrow & \searrow \\ B & & B' \\ & \searrow & \swarrow \\ & Y & \end{array} \quad \text{and}$$

the functor takes them to the object

$$\begin{array}{ccc} A \otimes B & & A' \otimes B' \\ & \searrow & \swarrow \\ & X \otimes Y & \end{array}$$

It is not difficult to show that this multiplication extends to the non–connective spectra, to give a map

$$Ext_C(A, A') \wedge Ext_{C'}(B, B') \longrightarrow Ext_D(A \otimes B, A' \otimes B')$$

Now consider the special case where  $C'$  is the category of chain complexes of free  $\mathbb{Z}$  modules, and  $C = D$ . Then there is an obvious tensor product

$$C \times C' \longrightarrow C$$

and hence an action of  $Ext_{C'}(\mathbb{Z}, \mathbb{Z})$  on  $Ext_C(A, A')$ . But the homotopy groups of the spectrum  $Ext_{C'}(\mathbb{Z}, \mathbb{Z})$  are  $Ext^i(\mathbb{Z}, \mathbb{Z})$ , that is are zero for all  $i \neq 0$ . It follows that  $Ext_{C'}(\mathbb{Z}, \mathbb{Z})$  is an Eilenberg–MacLane spectrum, in fact  $K(\mathbb{Z}, 0)$ , and that every  $Ext_C(A, A')$  is a module over it. This forces  $Ext_C(A, A')$  to be a wedge of Eilenberg–MacLane spectra.  $\square$

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