

Singular homology of abstract algebraic varieties.

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1 Introduction.

The main objective of the present paper is to construct a reasonable singular homology theory on the category of schemes of finite type over an arbitrary field k .

Let X be a CW-complex. The theorem of Dold and Thom [3] shows that $H_i(X, Z)$ coincide with π_i of the simplicial abelian group

$$Hom_{top}(\Delta_{top}, \prod_{d=0}^{\infty} S^d(X))^+,$$

where $S^d(X)$ is the d -th symmetric power of X , Δ_{top}^i is the usual i -dimensional topological simplex and for any abelian monoid M we denote by M^+ the associated abelian group.

Let now X be any scheme of finite type over a field k . Define $H_i^{sing}(X)$ as π_i of the simplicial abelian group

$$Hom(\Delta, \prod_{d=0}^{\infty} S^d(X))^+,$$

¹Supported by NSF grant DMS-9100383.

where this time Δ^i denotes the linear subvariety of A^{i+1} given by the equation

$$t_0 + \dots + t_i = 1.$$

Note further that for any normal connected scheme S the abelian monoid $Hom(S, \coprod_{d=0}^{\infty} S^d(X))$ coincides (after localization by $p = \text{exponential characteristic of } k$) with the monoid of effective cycles in $S \times X$ each component of which is finite and surjective over S . Thus, (after localization by p) $H_*^{sing}(X)$ coincide with

$$\pi_*(C_*(X)) = H_*(C_*(X), d = \sum (-1)^i \partial_i),$$

where $C_*(X)$ is the simplicial abelian group generated by closed integral subschemes $Z \subset \Delta^i \times X$ such that the projection $Z \rightarrow \Delta^i$ is finite and surjective.

This construction appeared first in the talk given by A.Suslin at the Luminy conference on algebraic K-theory (1987) as a part of a program towards the computation of higher algebraic K-theory of varieties over C . Two main conjectures relating singular algebraic homology to other homology theories were made.

Conjecture 1.1 *If X is a variety over C then the evident homomorphism*

$$Hom(\Delta, \coprod_{d=0}^{\infty} S^d(X))^+ \longrightarrow Hom_{top}(\Delta_{top}, \coprod_{d=0}^{\infty} S^d(X))^+$$

induces isomorphisms

$$H_i^{sing}(X, Z/n) \cong H_i(X(C), Z/n).$$

Conjecture 1.2 *Let X be a complete irreducible variety of dimension n over a field k . Then the evident embedding of complexes*

$$C_*(X) \longrightarrow \mathcal{Z}^n(X, *)$$

is a quasi-isomorphism, i.e. $H_i^{sing}(X) \cong CH^n(X, i)$, where $CH^(X, *)$ are the higher Chow groups introduced by S.Bloch [1].*

Here we prove that the first conjecture is true. The method of the proof is close to the methods developed previously by A.Suslin [19],[20],O.Gabber (unpublished), H.Gillet and R.Thomason [5] and R.Jardine [10] for the computation of algebraic K-theory of algebraically closed fields. In section 3 we prove a rather general version of the rigidity theorem of Suslin, Gabber, Gillet and Thomason (theorem (4.4) below). The crucial role in the application of this rigidity theorem to the problem in question is played by h-topology of Voevodsky [21]. One of the main results of the paper (theorem (7.6) and corollary (7.7) below) shows that if F is any qfh-sheaf on the category of schemes of finite type over an algebraically closed field k of characteristic zero, then

$$H_{sing}^*(F, Z/n) = Ext_{qfh}^*(F, Z/n).$$

Applying this theorem to the sheaf $Z_{qfh}(X)$ and using some topological computations parallel to theorem (7.6) we conclude the proof of conjecture 1 in section 7. The last section contains a generalization of the main theorem to cycles of positive dimension: we prove that for projective varieties over C algebraic Lawson homology with finite coefficients coincides with the usual one ([12]). This answers a question raised by E.Friedlander.

For reader's convenience we have included an appendix where we discuss briefly the definition and some of the main properties of h-cohomology (for more details see ([21]). There we give in particular a corrected version of the proof of the comparison theorem relating h- qfh- and etale cohomology (the proof of this theorem in ([21]) contains an error).

2 The relative Picard group.

Let X be a scheme and let Y be a closed subscheme of X . Set $U = X - Y$ and denote by $i : Y \rightarrow X, j : U \rightarrow X$ the corresponding closed and open embeddings.

Denote by $Pic(X, Y)$ the group, whose elements are isomorphism classes of pairs of the form (L, ϕ) , where L is a linear bundle on X and $\phi : L|_Y \cong \mathcal{O}_Y$ is a trivialization of L over Y and the operation is given by the tensor product. There is an evident exact sequence:

$$\Gamma(X, \mathcal{O}_X^*) \rightarrow \Gamma(Y, \mathcal{O}_Y^*) \rightarrow Pic(X, Y) \rightarrow Pic(X) \rightarrow Pic(Y). \quad (1)$$

Let G_X be the sheaf of invertible functions on X . The homomorphism

$G_X \longrightarrow i_*(G_Y)$ is surjective both in etale and Zariski topologies and we will denote by $G_{X,Y}$ its kernel.

Lemma 2.1

$$Pic(X, Y) = H_{Zar}^1(X, G_{X,Y}) = H_{et}^1(X, G_{X,Y}).$$

Proof: Any relative linear bundle (L, ϕ) is locally trivial in the Zariski topology and the automorphism group of the trivial relative bundle $(\mathcal{O}_U, Id : (\mathcal{O}_U)|_Y \cong \mathcal{O}_{Y \cap U})$ is canonically isomorphic to $\Gamma(U, G_{X,Y})$. This implies the first formula. The second follows from the Hilbert theorem 90 ([15][ch.3]) and the five lemma.

Corollary 2.2 *Assume that n is invertible on X , then*

$$Pic(X, Y)/nPic(X, Y) \subset H_{et}^2(X, j_!(\mu_n))$$

(here $j_!$ is the extension by zero functor - see [15]).

Proof: This follows from the lemma 2.1 in view of the exact sequence of etale sheaves:

$$0 \longrightarrow j_!(\mu_n) \longrightarrow G_{X,Y} \xrightarrow{n} G_{X,Y} \longrightarrow 0.$$

Assume that X is integral and denote by K the field of rational functions on X . The relative Cartier divisor on X is a Cartier divisor D such, that $Supp(D) \cap Y = \emptyset$. If D is a relative divisor and $Z = Supp(D)$, then $\mathcal{O}_X(D)|_{X-Z} = \mathcal{O}_{X-Z}$. Thus D defines an element in $Pic(X, Y)$. Denoting the group of relative Cartier divisors by $Div(X, Y)$ we get a homomorphism $cl : Div(X, Y) \longrightarrow Pic(X, Y)$. The image of this homomorphism consists of pairs (L, ϕ) such that ϕ admits an extension to a trivialization of L over an open neighborhood of Y . In particular this map is surjective provided that Y has an affine open neighborhood. Finally set

$$\begin{aligned} G &= \{f \in K^* : f \in \ker((\mathcal{O}_X^*)_y \longrightarrow (\mathcal{O}_Y^*)_y) \text{ for any } y \in Y\} = \\ &= \{f \in K^* : f \text{ is defined and equal to one at each point of } Y\}. \end{aligned}$$

Lemma 2.3 *Assume that Y has an affine open neighborhood in X . Then the following sequence is exact:*

$$0 \longrightarrow \Gamma(X, G_{X,Y}) \longrightarrow G \longrightarrow Div(X, Y) \longrightarrow Pic(X, Y) \longrightarrow 0.$$

Proof: Obvious.

Lemma 2.4 *Assume that U is regular (or factorial). Then $\text{Div}(X, Y)$ is a free abelian group generated by irreducible divisors $T \subset U$ which are closed in X .*

Proof: Easy.

Finally we will mention the homotopy invariance of the relative Picard group.

Lemma 2.5 *Assume that X is normal and Y is reduced. Then $\text{Pic}(X, Y) \cong \text{Pic}(X \times A^1, Y \times A^1)$.*

Proof: It follows from the exact sequence 1, the five lemma and the homotopy invariance of the Picard group for normal schemes.

3 Singular homology of curves.

Let k be a field. All schemes considered in this section are assumed to be of finite type over k . Denote by Δ^n the closed subscheme of A^{n+1} given by the equation $t_0 + \dots + t_n = 1$. There are obvious coface and codegeneracy maps

$$\begin{aligned} \delta^i &: \Delta^{n-1} \longrightarrow \Delta^n \\ \sigma^i &: \Delta^{n+1} \longrightarrow \Delta^n \end{aligned}$$

making Δ a cosimplicial scheme.

Suppose, that S is irreducible and X is any scheme over S . Denote by $C_n(X/S)$ the free abelian group generated by closed irreducible reduced subschemes $Z \subset X \times \Delta^n$ such that the projection $Z \longrightarrow S \times \Delta^n$ is finite and surjective. One verifies immediately that if Z is as above, then each component of $(\delta^i)^{-1}(Z) \subset X \times \Delta^{n-1}$ is finite and surjective over $S \times \Delta^{n-1}$ and hence has the “correct” dimension so, that the cycle-theoretic inverse image $(\delta^i)^*(Z)$ is well defined and lies in $C_{n-1}(X/S)$. This gives us the face operators

$$\partial_i = (\delta^i)^* : C_n(X/S) \longrightarrow C_{n-1}(X/S).$$

In the same way one defines degeneracy operators

$$s_i = (\sigma^i)^* : C_n(X/S) \longrightarrow C_{n+1}(X/S),$$

thus making $C_*(X/S)$ a simplicial abelian group. The homotopy groups of this simplicial abelian group, i.e. the homology of the complex $(C_*(X/S), d = \sum(-1)^i \partial^i)$ will be denoted by $H_*^{sing}(X/S)$.

Assume that S is a normal affine scheme and X is a smooth affine irreducible scheme of relative dimension one over S . It is said to admit a good compactification if there exists an open embedding of schemes over S , $X \rightarrow \bar{X}$ such that the following conditions hold:

1. The scheme $Y = \bar{X} - X$ has an affine open neighborhood in \bar{X} .
2. \bar{X} is normal,
3. $\bar{X} \rightarrow S$ is a proper morphism with fibers of dimension one.

Theorem 3.1 *Let S, X, \bar{X}, Y be as above, then*

$$H_0^{sing}(X/S) = Pic(\bar{X}, Y)$$

$$H_i^{sing}(X/S) = 0, \text{ for } i > 0.$$

Proof: Let $Z \subset X \times \Delta^n$ be a closed reduced and irreducible subscheme. The projection $Z \rightarrow S \times \Delta^n$ is finite and surjective if and only if Z is closed and of codimension one in $\bar{X} \times \Delta^n$. Since S is normal and X is smooth over S any such Z is in fact a Cartier divisor on X (see [9, 21.14.3]). Thus $C_n(X/S) = Div(\bar{X} \times \Delta^n, Y \times \Delta^n)$. Let U be an affine open neighborhood of Y in \bar{X} . Then $U \times \Delta^n$ is an affine open neighborhood of $Y \times \Delta^n$ in $\bar{X} \times \Delta^n$. According to the lemma 2.3 we have an exact sequence of simplicial abelian groups:

$$0 \rightarrow A_n \rightarrow G_n \rightarrow C_n(X/S) \rightarrow Pic(\bar{X} \times \Delta^n, Y \times \Delta^n) \rightarrow 0 \quad (2)$$

where

$$A_n = \Gamma(\bar{X} \times \Delta^n, G_{\bar{X} \times \Delta^n, Y \times \Delta^n})$$

$$G_n = \{f \in k(\bar{X} \times \Delta^n)^* : f \text{ is defined and equal to 1 at each point of } Y \times \Delta^n\}$$

and $k(\bar{X} \times \Delta^n)^*$ is the multiplicative group of the field of rational functions on the scheme $\bar{X} \times \Delta^n$.

Let us show that $A_n = 0$ for all n . Consider an element $f \in \Gamma(\bar{X} \times \Delta^n, G_{\bar{X} \times \Delta^n, Y \times \Delta^n})$. The restriction of f to any geometric fiber of $\bar{X} \times \Delta^n \rightarrow S \times \Delta^n$ is a regular function on a complete curve and hence has to be a constant on each component of the fiber. On the other hand each component contains at least one point of $Y \times \Delta^n$ where f is equal to one. This shows, that $f(x) = 1$ for any $x \in \bar{X} \times \Delta^n$ and hence $f = 1$ since $\bar{X} \times \Delta^n$ is reduced.

Consider now the simplicial abelian group G_\cdot . Let us show, that it is acyclic, i.e. $\pi_*(G_\cdot) = 0$. It suffices to check that for any $f \in G_n$ such that $\partial_i(f) = 1$ for $i = 0, \dots, n$ there exists $g \in G_{n+1}$ such, that $\partial_i(g) = 1$ for $i = 0, \dots, n$ and $\partial_{n+1}(g) = f$. Define functions $g_i \in G_{n+1}$ for $i = 0, \dots, n$ by means of the formula:

$$g_i = (t_{i+1} + \dots + t_{n+1}) + (t_0 + \dots + t_i)s_i(f)$$

. These functions satisfy the following equations:

$$\partial_j(g_i) = \begin{cases} 1 & \text{if } j \neq i, i+1 \\ (t_i + \dots + t_n) + (t_0 + \dots + t_{i-1})f & \text{if } j = i \\ (t_{i+1} + \dots + t_n) + (t_0 + \dots + t_i)f & \text{if } j = i+1 \end{cases}$$

In particular $\partial_0(g_0) = 1$, $\partial_{n+1}(g_n) = f$. Finally we set

$$g = g_n g_{n-1}^{-1} g_{n-2} \dots g_0^{(-1)^n}.$$

This function obviously satisfies the conditions we need. Note now that the lemma (2.5) implies that

$$Pic(\bar{X} \times \Delta^n, Y \times \Delta^n) = Pic(\bar{X}, Y)$$

and we conclude from the exact sequence (2) that

$$H_i^{sing}(X/S) = \begin{cases} Pic(\bar{X}, Y) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Theorem is proven.

Remark: A result similar to our theorem (3.1) was proved independently by S.Lichtenbaum [13].

4 The rigidity theorem.

Denote by Sch/k the category of separated schemes of finite type over k . Let $\Phi : Sch/k \rightarrow Ab$ be a presheaf of abelian groups on Sch/k , i.e. a contravariant functor from Sch/k to the category of abelian groups.

Definition 4.1 *A presheaf Φ is said to admit trace maps if for any finite surjective morphism $p : X \rightarrow S$ in Sch/k , where X is reduced and irreducible and S is irreducible and regular we are given a homomorphism:*

$$Tr_{X/S} : \Phi(X) \rightarrow \Phi(S)$$

such that the following conditions hold:

1. If p is an isomorphism then $Tr_{X/S} \circ p^* = Id$.
2. Let $V \subset S$ be a closed irreducible regular subscheme. Denote by W_i the components of $p^{-1}(V)$ and let

$$n_i = \sum_{k=0}^{\infty} (-1)^k l_{\mathcal{O}_{X,W_i}}(Tor_k^{\mathcal{O}_{S,V}}(\mathcal{O}_{X,W_i}, k(V)))$$

be the multiplicity of W_i . Then the following diagram commutes:

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{Tr_{X/S}} & \Phi(S) \\ \downarrow & & \downarrow \\ \bigoplus \Phi(W_i) & \xrightarrow{\sum n_i Tr_{W_i/V}} & \Phi(V). \end{array}$$

Let S be a regular irreducible scheme and let X be any scheme over S . We have a pairing

$$C_0(X/S) \otimes \Phi(X) \xrightarrow{\langle, \rangle} \Phi(S)$$

given by the formula

$$\langle Z, \phi \rangle = Tr_{Z/S}(\phi|_Z)$$

(here $\phi \in \Phi(X)$ and Z is a reduced irreducible subscheme of X which is finite and surjective over S).

Proposition 4.2 *Assume in addition that Φ is homotopy invariant, i.e. that for any X in Sch/k we have $\Phi(X \times A^1) = \Phi(X)$. Then the above pairing factors through $H_0^{sing}(X/S) \otimes \Phi(X)$.*

Proof: Let $W \subset X \times A^1$ be a closed reduced and irreducible subscheme such, that W is finite and surjective over $S \times A^1$. Denote by W^i the cycle $W \cap (X \times i)$. we have to show, that $\langle W^1 - W^0, \phi \rangle = 0$ for any $\phi \in \Phi(X)$. Let ψ denotes the inverse image of ϕ in $\Phi(X \times A^1)$. Consider $\langle W, \psi \rangle \in \Phi(S \times A^1)$. The homotopy invariance of Φ shows, that

$$\langle W, \psi \rangle_{|S \times 0} = \langle W, \psi \rangle_{|S \times 1}.$$

On the other hand the properties of the trace map imply that

$$\langle W, \psi \rangle_{|S \times i} = \langle W^i, \psi|_{X \times i} \rangle = \langle W_i, \psi \rangle.$$

Proposition is proven.

From now on we will assume that Φ is homotopy invariant. Extend Φ to pro-objects in Sch/k setting

$$\Phi(\varprojlim X_i) = \varinjlim(\Phi(X_i)).$$

Theorem 4.3 *Assume that $n\Phi = 0$ where n is prime to $\text{char}(k)$. Let S be a henselization of a smooth variety over k in a closed point. Let further X/S be a smooth affine scheme over S of relative dimension one and assume, that X admits a good compactification $\bar{X} \rightarrow S$.*

If $g_1, g_2 : S \rightarrow X$ are two sections which coincide in the closed point of S then $g_1^ = g_2^* : \Phi(X) \rightarrow \Phi(S)$.*

Proof: We may suppose, that X is irreducible and \bar{X} is normal. Let Y denote the closed reduced subscheme $\bar{X} - X$ of \bar{X} . The sections g_i are closed embeddings and we will denote by W_i the corresponding subschemes of X . The properties of the trace map imply that $g_i^*(\phi) = \langle W_i, \phi \rangle$ for any $\phi \in \Phi(X)$. Thus our statement is equivalent to the fact that $W_1 - W_2 \in C_0(X/S)$ is in the kernel of the above pairing. In view of the proposition (4.2) it is sufficient to show, that image of $W_1 - W_2$ in $H_0^{sing}(X/S)/n$ is zero. According to (2.2) and (3.1) we have:

$$H_0^{sing}(X/S)/n = \text{Pic}(\bar{X}, Y)/n \subset H_{et}^2(\bar{X}, j_!(\mu_n)).$$

The proper base change theorem implies that

$$H_{et}^2(\bar{X}, j_!(\mu_n)) = H_{et}^2(\bar{X}_0, (j_0)_!(\mu_n)),$$

where \bar{X}_0 is the closed fiber of \bar{X} and $j_0 : X \rightarrow \bar{X}_0$ is the corresponding open embedding. The diagram

$$\begin{array}{ccc} \text{Pic}(\bar{X}, Y)/n & \longrightarrow & H_{et}^2(\bar{X}, j_!(\mu_n)) \\ \downarrow & & \downarrow \\ \text{Pic}(\bar{X}_0, Y_0)/n & \longrightarrow & H_{et}^2(\bar{X}_0, (j_0)_!(\mu_n)) \end{array}$$

commutes. This shows that $\text{Pic}(\bar{X}, Y)/n \subset \text{Pic}(\bar{X}_0, Y_0)/n$. On the other hand the image of $W_1 - W_2$ in $\text{Pic}(\bar{X}_0, Y_0)$ is trivial since g_1, g_2 coincide in the closed point.

Theorem 4.4 *Let S_l be the henselization of A^l in 0. Then*

$$\Phi(S_l) = \Phi(\text{Spec}(k))$$

(still assuming that $n\Phi = 0$, where n is prime to $\text{char}(k)$).

Proof: We will proceed by induction on l . For $l = 0$ the statement is evident. In general case it is sufficient to show, that the homomorphism $\Phi(\text{Spec}(k)) \rightarrow \Phi(S_l)$ is surjective. According to our definition $\Phi(S_l) = \lim_{(X, x_0)} \Phi(X)$, where (X, x_0) runs through all etale neighborhoods of $(A^l, 0)$.

So it is sufficient to show, that if $f : (X, x_0) \rightarrow (A^l, 0)$ is an affine etale neighborhood, then the image of $\Phi(X)$ in $\Phi(S_l)$ lies in $\Phi(\text{Spec}(k))$. Let us show that there exists a linear projection $r : A^l \rightarrow A^{l-1}$ such that the curve $p' : X' \rightarrow S_k$ defined by the Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ p \downarrow & & \downarrow rf \\ S_l & \longrightarrow & A^l \xrightarrow{f} A^{l-1} \end{array}$$

admits a good compactification. In fact let $Z \subset A^l$ be a divisor such that $f : X \rightarrow A^l$ is finite over $A^l - Z$. Then there exists a linear projection $r : A^l \rightarrow A^{l-1}$ such that $r|_Z : Z \rightarrow A^{l-1}$ is quasifinite over the point $r(0) = 0$ in A^{l-1} . One can easily see that for any projection satisfying this property the curve $p : X' \rightarrow S_l$ admits a good compactification. Let now $s_1 : S_l \rightarrow X'$ be the section defined by the canonical morphism $S_l \rightarrow X$ and $s_2 : S_l \rightarrow X'$ be the section defined by the composition

$$S_l \rightarrow S_{l-1} \rightarrow S_l \rightarrow X.$$

They obviously coincide in the closed point of S_l and to finish the proof it is sufficient now to apply theorem 4.3.

Theorem 4.5 *Assume that k is an algebraically closed field of characteristic zero. Assume further that Φ is a homotopy invariant presheaf on Sch/k equipped with trace maps. Denote by Φ_h (resp. Φ_{qfh}, Φ_{et}) the sheaf associated with Φ in the h -topology (resp. in the qfh -topology, etale topology). Then for any $n > 0$ we have canonical isomorphisms:*

$$\begin{aligned} Ext_{et}^*(\Phi_{et}, Z/n) &= Ext_{qfh}^*(\Phi_{qfh}, Z/n) = Ext_h^*(\Phi_h, Z/n) = \\ &= Ext_{Ab}^*(\Phi(Spec(k)), Z/n). \end{aligned}$$

Proof: The first two isomorphisms follow from the corollary (10.11). To prove the last one denote $\Phi(Spec(k))$ by Φ_0 . We will use the same notation Φ_0 for the corresponding constant presheaf and constant h -sheaf. Consider the natural monomorphism of presheaves $\Phi_0 \rightarrow \Phi$ and denote by Φ' its cokernel. Applying theorem (4.4) to the presheaves Φ/n and ${}_n\Phi$ we conclude that

$$(\Phi'/n)(X_x^h) = 0 = ({}_n\Phi')(X_x^h)$$

for any smooth scheme X and any closed point x on X (here X_x^h denotes the henselization of X in x). Let now X be any object of Sch/k . Resolving singularities we construct a proper surjective morphism $Y \rightarrow X$ with Y smooth. If $u \in (\Phi'/n)(X)$ is any section, then vanishing of the groups $(\Phi'/n)(Y_y^h)$ implies that we can find an etale covering $\{Y_i \rightarrow Y\}_{i=1}^k$ such that $u|_{Y_i} = 0$. since $\{Y_i \rightarrow X\}$ is an h -covering of X this shows, that $(\Phi'/n)_h = 0$. The same argument shows that $({}_n\Phi')_h = 0$. Thus the sheaf $(\Phi')_h$ is uniquely n -divisible and hence $Ext_h^*((\Phi')_h, Z/n) = 0$. Now the exact sequence of sheaves

$$0 \rightarrow \Phi_0 \rightarrow \Phi_h \rightarrow (\Phi')_h \rightarrow 0$$

shows, that

$$Ext_h^*(\Phi_h, Z/n) = Ext_h^*(\Phi_0, Z/n) = Ext_{Ab}^*(\Phi_0, Z/n).$$

Theorem is proven.

Remark: Resolution of singularities for varieties over a field of characteristic $p > 0$ was announced recently by M.Spivakovsky. This implies that one can drop the restriction on $\text{char}(k)$ in theorem (4.5) (demanding instead that n is prime to $\text{char}(k)$).

5 qfh-sheaves.

The main objective of this section is to show that any qfh-sheaf admits trace maps satisfying the conditions of definition 4.1. Let \mathcal{F} be a qfh-sheaf of abelian groups and $X \rightarrow S$ be a finite surjective morphism such that X is integral and S is normal. let further Y be the normalization of X in a normal extension of $k(S)$ which contains $k(X)$. Denote the Galois group $\text{Gal}(k(Y)/k(S))$ by G . One checks immediately (lemma 5.8) that $\mathcal{F}(S)$ coincides with $\mathcal{F}(Y)^G$. This enables us to define the trace map

$$\text{Tr}_{X/S} : \mathcal{F}(X) \rightarrow \mathcal{F}(S)$$

by the following formula

$$\text{Tr}_{X/S}(a) = [F(X) : F(S)]_i \sum_{q \in \text{Hom}_S(Y, X)} q^*(a).$$

The main difficulty is to check that these trace maps are compatible with the base change homomorphisms, i.e. satisfy the property 4.1(2). To do so we have to interpret the multiplicities n_i appearing in 4.1(2) in similar terms. Actually we do it in more general situation. Suppose that S'/S is any integral scheme over S . We ascribe using elementary Galois theory, certain multiplicities n_i to all irreducible components X'_i of $X' = X \times_S S'$ which, in the case of S being regular coincide with multiplicities given by the Tor-formula and (in the case when S' is also normal) make the following diagram commutative

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \coprod \mathcal{F}(X_i) \\ \text{Tr}_{X/S} \downarrow & & \downarrow \qquad \sum n_i \text{Tr}_{X'_i/S'} \\ \mathcal{F}(S) & \longrightarrow & \mathcal{F}(S'). \end{array}$$

The explicite formula for multiplicities n_i shows that they belong to $Z[1/p]$ where p is the exponential characteristic of k . Moreover $n_i \in Z$ if

S is a regular scheme. However, A.S. Merkurjev produced recently an example showing that for an arbitrary normal S the multiplicities need not be integers.

In the next section this construction is used to show that if X is any scheme of finite type over k and S is a normal irreducible scheme then $C_0(X \times S/S) \otimes Z[1/p]$ coincides with the group of sections of the free qfh-sheaf $Z[1/p](X)$ of $Z[1/p]$ -modules generated by X .

Fix a field k . All schemes considered in this section are assumed to be of finite type over k . We denote by p the exponential characteristic of k .

Assume that S is a normal connected scheme. Let E be a finite normal extension of the field $k(S)$ with Galois group G , let Y denote the normalization of S in E and let $f : Y \rightarrow S$ denote the natural morphism.

Lemma 5.1 *For any $s \in S$ the group G acts transitively on $f^{-1}(s)$. Furthermore if $y \in f^{-1}(s)$ then the extension of fields $k(y)/k(s)$ is normal and the canonical homomorphism*

$$G_y = \{\sigma \in G : y^\sigma = y\} \longrightarrow \text{Gal}(k(y)/k(s))$$

is surjective.

Proof: See [2][ch.5,§2,n.3].

Corollary 5.2 *Let Ω be an algebraically closed field. For any geometric point $\omega : \text{Spec}(\Omega) \rightarrow S$ the action of G on the set of geometric points of Y over ω is transitive.*

Corollary 5.3 *The canonical morphism $Y \times G \rightarrow Y \times_S Y$ is surjective.*

Corollary 5.4 *Let S' be any scheme over S . Set $Y' = Y \times_S S'$, $f' = f \times_S \text{Id}_{S'}$. For any $s' \in S'$ the action of G on $(f')^{-1}(s')$ is transitive. Furthermore if $y' \in (f')^{-1}(s')$ then the extension $k(y')/k(s')$ is normal and the homomorphism*

$$\{\sigma \in G : (y')^\sigma = y'\} \longrightarrow \text{Gal}(k(y')/k(s'))$$

is surjective.

Corollary 5.5 *In the notations of (5.4) assume further that S' is irreducible. The group G acts transitively on the set of irreducible components of Y' . Hence any irreducible component of Y' maps surjectively onto S' .*

Let $p : X \rightarrow S$ be a finite surjective morphism from an integral scheme X to a normal connected scheme S . Denote by Y the normalization of X in the normal closure of $k(X)$ over $k(S)$. Let further V be an integral scheme over S . Denote by $W_i, i = 1, \dots, k$ (resp. $U_j, j = 1, \dots, l$) the irreducible components of $X \times_S V$ (resp. of $Y \times_S V$). The morphism $Y \times_S V \rightarrow X \times_S V$ is finite and surjective and hence for any i there exist components U_j lying over W_i . Denote the number of these components by $t(i)$. According to (5.5) $\dim(U_j) = \dim(V)$ for any j and hence $\dim(W_i) = \dim(V)$ for any i (i.e. each component W_i maps surjectively onto V).

We will consider W_i (resp. U_j) as reduced closed subschemes of $X \times_S V$ (resp. $Y \times_S V$) and we will denote by \tilde{U}_j the normalizations of U_j . Fix a component $U \in \{U_j\}_{j=1}^l$. For any S -morphism $q : Y \rightarrow X$ let $i(q)$ be defined by the formula $q \times_S Id_V(U) = W_{i(q)}$ and let $\bar{q} : \tilde{U} \rightarrow W_{i(q)}$ denotes the induced morphism.

Lemma 5.6 *Let $q : \tilde{U} \rightarrow W_i$ be any V -morphism. Then the number of morphisms $q : Y \rightarrow X$ such that $i(q) = i$ and $\bar{q} = g$ is equal to*

$$\frac{[k(X) : k(S)]_{sep} t(i)}{[k(W_i) : k(V)]_{sep} l}.$$

Proof: Set $G = Gal(k(Y)/k(S))$. Denote by $q_0 : Y \rightarrow X$ the canonical morphism. For any j define $i(j)$ by the formula $(q_0) \times_S Id_V(U_j) = W_{i(j)}$ and denote by \bar{q}_0^j the induced morphism $\tilde{U}_j \rightarrow W_{i(j)}$. The family $\{q_0 \sigma\}_{\sigma \in G}$ contains all S -morphisms $Y \rightarrow X$, each one with the multiplicity $[k(Y) : k(X)]_{sep}$. The morphism $q = q_0 \sigma$ has the desired properties if and only if the following conditions hold:

1. $\sigma(U) = U_j$ where $i(j) = i$.
2. The composition $\tilde{U} \xrightarrow{\sigma} \tilde{U}_j \xrightarrow{\bar{q}_0^j} W_i$ coincides with g .

Fix j such, that $i(j) = i$ and $\sigma_0 \in G$ such that $\sigma_0(U) = U_j$. Any σ with the property $\sigma(U) = U_j$ has the form $\sigma = \sigma_0 \tau$, where $\tau \in Stab_G(U) = H$. According to (5.4) the extension $k(U)/k(V)$ is normal and the homomorphism

$H \longrightarrow \text{Gal}(k(U)/k(V))$ is surjective. The Galois group acts transitively on the set of V -morphisms $\tilde{U} \longrightarrow U_j$. Thus the number of $\tau \in H$ such that $\bar{g}_0^j \sigma_0 \tau = g$ is equal to

$$\frac{|H|}{|\text{Gal}(k(U)/k(V))|} \frac{|\text{Gal}(k(U)/k(V))|}{[k(W_i) : k(V)]_{\text{sep}}} = \frac{|G|}{l[k(W_i) : k(V)]_{\text{sep}}}.$$

Thus the number of σ such that $q = q_0 \sigma$ has the desired properties is equal to

$$\frac{[k(Y) : k(S)]_{\text{sept}} t(i)}{l[k(W_i) : k(V)]_{\text{sep}}}$$

and finally the number of q with the desired properties is equal to

$$\frac{[k(Y) : k(S)]_{\text{sept}} t(i)}{l[k(W_i) : k(V)]_{\text{sep}} [k(Y) : k(X)]_{\text{sep}}} = \frac{[k(X) : k(S)]_{\text{sept}} t(i)}{[k(W_i) : k(V)]_{\text{sep}} l}.$$

In the above situation define the multiplicity n_i of the component W_i by means of the formula

$$n_i = \frac{[k(X) : k(S)] t(i)}{[k(W_i), k(V)] l}.$$

Lemma (5.6) shows in particular that $n_i \in Z[\frac{1}{p}]$ (apparently these multiplicities are always integers, we did not care to check since we did not need that). Note the following evident formula:

$$\sum_{i=1}^k n_i [k(W_i) : k(V)] = [k(X) : k(S)]. \quad (3)$$

Lemma 5.7 *Assume that S is regular and V is a closed subscheme of S . Then n_i coincides with the multiplicity with which $[W_i]$ appears in the cycle $p^*([V])$.*

Proof: Denote by $q : Y \longrightarrow X$ the canonical morphism and set $f = pq : Y \longrightarrow S$. The projection formula ([17][ch.5]) shows that $q_*(f^*([V])) = [k(Y) : k(X)] p^*([V])$. All components U_j appear with the same multiplicity m in the cycle $f^*([V])$. Applying once again the projection formula we get

$$[k(Y) : k(S)] [V] = f_*(f^*([V])) = f_*(m \sum_{j=1}^l [U_j]) = ml [k(U) : k(V)] [V].$$

Hence

$$m = \frac{[k(Y) : k(S)]}{[k(U) : k(V)]l}.$$

Thus the multiplicity of W_i in $p^*([V])$ is equal to

$$\begin{aligned} \frac{1}{[k(Y) : k(X)]} \sum_{U_j/W_i} [k(U_j) : k(W_i)] &= \frac{[k(Y) : k(S)][k(U) : k(W_i)]t(i)}{[k(Y) : k(X)][k(U) : k(V)]l} = \\ &= \frac{[k(X) : k(S)]t(i)}{[k(W_i) : k(V)]l} = n_i. \end{aligned}$$

Lemma 5.8 *Let $f : Y \rightarrow S$ be a finite surjective morphism of normal connected schemes. Assume that $k(Y)$ is a normal extension of $k(S)$ and denote by G the corresponding Galois group. For any qfh-sheaf of sets \mathcal{F} the map $\mathcal{F}(S) \rightarrow \mathcal{F}(Y)^G$ is a bijection.*

Proof: The canonical morphism $Y \times G \rightarrow Y \times_S Y$ is finite and surjective (see (5.3)) and hence is a qfh-covering. Thus $\mathcal{F}(Y \times_S Y) \subset \mathcal{F}(Y \times G) = \mathcal{F}(Y) \times G$. The compositions

$$\mathcal{F}(Y) \xrightarrow{pr_i^*} \mathcal{F}(Y \times_S Y) \subset \mathcal{F}(Y) \times G$$

are given by the formulas

$$\begin{aligned} a &\longrightarrow \{a\}_{\sigma \in G} \\ a &\longrightarrow \{a^\sigma\}_{\sigma \in G}. \end{aligned}$$

The map f^* identifies $\mathcal{F}(S)$ with the equalizer of the maps pr_1^*, pr_2^* i.e. with $\mathcal{F}(Y)^G$.

Let $p : X \rightarrow S$ be a finite surjective morphism with X integral and S normal. Denote by Y the normalization of X in the normal closure of $k(X)$ over $k(S)$ and denote by $q_0 : Y \rightarrow X$ the canonical morphism. Set $G = Gal(k(Y)/k(S)), H = Gal(k(Y)/k(X))$. For any qfh-sheaf of abelian groups \mathcal{F} and any $a \in \mathcal{F}(X)$ set:

$$\begin{aligned} Tr_{X/S}(a) &= [k(X) : k(S)]_i \sum_{q \in Hom_S(Y, X)} q^*(a) = \\ &= [k(X) : k(S)]_i \sum_{\sigma \in G/H} (q_0^*(a))^\sigma \in \mathcal{F}(Y)^G = \mathcal{F}(X). \end{aligned}$$

Lemma 5.9 *Let V be any normal connected scheme over S . Denote by W_i the components of $X \times_S V$ and by n_i denote the corresponding multiplicities. If \mathcal{F} is a qfh-sheaf of $Z[1/p]$ -modules then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{Tr_{X/S}} & \mathcal{F}(S) \\ \downarrow & & \downarrow \\ \bigoplus \mathcal{F}(W_i) & \xrightarrow{\sum n_i Tr_{W_i/V}} & \mathcal{F}(V) \end{array}$$

Proof: Let Y be the normalization of X in the normal closure of $k(X)$ over $k(S)$, let U be any component of $Y \times_S V$ and let \tilde{U} be the normalization of U . For any $a \in \mathcal{F}(X)$ we have:

$$\begin{aligned} [Tr_{X/S}(a)|_V]_{|\tilde{U}} &= [Tr_{X/S}(a)|_Y]_{|\tilde{U}} = \\ &= [k(X) : k(S)]_i ((pr_1)_{|\tilde{U}})^* \left(\sum_{q \in Hom_S(Y, X)} q^*(a) \right). \end{aligned}$$

According to the lemma (5.6) this expression is equal to

$$\begin{aligned} [k(X) : k(S)]_i \sum_{i=1}^k \frac{[k(X) : k(S)]_{sep^t(i)}}{[k(W_i) : k(V)]_{sep^l}} \sum_{f \in Hom_V(\tilde{U}, W_i)} f^*(a|_{W_i}) = \\ = \sum_{i=1}^k n_i (Tr_{W_i/V}(a|_{W_i}))_{|\tilde{U}}. \end{aligned}$$

Since the map $\mathcal{F}(V) \rightarrow \mathcal{F}(\tilde{U})$ is injective this concludes the proof.

6 Free qfh-sheaves.

We preserve the assumptions and notations of the previous section. For any scheme Z and any ring Λ denote by $\Lambda(Z)$ the qfh-sheaf associated with the presheaf $X \rightarrow \Lambda[Hom(X, Z)]$.

Consider now the following functor on the category of normal connected schemes:

$$L_Z(S) = C_0(Z \times S/S)[1/p] = \left(\begin{array}{l} \text{free } Z[1/p] \text{ - module generated by} \\ \text{closed integral subschemes } X \subset Z \times S \\ \text{for which the projection } X \rightarrow S \\ \text{is finite and surjective.} \end{array} \right)$$

Assume that $g : V \rightarrow S$ is a morphism of normal connected schemes and $X \subset Z \times S$ is a closed integral subscheme for which the projection $X \rightarrow S$ is a finite surjective morphism. Denote by W_i the components of $X \times_{Z \times S} (V \times S) = X \times_S V$, by n_i the multiplicity of W_i and set $g^*([X]) = \sum n_i [W_i]$. One checks easily (using lemma (5.6)) that $(gh)^* = h^*g^*$, so that L_Z is a contravariant functor from the category of normal connected schemes to the category of $Z[1/p]$ -modules.

Lemma 6.1 *Assume that V is a normalization of S in a finite normal extension of $k(S)$ with Galois group G . Let X be a closed integral subscheme of $S \times Z$ finite and surjective over S . Denote by $W_i, i = 1, \dots, k$ the components of $V \times_S X$ and by n_i the corresponding multiplicities. Then the group G acts transitively on the set $\{W_i\}_{i=1}^k$ and for each i we have*

$$n_i = \frac{[k(V) : k(S)]_i}{[k(W_i) : k(X)]_i}.$$

Proof: To give a component W of $V \times_S X$ is the same as to give a point of $V \times_S X$ over the generic point of X (and hence also over the generic point of V). This is equivalent to giving a composite $k(W)$ of the fields $k(V)$ and $k(X)$ over $k(S)$. Since $k(V)$ is normal over $k(S)$ the group G acts transitively on the set of composites and the total number of composites is equal to $\frac{[k(V):k(S)]_{sep}}{[k(W):k(X)]_{sep}}$. This shows that the group G acts transitively on the set $\{W_i\}_{i=1}^k$ and hence multiplicities n_i are all equal. Denoting the common value of n_i by n and taking into account that

$$k = \frac{[k(V) : k(S)]_{sep}}{[k(W) : k(X)]_{sep}}$$

we get from (3)

$$n = \frac{[k(X) : k(S)]}{k[k(W) : k(V)]} = \frac{[k(X) : k(S)][k(W) : k(X)]_{sep}}{[k(W) : k(V)][k(V) : k(S)]_{sep}} = \frac{[k(V) : k(S)]_i}{[k(W) : k(X)]_i}.$$

Corollary 6.2 *In the situation of the lemma (6.1) denote the natural morphism $V \rightarrow S$ by g . Then the map $g^* : L_Z(S) \rightarrow L_Z(V)^G$ is an isomorphism.*

Proof: Since $g : V \longrightarrow S$ is finite and surjective we have a well defined direct image homomorphism $g_* : L_Z(V) \longrightarrow L_Z(S)$. The composition g_*g^* coincides with multiplication by $[k(V) : k(S)]$. This shows that g^* is injective. The $Z[1/p]$ -module $L_Z(V)^G$ is generated by expressions of the form

$$\sum_{\sigma \in G/H} [W^\sigma]$$

where W is a closed integral subscheme of $Z \times V$ finite and surjective over V and $H = \text{Stab}_G(W)$. Denote by X the image of W in $Z \times S$. Thus X is a closed integral subscheme of $Z \times S$ finite and surjective over S . It is clear that W is a component of $V \times_S X$. According to the lemma (6.1) we conclude that

$$g^*([X]) = \frac{[k(V) : k(S)]_i}{[k(W) : k(X)]_i}.$$

This concludes the proof since

$$\frac{[k(V) : k(S)]_i}{[k(W) : k(X)]_i} \in Z[1/p]^*.$$

Theorem 6.3 *Assume that Z is separated, then*

$$\Gamma(S, Z[1/p](Z)) = L_Z(S)$$

for any normal connected S .

Proof: Denote by \mathcal{F} the presheaf $X \longrightarrow Z[1/p](\text{Hom}(X, Z))$. The sheaf $\underline{\mathcal{F}} = Z[1/p](Z)$ is the qfh-sheaf associated with \mathcal{F} and is constructed in two steps. First one defines a separated presheaf \mathcal{F}' setting:

$$\mathcal{F}'(X) = \mathcal{F}(X) / \left(\begin{array}{l} \text{subgroup of } u \in \mathcal{F}(X) \text{ for which there} \\ \text{exists a qfh-covering } \{X_i \longrightarrow X\}_{i \in I} \text{ such that} \\ u|_{X_i} = 0 \text{ for all } i. \end{array} \right)$$

Next one defines the sheaf $\underline{\mathcal{F}}$ by means of the formula

$$\underline{\mathcal{F}}(X) = \check{H}^0(X, \mathcal{F}') = \varinjlim_{\mathcal{V}} \check{H}^0(\mathcal{V}, \mathcal{F}')$$

where \mathcal{V} runs through all *qfh*-coverings of X . If X is reduced irreducible and a morphism $X' \rightarrow X$ is dominant then $Hom(X, Z) \subset Hom(X', Z)$ and hence $\mathcal{F}(X) \subset \mathcal{F}(X')$. This implies that $\mathcal{F}'(X) = \mathcal{F}(X)$ for any reduced irreducible X .

Any *qfh*-covering of a normal connected scheme admits a refinement of the form $\mathcal{V} = \{V_i \rightarrow V \rightarrow S\}$ where V is a normalization of S in a finite normal extension of the field $k(S)$ and $\{V_i\}_{i \in I}$ is a Zariski open covering of V (see (10.4)). Denote by G Galois group $Gal(k(V)/k(S))$.

Lemma 6.4 $\check{H}^0(\mathcal{V}, \mathcal{F}') = \mathcal{F}(V)^G$.

Proof: The family of morphisms $f_\sigma = (Id_V, \sigma) : V \rightarrow V \times_S V$ is a *qfh*-covering of $V \times_S V$. Since $f_\sigma^{-1}(V_i \times_S V_j) = V_i \cap V_j^{\sigma^{-1}}$ we conclude that

$$\{f_\sigma : V_i \cap V_j^{\sigma^{-1}} \rightarrow V_i \times_S V_j\}_{\sigma \in G}$$

is a *qfh*-covering of $V_i \times_S V_j$. Thus

$$\prod_{i,j} \mathcal{F}'(V_i \times_S V_j) \subset \prod_{i,j,\sigma} \mathcal{F}'(V_i \cap V_j^{\sigma^{-1}}) = \prod_{i,j,\sigma} \mathcal{F}(V_i \cap V_j^{\sigma^{-1}})$$

and hence

$$\check{H}^0(\mathcal{V}, \mathcal{F}') = \ker\left(\prod_i \mathcal{F}(V_i) \rightarrow \prod_{i,j,\sigma} \mathcal{F}(V_i \cap V_j^{\sigma^{-1}})\right).$$

The family $\{s_i \in \mathcal{F}(V_i)\}_{i \in I}$ is compatible if and only if the restrictions of s_i and $s^{\sigma^{-1}}$ to $V_i \cap V_j^{\sigma^{-1}}$ coincide for any i, j, σ . This implies in particular that $(s_i)|_{V_i \cap V_j} = (s_j)|_{V_i \cap V_j}$. Since \mathcal{F} restricts to a sheaf in the Zariski topology on V this shows that $s_i = s|_{V_i}$ for a unique $s \in \mathcal{F}(V)$.

The same reasoning shows that the family $\{s|_{V_i}\}$ is compatible if and only if $s \in \mathcal{F}(V)^G$. Lemma is proven.

According to this lemma we have

$$\Gamma(S, Z[1/p](Z)) = \varinjlim \mathcal{F}(V)^G$$

where the limit is taken over all finite normal extensions E of $k(S)$, V denotes the normalization of S in E and $G = Gal(E/k(S))$. Identifying any

morphism $V \rightarrow Z$ with its graph we get a canonical injective homomorphism $\mathcal{F} \rightarrow L_Z$ of functors on the category of normal connected schemes. Corollary (6.2) shows that

$$\Gamma(S, Z[1/p](Z)) = \varinjlim \mathcal{F}(V)^G \subset \varinjlim L_Z(V)^G = L_Z(S).$$

To finish the proof it is sufficient to note that if $X \subset Z \times S$ is a closed integral subscheme which is finite and surjective over S and if V denotes the normalization of X in the normal closure of $k(X)$ over $k(S)$ then each component of $X \times_S V$ maps isomorphically onto V and hence $[X]_{|V} \in \mathcal{F}(V)^G$. This shows that the map $\Gamma(S, Z[1/p](Z)) \rightarrow L_Z(S)$ is surjective.

Finally we want to establish a relation between cycles and morphisms to symmetric powers. Assume that $X \rightarrow Y$ is a finite flat morphism of constant degree d . Thus $X = \text{Spec}(\mathcal{A})$ where \mathcal{A} is a locally free sheaf of \mathcal{O}_Y -algebras of constant rank d . Multiplication in $\mathcal{A}^{\otimes d}$ defines an \mathcal{O}_Y -bilinear pairing

$$(\mathcal{A}^{\otimes d})^{\Sigma_d} \otimes_{\mathcal{O}_Y} \wedge^d \mathcal{A} \rightarrow \wedge^d \mathcal{A},$$

making $\wedge^d \mathcal{A}$ a module over an \mathcal{O}_Y -algebra $(\mathcal{A}^{\otimes d})^{\Sigma_d}$. In this way we get a homomorphism of \mathcal{O}_Y -algebras

$$(\mathcal{A}^{\otimes d})^{\Sigma_d} \rightarrow \text{End}_{\mathcal{O}_Y}(\wedge^d \mathcal{A}) = \mathcal{O}_Y$$

and hence a canonical section for the projection $S^d(X/Y) \rightarrow Y$. Assume now, that $X \rightarrow Y$ is a finite surjective morphism, X is integral and Y is normal connected. Set $d = [k(X) : k(Y)]$ and $X = \text{Spec}(\mathcal{A})$. According to the above construction we get a canonical homomorphism of $k(Y)$ -algebras $(k(X)^{\otimes d})^{\Sigma_d} \rightarrow k(Y)$. For any $y \in Y$ the image of $(\mathcal{A}_y^{\otimes d})^{\Sigma_d}$ in $k(Y)$ is integral over \mathcal{O}_Y and hence is contained in \mathcal{O}_Y . This shows that situation we still have a canonical homomorphism of \mathcal{O}_Y -algebras $(\mathcal{A}^{\otimes d})^{\Sigma_d} \rightarrow \mathcal{O}_Y$ and hence a canonical section for the projection $S^d(X/Y) \rightarrow Y$.

Let Z be a scheme of finite type over k such, that any finite subset of Z is contained in an affine open set, so that one can consider symmetric powers $S^d(Z)$ of Z . Let S be a normal connected scheme and let $X \subset Z \times S$ be a closed integral subscheme for which the projection $X \rightarrow S$ is finite and surjective. Set $d = [k(X) : k(S)]$. The subscheme X defines an S -morphism

$$S \rightarrow S^d(X/S) \rightarrow S^d(Z \times S/S) = S^d(Z) \times S/S$$

and hence a morphism $f_X : S \rightarrow S^d(Z)$. In this way we get a canonical homomorphism of monoids

$$C_0^{eff}(Z \times S/S) \rightarrow \text{Hom}(S, \prod_{d=0}^{\infty} S^d(Z))$$

where $C_0^{eff}(Z \times S/S) \rightarrow C_0(Z \times S/S)$ is the submonoid of effective cycles. Localizing at p we finally get a homomorphism

$$L_Z^{eff}(S) \rightarrow \text{Hom}(S, \prod_{d=0}^{\infty} S^d(Z))[1/p].$$

Theorem 6.5 *If Z is separated then the above homomorphism*

$$L_Z^{eff}(S) \rightarrow \text{Hom}(S, \prod_{d=0}^{\infty} S^d(Z))[1/p]$$

is an isomorphism for any normal connected S .

Proof: Set $Y = Z^d$. The group Σ_d acts admissibly on Y and $Y/\Sigma_d \cong S^d(Z)$. If Ω is an algebraically closed field and $\omega : \text{Spec}(\Omega) \rightarrow S^d(Z)$ is any Ω -point then Σ_d acts transitively on the set of Ω -points of Y over ω . This implies that the natural morphism $Y \times \Sigma_d \rightarrow Y \times_{S^d(Z)} Y$ is surjective (and finite) and hence defines a qfh-covering of $Y \times_{S^d(Z)} Y$. This shows that for any qfh-sheaf \mathcal{F} one has $\mathcal{F}(S^d(Z)) = \mathcal{F}(Y)^{\Sigma_d}$.

Let $p_i : Y \rightarrow Z$ be the projection on the i -th factor. The element

$$\sum_{i=1}^d p_i \in \Gamma(Y, Z[1/p](Z))$$

is Σ_d -invariant and hence defines a canonical element

$$u_d \in \Gamma(S^d(Z), Z[1/p](Z)).$$

Let now $f : S \rightarrow S^d(Z)$ be any morphism. It defines an element $f^*(u_d) \in \Gamma(S, Z[1/p](Z)) = L_Z(S)$. One verifies easily that $\mathcal{F}^*(u_d) \in L_Z^{eff}(S)$ and that the resulting map

$$\text{Hom}(S, \prod_{d=0}^{\infty} S^d(Z))[1/p] \rightarrow L_Z^{eff}(S)$$

is inverse to the one constructed before.

7 Singular homology of qfh-sheaves.

Denote by $(Sch/k)_h$ (resp. by $(Sch/k)_{qfh}, (Sch/k)_{et}$) the site of schemes of finite type over k with h-topology (resp. qfh-topology, etale topology). Wherever no topology is specified it means that the corresponding result holds in any of the above topologies.

Let \mathcal{F} be a presheaf of abelian groups. Applying \mathcal{F} to the cosimplicial scheme Δ we get a simplicial abelian group $C_*(\mathcal{F})$. The homotopy groups of $C_*(\mathcal{F})$ coincide with the homology of the complex $(C_*(\mathcal{F}), d = \sum (-1)^i \partial_i)$ and will be denoted $H_*^{sing}(\mathcal{F})$. For any abelian group A we set further:

$$H_*^{sing}(\mathcal{F}, A) = H_*(C_*(\mathcal{F}) \overset{L}{\otimes} A)$$

$$H_{sing}^*(\mathcal{F}, A) = H^*(RHom(C_*(\mathcal{F}), A)).$$

Note that if \mathcal{F} has no torsion then $C_*(\mathcal{F}) \overset{L}{\otimes} Z/n = C_*(\mathcal{F}) \otimes Z/n$ and $RHom(C_*(\mathcal{F}), Z/n) = Hom(C_*(\mathcal{F}), Z/n)$ for any $n > 0$.

Define a presheaf \mathcal{F}_q by the formula $\mathcal{F}_q(X) = \mathcal{F}(X \times \Delta^q)$. Thus \mathcal{F}_* is a simplicial presheaf of abelian groups and the complex $C_*(\mathcal{F})$ coincides with the complex of global sections of the corresponding complex of presheaves \mathcal{F}_* . Applying the functor of associated sheaf to this complex of presheaves we get a complex of sheaves $\underline{\mathcal{F}}_*$. Note that \mathcal{F}_q is a sheaf provided that \mathcal{F} is, however $\underline{\mathcal{F}}_q$ need not coincides with the sheaf $(\underline{\mathcal{F}})_q$.

Suppose, that n is prime to $char(k)$ and consider the hypercohomology spectral sequences corresponding to the complex $\underline{\mathcal{F}}_*$

$$I_1^{p,q} = Ext^p(\underline{\mathcal{F}}_q, Z/n) \Rightarrow Ext^{p+q}(\underline{\mathcal{F}}_*, Z/n)$$

$$II_2^{p,q} = Ext^p(H_q(\underline{\mathcal{F}}_*), Z/n) \Rightarrow Ext^{p+q}(\underline{\mathcal{F}}_*, Z/n).$$

The following theorem shows that the first spectral sequence degenerates.

Theorem 7.1 *For any \mathcal{F} and any q the canonical homomorphism $\underline{\mathcal{F}} \rightarrow \underline{\mathcal{F}}_q$ induces isomorphisms on $Ext^*(-, Z/n)$.*

Proof: Using the fact that $\Delta^q \cong A^q$ and proceeding by induction on q we reduce the general case to the case $q = 1$.

Lemma 7.2 *The natural homomorphism of sheaves $Z(A^1) \longrightarrow Z$ induces isomorphisms*

$$\underline{Ext}^*(Z, Z/n) \longrightarrow \underline{Ext}^*(Z(A^1), Z/n).$$

In other words $\underline{Hom}(Z(A^1), Z/n) = Z/n$ and $\underline{Ext}^i(Z(A^1), Z/n) = 0$ for $i > 0$.

Proof: By definition the sheaf $\underline{Ext}^i(Z(A^1), Z/n)$ is the sheaf associated with the presheaf

$$X \longrightarrow Ext_{Sch/X}^i(Z(A^1)|_{Sch/X}, Z/n).$$

The sheaf $Z(A^1)|_{Sch/X}$ coincides with $Z(A_X^1/X)$ and hence we have

$$Ext_{Sch/X}^i(Z(A^1)|_{Sch/X}, Z/n) = H^i(A_X^1, Z/n) = H^i(X, Z/n).$$

It is clear that the associated sheaf is isomorphic to Z/n for $i = 0$ and is trivial for $i > 0$.

This lemma implies easily that for any sheaf of abelian groups G the natural homomorphism $G \otimes Z(A^1) \longrightarrow G$ induces isomorphisms

$$Ext^*(G, Z/n) \longrightarrow Ext^*(G \otimes Z(A^1), Z/n).$$

Consider the embeddings $i_0, i_1 : G \longrightarrow G \otimes Z(A^1)$ defined by the points $0, 1 \in A^1$ respectively. These morphisms induce the same isomorphism

$$i_0^* = i_1^* : Ext^*(G \otimes Z(A^1), Z/n) \longrightarrow Ext^*(G, Z/n).$$

We will say that the sheaf G is contractible if there exists a homomorphism $\phi : G \otimes Z(A^1) \longrightarrow G$ such, that $\phi \circ i_0 = 0$ and $\phi \circ i_1 = Id_G$. We conclude immediately from previous remarks that for any contractible sheaf G we have $Ext^*(G, Z/n) = 0$. To finish the proof it is sufficient to note that the kernel of the natural morphism of sheaves $\underline{\mathcal{F}}_1 \longrightarrow \mathcal{F}$ is contractible.

Corollary 7.3

$$I_2^{p,q} = \begin{cases} Ext^p(\underline{\mathcal{F}}, Z/n) & q = 0 \\ 0 & q > 0 \end{cases}$$

and hence $Ext^p(\underline{\mathcal{F}}_*, Z/n) = Ext^p(\underline{\mathcal{F}}, Z/n)$.

Now let us investigate the second spectral sequence. From now on we will assume that \mathcal{F} is equipped with trace maps $Tr_{X/S}$ satisfying the properties (4.1). The sheaf $H_q(\mathcal{F}_*)$ is associated with the presheaf $X \longrightarrow H_q(\mathcal{F}(X \times \Delta^*))$. We will denote this presheaf by \mathcal{H}_q .

Lemma 7.4 *For any scheme X consider the embeddings $i_0, i_1 : X \longrightarrow X \times A^1$ defined by the points $0, 1$ of A^1 . The induced homomorphisms of complexes*

$$i_0^*, i_1^* : \mathcal{F}(X \times A^1 \times \Delta^\cdot) \longrightarrow \mathcal{F}(X \times \Delta^\cdot)$$

are homotopic.

Proof: One can use the usual topological homotopy operator (see [14][ch.2,§8]). Define a homomorphism

$$s_p : \mathcal{F}(X \times A^1 \times \Delta^p) \longrightarrow \mathcal{F}(X \times \Delta^{p+1})$$

by the formula

$$s_p = \sum_{i=0}^p (-1)^i (Id_X \times \psi_i)^*$$

where $\psi_i : \Delta^{p+1} \longrightarrow \Delta^p \times A^1$ is the linear isomorphism taking v_j to $v_j \times 0$ if $j \leq i$ or to $v_{j-1} \times 1$ if $j > i$ (here $v_j = (0, \dots, 1, \dots, 0)$ is the j -th vertex of Δ^{p+1} (resp. Δ^p)). A straightforward computation shows that $sd + ds = i_1^* - i_0^*$.

The usual reasoning now proves the following:

Corollary 7.5 *The presheaf \mathcal{H}_q is homotopy invariant.*

Assume that $f : X \longrightarrow S$ is a finite surjective map with X integral and S regular. For any i we have a homomorphism

$$Tr_{X \times \Delta^i / S \times \Delta^i} : \mathcal{F}(X \times \Delta^i) \longrightarrow \mathcal{F}(S \times \Delta^i).$$

The properties of trace maps imply in particular that the resulting map

$$Tr_{X/S} : \mathcal{F}(X \times \Delta^*) \longrightarrow \mathcal{F}(S \times \Delta^*)$$

is a homomorphism of complexes and hence induces homomorphisms on homology groups

$$Tr_{X/S} : \mathcal{H}_q(X) \longrightarrow \mathcal{H}_q(S).$$

It is clear that these maps satisfy the conditions of definition (4.1). Assuming now that our base scheme is the spectrum of an algebraically closed field of characteristic zero we deduce from theorem (4.5) that

$$II_2^{p,q} = Ext_{Ab}^p(H_q^{sing}(\mathcal{F}), Z/n).$$

Comparing this spectral sequence to the hyperhomology spectral sequence corresponding to the complex of abelian groups $C_*(\mathcal{F})$ we conclude that (identifying $C_*(|calF)$ with the corresponding complex of constant sheaves) the natural homomorphism $C_*(\mathcal{F}) \rightarrow \underline{\mathcal{F}}_*$ induces isomorphisms on $Ext^*(-, Z/n)$. Thus we have proven the main result of this paper.

Theorem 7.6 *Let k be an algebraically closed field of characteristic zero. Let \mathcal{F} be a presheaf on $Sch/Spec(k)$ which admits trace maps. Then both arrows in the diagram*

$$C_*(\mathcal{F}) \longrightarrow \underline{\mathcal{F}}_* \longleftarrow \underline{\mathcal{F}}$$

induce isomorphisms on $Ext^(-, Z/n)$. In particular for any $n > 0$ we have canonical isomorphisms*

$$H_{sing}^*(\mathcal{F}, Z/n) = Ext^*(\underline{\mathcal{F}}, Z/n).$$

Corollary 7.7 *For any qfh-sheaf \mathcal{F} we have canonical isomorphisms*

$$H_{sing}^*(\mathcal{F}, Z/n) = Ext_{qfh}^*(\mathcal{F}, Z/n).$$

Proof: One has only to note that qfh-sheaves admit trace maps according to results of §5.

Applying corollary (7.7) to the free qfh-sheaf $Z(X)$ and using the theorem (6.3) we come to the following result.

Corollary 7.8 *Let X be a separated scheme of finite type over an algebraically closed field k of characteristic zero. Then*

$$H_{sing}^*(X, Z/n) = H_{qfh}^*(X, Z/n) = H_{et}^*(X, Z/n).$$

Remark: Resolution of singularities for varieties over an algebraically closed field of characteristic $p > 0$ would imply that all results of this section are valid in positive characteristic as well (with n prime to p).

8 Singular homology of varieties over C .

Denote by CW the category of topological spaces which admit a triangulation. Note that the product in CW is the usual product equipped with the compactly generated topology. We will be considering CW as a site with the Grothendieck topology defined by local homeomorphisms. To distinguish the usual topological simplices from the schemes Δ^i used above we will use the notation Δ_{top}^i for the topological simplices.

The spaces Δ_{top}^i form a cosimplicial topological space. If \mathcal{F} is any presheaf of abelian groups on CW then we will denote by $C_*(\mathcal{F})$ the simplicial abelian group $\mathcal{F}(\Delta_{top}^*)$ and for any abelian group A we will denote by $H_*^{sing}(\mathcal{F}, A)$ (resp. $H_{sing}^*(\mathcal{F}, A)$) the homology of the complex $C_*(\mathcal{F}) \otimes^L A$ (resp. $RHom(C_*(\mathcal{F}), A)$). For any presheaf \mathcal{F} we will be also considering (the same as above) the presheaves \mathcal{F}_q defined by the formula $\mathcal{F}_q(X) = \mathcal{F}(X \times \Delta_{top}^q)$. Thus \mathcal{F} is a simplicial presheaf of abelian groups and $C_*(\mathcal{F})$ coincides with the complex of global sections of the associated complex of presheaves. Our main theorem (7.6) has the following easy topological version.

Theorem 8.1 *For any presheaf \mathcal{F} on CW and any abelian group A both arrows in the diagram*

$$C_*(\mathcal{F}) \longrightarrow \underline{\mathcal{F}}_* \longleftarrow \underline{\mathcal{F}}$$

induce isomorphisms on $Ext^(-, A)$, so that*

$$Ext^*(\underline{\mathcal{F}}, A) = H_{sing}^*(\mathcal{F}, A).$$

Proof: The proof of this theorem is strictly parallel to that of theorem (7.6). One considers the hypercohomology spectral sequences

$$I_1^{pq} = Ext^p(\underline{\mathcal{F}}_q, A) \Rightarrow Ext^{p+q}(\underline{\mathcal{F}}, A)$$

$$II_2^{pq} = Ext^p(H_q(\underline{\mathcal{F}}_*), A) \Rightarrow Ext^{p+q}(\underline{\mathcal{F}}, A).$$

Repeating the argument of (7.1) one shows that $Ext^*(\underline{\mathcal{F}}_*, A) = Ext^*(\underline{\mathcal{F}}, A)$ and hence the first spectral sequence degenerates and provides isomorphisms $Ext^*(\underline{\mathcal{F}}_*, A) \longrightarrow Ext^*(\underline{\mathcal{F}}, A)$. Furthermore the sheaf $H_q(\underline{\mathcal{F}}_*)$ is associated with the presheaf \mathcal{H}_q of the form $\mathcal{H}_q(X) = H_q(\mathcal{F}(X \times \Delta_{top}^*))$. The presheaf \mathcal{H}_q is homotopy invariant and hence $\mathcal{H}_q(X) = \mathcal{H}_q(pt) = H_q^{sing}(\mathcal{F})$ for any contractible space X . Since any object of CW is locally contractible this

implies that the sheaf $H_q(\underline{\mathcal{F}}_*)$ is a constant sheaf equal to $H_q^{sing}(\mathcal{F})$. Thus $C_*(\mathcal{F}) \rightarrow \underline{\mathcal{F}}_*$ is a quasi-isomorphism and in particular induces isomorphisms on any Ext-groups.

For any abelian monoid M denote by M^+ the associated abelian group. If M is a simplicial abelian group then applying the functor $^+$ componentwise we get a simplicial abelian group M^+ .

Let X be an object of CW . Consider the abelian topological monoid $\coprod_{d=0}^{\infty} S^d(X)$ and the corresponding simplicial abelian monoid $Sin.(\coprod_{d=0}^{\infty} S^d(X))$. The following result is a variant of the Dold-Thom theorem [3].

Theorem 8.2 *The evident embedding*

$$Z(Sin.(X)) \longrightarrow Sin.(\coprod_{d=0}^{\infty} S^d(X))^+$$

is a weak equivalence of simplicial abelian groups, i.e.

$$\pi_*(Sin.(\coprod_{d=0}^{\infty} S^d(X))^+) = \pi_*(Z(Sin.(X))) = H_*(X, Z).$$

Denote by $j : CW \rightarrow (Sch/C)$ the canonical morphism of sites such that $j^{-1}(X) = X(C)$ for any object X of Sch/C . The classical comparison theorem [7] states that $j_*(Z/n) = Z/n$, $R^q i_*(Z/n) = 0$ for $q > 0$ and hence the natural map

$$Ext^*(\mathcal{F}, Z/n) \longrightarrow Ext^*(j^*(\mathcal{F}), Z/n)$$

is an isomorphism for any etale sheaf of abelian groups \mathcal{F} on Sch/C .

Let Z be an object of Sch/C . Denote by \mathcal{M} (resp. \mathcal{M}_{top}) the etale (resp. topological) sheaf represented by the ind-scheme $\coprod_{d=0}^{\infty} S^d(Z)$ (resp. by the topological space $\coprod_{d=0}^{\infty} S^d(X(C))$). Since j^* commutes with direct limits one sees immediately that $j^*(\mathcal{M}) = \mathcal{M}_{top}$. Let further \mathcal{F} (resp. \mathcal{F}_{top}) denote the presheaf of abelian groups $X \rightarrow \mathcal{M}(X)^+$ (resp. $X \rightarrow \mathcal{M}_{top}(X)^+$). Let finally \mathcal{M}^+ (resp. \mathcal{M}_{top}^+) be the sheaf of abelian groups associated with the presheaf \mathcal{F} (resp. \mathcal{F}_{top}). The homomorphism $\mathcal{M} \rightarrow \mathcal{M}^+$ (resp. $\mathcal{M}_{top} \rightarrow \mathcal{M}_{top}^+$) is universal for homomorphisms from \mathcal{M} (resp. \mathcal{M}_{top}) to sheaves of abelian groups and the universal property of j^* shows immediately that $j^*(\mathcal{M}^+) = j^*(\mathcal{M})^+ = \mathcal{M}_{top}^+$. The canonical homomorphism of sheaves

$M \longrightarrow j_*(\mathcal{M}_{top})$ together with an evident embedding of cosimplicial spaces $\Delta_{top} \longrightarrow \Delta \cdot(C)$ defines a homomorphism of simplicial abelian monoids

$$\mathcal{M}(\Delta \cdot) \longrightarrow j_*(\mathcal{M}_{top})(\Delta \cdot) = \mathcal{M}_{top}(\Delta \cdot(C)) \longrightarrow \mathcal{M}_{top}(\Delta_{top}).$$

Applying the functor $+$ we get further a homomorphism of simplicial abelian groups (and hence of the corresponding complexes of abelian groups)

$$C_*(Z) = \mathcal{M}(\Delta \cdot)^+ \longrightarrow \text{Sin.}(\prod_{d=0}^{\infty} S^d(Z(C)))^+.$$

Thus for any $n \geq 0$ we have the induced homomorphisms on homology and cohomology with Z/n -coefficients

$$H_*^{sing}(Z, Z/n) \longrightarrow H_*(Z(C), Z/n)$$

$$H^*(Z(C), Z/n) \longrightarrow H_{sing}^*(Z, Z/n).$$

Theorem 8.3 *For any $Z \in \text{Sch}/C$ the above homomorphisms are isomorphisms.*

Proof: Note first of all that the presheaf \mathcal{F} admits trace maps. In fact if S is any normal connected scheme then theorems (6.3) and (6.5) show that $\mathcal{F}(S) = \Gamma(S, Z_{qfh}(Z))$. Let now $X \longrightarrow S$ be a finite surjective map with X integral and S regular. Denote the normalization of X by \tilde{X} and define the trace map $Tr_{X/S} : \mathcal{F}(X) \longrightarrow \mathcal{F}(S)$ as the composition

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(\tilde{X}) = \Gamma(\tilde{X}, Z_{qfh}(Z)) \xrightarrow{Tr_{\tilde{X}/S}} \Gamma(S, Z_{qfh}(Z)) = \mathcal{F}(S).$$

One checks easily that these trace maps satisfy the conditions (4.1). The theorem (7.6) implies now that both arrows in the diagram

$$C_*(Z) = C_*(\mathcal{F}) \longrightarrow \underline{\mathcal{F}}_* \longleftarrow \underline{\mathcal{F}} = \mathcal{M}^+$$

are quasi-isomorphisms. Applying the functor j^* we get a diagram of sheaves on CW which has the same property. Consider finally the commutative diagram

$$\begin{array}{ccccccc} j^*(C_*(Z)) & \longrightarrow & j^*(\underline{\mathcal{F}}_*) & \longleftarrow & j^*(\mathcal{M}^+) & = & \mathcal{M}_{top}^+ \\ \downarrow & & \downarrow & & & & \downarrow = \\ C_*(\mathcal{F}_{top}) & \longrightarrow & \underline{\mathcal{F}}_{top*} & \longleftarrow & \underline{\mathcal{F}}_{top} & = & \mathcal{M}_{top}^+. \end{array}$$

All horizontal arrows induce isomorphisms on $Ext^*(-, Z/n)$. The right hand side vertical arrow is an isomorphism. Thus the left hand side vertical arrow induces isomorphisms on $Ext^*(-, Z/n)$. In other words the homomorphism of complexes of abelian groups

$$C_*(Z) \longrightarrow \text{Sin}_*(\prod_{d=0}^{\infty} S^d(Z(C)))^+$$

induces isomorphism on cohomology and hence also on homology with finite coefficients.

9 Algebraic Lawson homology.

Let $Z \subset P^N$ be an integral projective scheme over C . Denote by $C_{r,d}(Z)$ the (not necessary connected) projective algebraic set of effective cycles of dimension r and degree d on Z [18]. We will be considering $C_{r,d}(Z)$ as a reduced projective scheme. The Chow monoid of effective r -cycles on Z is a disjoint union

$$C_r(Z) = \prod_{d=0}^{\infty} C_{r,d}(Z)$$

provided with the operation determined by addition of cycles. Thus $C_r(Z)$ is an abelian topological monoid (to simplify notations we do not distinguish between a reduced scheme over C and the topological space of its C -valued points).

The Lawson homology of Z are defined by means of the formula

$$L_r H_{2r+i}(Z) = \pi_i(C_r(Z)^+)$$

where for any topological monoid M we denote by M^+ its group completion.

Let M be any abelian topological monoid. The canonical monoid map $|\text{Sin.}(M)| \longrightarrow M$ is a weak equivalence and hence induces an equivalence of group completions $|\text{Sin.}(M)|^+ \longrightarrow M^+$. On the other hand $|\text{Sin.}(M)|^+$ is canonically homotopy equivalent to $|\text{Sin.}(M)^+|$ where $\text{Sin.}(M)^+$ is the componentwise group completion of the simplicial abelian monoid $\text{Sin.}(M)$. Thus we get a weak equivalence $|\text{Sin.}(M)^+| \longrightarrow M^+$. This shows in particular that the groups $L_r H_{2r+i}(Z)$ coincide with the homotopy groups of

a simplicial abelian group $\text{Sin.}(C_r(Z))^+$. One can consider also the algebraic version of this construction: consider the simplicial abelian monoid $\text{Hom}(\Delta, C_r(Z))$ apply to it the group completion functor and finally set

$$L_r H_{2r+i}^{\text{alg}}(Z) = \pi_i(\text{Hom}(\Delta, C_r(Z))^+).$$

Since every algebraic morphism $\Delta^q \rightarrow C_r(Z)$ defines by restriction a continuous map $\Delta_{\text{top}}^q \rightarrow C_r(Z)$ we get a homomorphism of simplicial abelian groups

$$\text{Hom}(\Delta, C_r(Z))^+ \rightarrow \text{Sin.}(C_r(Z))^+$$

and hence the induced maps from algebraic to topological Lawson homology with finite coefficients

$$L_r H_{2r+i}^{\text{alg}}(Z, Z/n) \rightarrow L_r H_{2r+i}(Z, Z/n).$$

The simplicial abelian groups used to define $L_r H_{2r+i}^{\text{alg}}(Z)$ may be given a slightly different description. E.Friedlander and H.Lawson [4] proved that for any normal connected scheme S the abelian monoid $\text{Hom}(S, C_r(Z))$ coincides with the monoid of effective cycles on $S \times Z$ every component of which is equidimensional of relative dimension r over S . Thus $\text{Hom}(\Delta^q, C_r(Z))^+$ coincides with the group of cycles in $\Delta^q \times Z$ every component of which is equidimensional of relative dimension r over Δ^q .

Theorem 9.1 *For any Z and any $n > 0$ we have a canonical isomorphism*

$$L_r H_{2r+i}^{\text{alg}}(Z, Z/n) \cong L_r H_{2r+i}(Z, Z/n).$$

Proof: Let \mathcal{M} (resp. \mathcal{M}_{top}) denotes the etale (resp. topological) sheaf of abelian groups represented by $C_r(Z)$. Let further \mathcal{F} denote the presheaf of abelian groups $X \rightarrow \mathcal{M}(X)^+$ and let \mathcal{M}^+ denote the sheaf associated with \mathcal{F} . It is clear that $j^*(\mathcal{M}) = \mathcal{M}_{\text{top}}$ and hence $j^*(\mathcal{M}^+) = \mathcal{M}_{\text{top}}^+$. Notice further that \mathcal{M} (and hence also \mathcal{F}) admit trace maps: suppose that we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & C_r(Z) \\ f \downarrow & & \\ S & & \end{array}$$

with f finite and surjective, X integral and S normal. Set $d = [k(X) : k(S)]$. According to §6 f defines a morphism $S \rightarrow S^d(X/S)$. Define $Tr_{X/S}(g)$ as the composition

$$S \rightarrow S^d(X/S) \rightarrow S^d(X) \xrightarrow{S^d(g)} S^d(C_r(Z)) \rightarrow C_r(Z)$$

where the last morphism is defined by the monoid structure on $C_r(Z)$. It is easy to check that

$$Tr_{X/S} : Hom(X, C_r(Z)) \rightarrow Hom(S, C_r(Z))$$

is a monoid homomorphism and further that it satisfies the conditions (4.1). Proceeding now exactly as in the proof of the theorem (8.3) we get a commutative diagram

$$\begin{array}{ccccc} C_*(\mathcal{F}) & \longrightarrow & j^*(\underline{\mathcal{F}}_*) & \longleftarrow & j^*(\mathcal{M}^+) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(\mathcal{F}_{top}) & \longrightarrow & \underline{\mathcal{F}}_{top*} & \longleftarrow & \mathcal{M}_{top}^+ \end{array}$$

All the horizontal arrows and the right hand side vertical arrow induce isomorphisms on $Ext^*(-, Z/n)$. Thus the homomorphism of complexes of abelian groups

$$Hom(\Delta, C_r(Z))^+ = C_*(\mathcal{F}) \rightarrow C_*(\mathcal{F}_{top}) = Sin.(C_r(Z))^+$$

induces isomorphisms on cohomology with finite coefficients and hence also on homology with finite coefficients.

10 Appendix: h-cohomology.

A morphism of schemes $p : X \rightarrow Y$ is called a topological epimorphism if the underlying Zariski topological space of Y is a quotient space of the underlying Zariski topological space of X (i.e. p is surjective and a subset A of Y is open if and only if $p^{-1}(A)$ is open in X), p is called a universal topological epimorphism if for any Z/Y the morphism $p_Z : X \times_Y Z \rightarrow Z$ is a topological epimorphism.

An h-covering of a scheme X is a finite family of morphisms of finite type $\{p_i : X_i \rightarrow X\}$ such that $\coprod p_i : \coprod X_i \rightarrow X$ is a universal topological epimorphism.

A qfh-covering of a scheme X is an h-covering $\{p_i\}$ such that all the morphisms p_i are quasi-finite.

h-coverings (resp. qfh-coverings) define a pretopology on the category of schemes, h-topology (resp. qfh-topology) is the associated topology.

Using the theorem of Chevalley (see [6, p.24]) one proves easily the following lemma.

Lemma 10.1 *Let X be a normal connected scheme and let $\{f_i : X_i \rightarrow X\}_{i \in I}$ be a finite family of quasi-finite morphisms of finite type. Assume that all X_i are irreducible and denote by J the set of those i for which X_i dominate X . The above family is a qfh-covering if and only if $X = \cup_{i \in J} f_i(X_i)$.*

Let S be a fixed base scheme (which will be assumed noetherian). Denote by Sch/S the category of schemes of finite type over S and consider Sch/S as a site in h- qfh- or etale topology. We have evident morphisms of sites

$$(Sch/S)_h \xrightarrow{\alpha} (Sch/S)_{qfh} \xrightarrow{\beta} (Sch/S)_{et}.$$

Lemma 10.2 *Let k be a separably closed field. Then for any qfh-sheaf of abelian groups \mathcal{F} on $Sch/Spec(k)$ and any $i > 0$ one has:*

$$H_{qfh}^i(Spec(k), \mathcal{F}) = 0.$$

Proof: See [21].

Theorem 10.3 *Let n be any integer, then $\beta_*(Z/n) = Z/n, R^q \beta_*(Z/n) = 0$ for $q > 0$.*

Proof: It is sufficient to show that if X is a strictly henselian scheme then $H_{qfh}^0(X, Z/n) = Z/n, H_{qfh}^q(X, Z/n) = 0$ for $q > 0$. The first formula is evident.

Denote by $Finite(X)$ the site which objects are schemes finite over X and coverings are surjective families of morphisms. We have an obvious morphism of sites

$$\gamma : (Sch/X)_{qfh} \rightarrow Finite(X).$$

Lemma (10.1) together with the standard properties of quasi-finite schemes over strictly henselian schemes shows that for any qfh-sheaf of abelian groups \mathcal{F} on Sch/X this morphism of sites induces isomorphisms

$$H_{finite}^i(X, \gamma_*(\mathcal{F})) = H_{qfh}^i(X, \mathcal{F}).$$

Hence it is sufficient to show that $H_{finite}^i(X, Z/n) = 0$ for $i > 0$.

Let $x : Spec(k) \rightarrow X$ be the closed point of X . For any finite morphism $Y \rightarrow X$ the scheme Y is a disjoint union of strictly henselian schemes (see ([15])) and hence the number of connected components of Y coincide with the number of connected components of the fiber $Y_x \rightarrow Spec(k)$. It implies that the canonical morphism

$$Z/n \rightarrow x_*(Z/n)$$

of sheaves on the finite sites is an isomorphism. Lemma (10.2) implies now that one has:

$$H_{finite}^i(X, Z/n) = H_{finite}^i(Spec(k), Z/n) = H_{qfh}^i(Spec(k), Z/n) = 0$$

for any $i > 0$.

Remark: The same argument shows that for any abelian group A we have $\beta_*(A_{qfh}) = A_{et}$ and $R^q\beta_*(A_{qfh}) = 0$ for $q > 0$ (here A_{qfh} and A_{et} are the corresponding constant sheaves in qfh- and etale topologies).

Lemma 10.4 *Let X be a normal connected excellent scheme. Any qfh-covering of X admits a refinement of the form $\{V_i \rightarrow V \rightarrow X\}_{i \in I}$ where V is the normalization of X in a finite normal extension of its field of functions and $\{V_i \rightarrow V\}$ is a Zariski open covering of V .*

Proof: Let $\{p_j : U_j \rightarrow X\}_{j \in J}$ be a qfh-covering of X . Replacing each U_j by the family of its irreducible components we may assume that all U_j are integral. Using lemma (10.1) we may assume that U_j dominate X for any j . According to the Zariski main theorem each p_j admits a factorization of the form $U_j \rightarrow \bar{U}_j \xrightarrow{\bar{p}_j} X$, where \bar{U}_j is integral, $U_j \rightarrow \bar{U}_j$ is an open embedding and \bar{p}_j is finite and surjective. Let E be the composite of normal closures of the fields $k(\bar{U}_j)$ over $k(X)$. Let V denote the normalization of X in E and let $q : V \rightarrow X$ be the canonical morphism. For each j the morphism q factors through \bar{U}_j and we will denote by V_j the inverse image of U_j in V . Finally set $G = Gal(E/k(X))$, $I = J \times G$ and for $i = (j, \sigma) \in I$ set $V_i = V_j^\sigma$.

An h-covering $\{p_i : U_i \rightarrow X\}$ is said to be of normal form if the morphisms p_i admit a factorization of the form $p_i = s \circ f \circ in_i$, where $\{in_i : U_i \rightarrow U\}$

is a Zariski open covering of U , $f : U \rightarrow Y$ is a finite surjective morphism and $s : Y \rightarrow X$ is a blowing up of a closed subscheme of X .

Lemma (10.4) and the “platification par eclatement” theorem of [16] imply easily:

Corollary 10.5 *Any h -covering of a reduced noetherian excellent scheme admits a refinement of normal form.*

Lemma 10.6 *Assume that k is a separably closed field and Y/k is a non-empty proper scheme over k . Then for any prime p the following sequence is exact*

$$0 \longrightarrow H_{et}^*(\text{Spec}(k), Z/p) \longrightarrow H_{et}^*(Y, Z/p) \longrightarrow H_{et}^*(Y \times_k Y, Z/p) \longrightarrow \dots$$

Proof: Denote the graded Z/p -algebra $H_{et}^*(Y, Z/p)$ by R . According to the Kunneth formula ([15][ch.6,§8]) we have $H_{et}^*(Y^k, Z/p) = R^{\otimes k}$ so that the above sequence takes the form

$$0 \longrightarrow Z/p \longrightarrow R \xrightarrow{d} R^{\otimes 2} \xrightarrow{d} \dots$$

where

$$d(r_1 \otimes \dots \otimes r_k) = \sum_{i=0}^k (r_1 \otimes \dots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \dots \otimes r_k).$$

Choose a closed point $y \in Y$ and consider the augmentation

$$e : R = H_{et}^*(Y, Z/p) \longrightarrow H_{et}^*(k(y), Z/p) = Z/p.$$

Finally define $s : R^{\otimes k} \rightarrow R^{\otimes(k-1)}$ by means of the formula

$$s(r_1 \otimes \dots \otimes r_k) = e(r_1)(r_2 \otimes \dots \otimes r_k).$$

A straightforward verification shows that s is a contracting homotopy for the complex in question.

Corollary 10.7 *Assume that X is a strictly henselian scheme and $f : Y \rightarrow X$ is a surjective proper morphism. Then for any prime p the following sequence is exact*

$$0 \longrightarrow H_{et}^*(X, Z/p) \longrightarrow H_{et}^*(Y, Z/p) \longrightarrow H_{et}^*(Y \times_X Y, Z/p) \longrightarrow \dots$$

Proof: This follows immediately from the proper base change theorem ([15][ch.6,§2]) and lemma (10.6).

Theorem 10.8 *Assume that S is excellent and $n > 0$. Then $(\beta\alpha)_*(Z/n) = Z/n$ and $R^q(\beta\alpha)_*(Z/n) = 0$ for $q > 0$.*

Proof: We may assume that $n = p$ is a prime integer. The first formula is evident, to prove the second one we will use induction on q . Assume that $R^i(\beta\alpha)_*(Z/n) = 0$ for $1 \leq i \leq q-1$. The hypercohomology spectral sequence gives (for any $X \in \text{ob}(\text{Sch}/S)$) an exact sequence

$$0 \longrightarrow H_{\text{et}}^q(X, Z/p) \longrightarrow H_h^q(X, Z/p) \longrightarrow H_{\text{et}}^0(X, R^q(\beta\alpha)_*(Z/p)).$$

This shows that $X \longrightarrow H_h^q(X, Z/p)/H_{\text{et}}^q(X, Z/p)$ is a separated presheaf in étale topology and implies:

Lemma 10.9 *Suppose that $u \in H_h^q(X, Z/p)$ and there exists an étale covering $\{X_i \longrightarrow X\}$ such, that $u|_{X_i} \in H_{\text{et}}^q(X_i, Z/p)$, then $u \in H_{\text{et}}^q(X, Z/p)$.*

To prove that $R^q(\beta\alpha)_*(Z/p) = 0$ it is sufficient to show that if X is an excellent noetherian scheme, then $H_h^q(X, Z/p) = 0$. Fix an element $u \in H_{\text{et}}^q(X, Z/p)$ and find (using (10.5)) a proper surjective morphism $Y \longrightarrow X$ and an open covering $Y = \bigcap Y_i$ such that $u|_{Y_i} = 0$. Lemma (10.9) shows that $u|_Y \in H_{\text{et}}^q(Y, Z/p)$. Since

$$pr_1^*(u|_Y) - pr_2^*(u|_Y) = 0 \in H_{\text{et}}^q(Y \times_X Y, Z/p) \subset H_h^q(Y \times Y, Z/p)$$

corollary (10.7) implies that $u|_Y = 0$. Finally consider the spectral sequence

$$E_1^{ij} = H_h^i(Y \times_X \dots \times_X Y, Z/p) \Rightarrow H_h^{i+j}(X, Z/p).$$

Induction hypothesis and corollary (10.7) show that

$$E_2^{ij} = 0 \text{ for } 0 \leq i \leq q-1, (i, j) \neq 0.$$

This spectral sequence shows that $H_h^q(X, Z/p) \subset H_h^q(Y, Z/p)$ and hence $u = 0$.

Corollary 10.10

$$\begin{aligned} \alpha_*(Z/n) &= Z/n \\ R^q\alpha_*(Z/n) &= 0 \text{ for } q > 0. \end{aligned}$$

Corollary 10.11 *Let \mathcal{F} be an étale sheaf and let \mathcal{G} be a qfh-sheaf, then*

$$Ext_{et}^*(\mathcal{F}, Z/n) = Ext_{qfh}^*(\beta^*\mathcal{F}, Z/n)$$

$$Ext_{qfh}^*(\mathcal{G}, Z/n) = Ext_h^*(\alpha^*\mathcal{G}, Z/n).$$

Let γ denote any of the morphisms of sites $\alpha, \beta, \alpha\beta$. For sheaves \mathcal{F} and \mathcal{G} on the corresponding sites we have canonical adjunction homomorphisms

$$f_{\mathcal{F}} : \mathcal{F} \longrightarrow \gamma_*\gamma^*(\mathcal{F})$$

$$g_{\mathcal{G}} : \gamma^*\gamma_*(\mathcal{G}) \longrightarrow \mathcal{G}.$$

The homomorphism $g_{\mathcal{G}}$ clearly is an isomorphism and the homomorphism $f_{\mathcal{F}}$ has the following property.

Corollary 10.12 *The homomorphism $f_{\mathcal{F}}$ induces isomorphisms on $Ext^*(-, Z/n)$ for any $n > 0$.*

Proof: The composition

$$\gamma^*(\mathcal{F}) \xrightarrow{\gamma^*(f_{\mathcal{F}})} \gamma^*\gamma_*\gamma^*(\mathcal{F}) \xrightarrow{g_{\gamma^*(\mathcal{F})}} \gamma^*(\mathcal{F})$$

is the identity morphism. Hence $\gamma^*(f_{\mathcal{F}})$ is an isomorphism and in particular induces isomorphisms on $Ext^*(-, Z/n)$. Our statement follows now from the commutativity of the diagram:

$$\begin{array}{ccc} Ext^*(\gamma_*\gamma^*(\mathcal{F}), Z/n) & \longrightarrow & Ext^*(\mathcal{F}, Z/n) \\ \downarrow & & \downarrow \\ Ext^*(\gamma^*\gamma_*\gamma^*(\mathcal{F}), Z/n) & \longrightarrow & Ext^*(\gamma^*(\mathcal{F}), Z/n) \end{array}$$

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