Dear Sasha,
I am writing this letter to put in order at least some of the intuitive ideas about the theory of mixed motives, which I have in mind. What I really would like to discuss with you is the hypothesis (0.0.25), but unfortunately to formulate it I have to start with more elementary matters.

In this text there are four types of statements:

Theorem is something I've proven and really wrote down.

Pretheorem is a statment which I believe I've proven but which is not carefully written yet.

Conjecture is a statement which I believe I know how to prove.

Hypothesis is something I just would like to be true.

All through this text I consider only the case of rational coefficients, i.e. all additive categories are supposed to be $\mathbf{Q}$-linear.

For a cohomological functor $H^i : \mathcal{D} \longrightarrow \mathcal{A}$ from a triangulated category $\mathcal{D}$ to an abelien category $\mathcal{A}$ I denote by H_i the functors H^{-i}.

1.Homological theories and the category DM_k^{ft}. Let k be a field of characteristic zero. Denote by Sch/k the category of separated schemes of finite type over $Spec(k)$. In fact, almost everything can be reformulated for a base scheme which is noetherian, separated and finite dimensional but since it does not really change anything I prefer to work in the case of a field.

Let $\partial\Delta^n$ be the scheme defined by the equations

$$\sum_{i=0}^{n} x_i = 1$$

$$\prod_{i=0}^{n} x_i = 0$$

in the affine space $\mathbf{A}_k^{n+1}$. We will consider it as $(n-1)$-dimensional sphere over k.

A *homological theory* over k is a functor $\mathcal{H} : Sch/k \longrightarrow \mathcal{D}$ from the category Sch/S to a triangulated category $\mathcal{D}$ together with a family of natural isomorphisms:

$$\sigma_X : \mathcal{H}(X \times \partial\Delta^2) \cong \mathcal{H}(X) \oplus \mathcal{H}(X)[1].$$

These data should satisfy some conditions which we will not need in the precise form here. Morally, this conditions are:

(H1) Homotopy invariance with respect to the affine line $\mathbf{A}^1$.

(H2) Mayer-Vietoris exact triangle for open coverings of the form $X = U \cup V$.

(H3) An exact triangle for blowups.

(H4) The transfer for flat finite morphisms.

One can define morphisms between homological theories and 2-morphism between morphisms and get a 2-category $\mathcal{HOM}(k)$ of homological theories over k.

Examples:

1. The algebraic K-theory with rational coefficients, which we consider as a functor which takes values in the category opposite to the derived category of $\mathbf{Q}$-vector spaces has a natural structure of a homological theory over k for any k. Actually to satisfy our conditions we have to consider the *homotopy invariant* K-theory defined by Weibel,which coincide with the usual one for regular schemes but is in general different for singular schemes.

2. The l-adic homologies which we consider as a functor which takes values in the derived category of the l-adic $Gal(\bar{k}/k)$-representations has a canonical structure of a homological theory. One can also define the theory of l-adic cohomologies.

3. The Hodge homologies (and cohomologies) associated with a complex embedding of our base field k which is a functor from the category Sch/k to the derived category of mixed Hodge structures (or any reasonable variant of this category) has a canonical structure of a homological theory.

Theorem 0.0.1 *There exists an initial object (DM_k^{ft}, M) of the 2-category $\mathcal{HOM}(k)$ (here DM_k^{ft} is a triangulated category and $M : Sch/k \longrightarrow DM_k^{ft}$ is a functor).*

It means basically that for any homological theory $(\mathcal{D}, \mathcal{H})$ over k there exists a unique up to a (unique) canonical isomorphism exact functor

$$\mathcal{H}' : DM_k^{ft} \longrightarrow \mathcal{D}$$

such that $\mathcal{H}(X) = \mathcal{H}'(M(X))$ for any object X of the category Sch/k.

I will call DM_k^{ft} the *triangulated category of effective mixed motives* over k.

It can be shown that for any homological theory $\mathcal{H} : Sch/k \longrightarrow \mathcal{D}$ there is the associated reduced theory $\tilde{\mathcal{H}} : Sch/k \longrightarrow \mathcal{D}$ such, that for any $X \in ob(Sch/k)$ there is a canonical exact triangle in $\mathcal{D}$ of the form:

$$\tilde{\mathcal{H}}(X) \longrightarrow \mathcal{H}(X) \longrightarrow \mathcal{H}(Spec(k)) \longrightarrow \tilde{\mathcal{H}}(X)[1].$$

In particular there is the reduced theory $\tilde{M} : Sch/k \longrightarrow DM_k^{ft}$ which takes a scheme to the corresponding "reduced motive".

Theorem 0.0.2 *Any object of DM_k^{ft} is isomorphic to an object of the form $\tilde{M}(X)[n]$, where X is an object of Sch/k. Moreover we may suppose, that X is affine and $n \leq 0$.*

Theorem 0.0.3 *The category DM_k^{ft} has a natural structure of a tensor triangulated category such, that for any $X, Y \in ob(Sch/k)$ one has canonical isomorphism*

$$M(X \times_k Y) = M(Y) \otimes M(Y).$$

I will denote the unite object of DM_k^{ft} with respect to this tensor structure by $\mathbf{Q}$. One has a canonical isomorphism $M(Spec(k)) = \mathbf{Q}$.

Consider the object $X = \mathbf{A}_k^1 - \{0\}$ of the category Sch/k. The point 1 of X defines a decomposition $M(X) = ? \oplus \mathbf{Q}$. We denote the object $?[-1]$ by $\mathbf{Q}(1)$ and call it the *Tate object*. We denote the n-th tensor power $\mathbf{Q}(1)^{\otimes n}$ by $\mathbf{Q}(n)$. In particular $\mathbf{Q}(0) = \mathbf{Q}$. For any object X in DM_k^{ft} we denote by $X(n)$ the object $X \otimes \mathbf{Q}(n)$.

Pretheorem 0.0.4 *Let X be an object of Sch/k which is a regular scheme. Then for any $p, q \in \mathbf{Z}$ one has canonical isomorphisms:*

$$Hom_{DM_k^{ft}}(M(X), \mathbf{Q}(p)[q]) = gr_p K^{2p-q}(X) \otimes \mathbf{Q}.$$

2.Sheaves and the category DM_k. I am going now to describe a bigger triangulated category which contains the category DM_k^{ft} as a full triangulated subcategory and which admits more explicite description than just the universal one.

We have to use the h-topology now. I am not going to repeat the definition here, but it is important to understand that the following types of morphisms are h-coverings:

1. The blow up $p : X_Z \longrightarrow X$ of a closed subscheme Z in X is an h-covering if p is a surjective morphism.

2. Any finite surjective morphism is an h-covering.

3. Any open covering or etale covering is an h-covering.

Denote by D^- the derived category of bounded above complexes of $\mathbf{Q}$-vector spaces on the site $(Sch/k)_h$. One can easily see that it is a tensor triangulated category. I will denote the tensor products in D^- just by $\otimes$ (instead of usual $\overset{L}{\otimes}$).

For any object X of Sch/k one can define the sheaf $\mathbf{Q}(X)$ of $\mathbf{Q}$-vector spaces freely generated by the sheaf of sets representable by X. It defines a functor from the category Sch/k to the category D^-. Denote by I^1 the sheaf

$$ker(\mathbf{Q}(\mathbf{A}_k^1) \longrightarrow \mathbf{Q} = \mathbf{Q}(Spec(k))).$$

We define the category DM_k to be the localization of the category D^- with respect to the minimal thick subcategory which contains all the objects of the form $X \otimes I^1, X \in ob(D^-)$ and is closed under arbitrary direct sums and countable inductive limits.

Denote (for a while) the functor which takes an object X of Sch/k to the object of DM_k which corresponds to the sheaf $\mathbf{Q}(X)$ by M_0.

Theorem 0.0.5 *The pair (DM_k, M_0) has a structure of a homological category over k such, that the following conditions hold:*

1. *The corresponding functor $DM_k^{ft} \longrightarrow DM_k$ is a full embedding.*

2. *Any object Y of the category DM_k is isomorphic to an object of the form:*

$$\lim_{\overrightarrow{n}}(\bigoplus_{i \in I_n} X_i)$$

where $n \in \mathbf{N}$ and the objects X_i belong to the image of DM_k^{ft}.

3. *An object Y of DM_k belongs to the image of DM_k^{ft} if and only if it is of finite type in DM_k.*

4. *The image of the category DM_k^{ft} in DM_k is generated as a triangulated category by objects of the form $M_0(X)[n]$ where $X \in ob(Sch/k)$ is a smooth projective scheme.*

Therefore the category DM_k is a "closure" of our triangulated category of effective mixed motives over k with respect to the direct sums and inductive limits.

3.Homotopy t-structure on DM_k. Let me construct now a t-structure on the category DM_k which I call the *homotopy t-structure*. Note that it does not coincide with the motivic t-structure if this last one exists.

Let F be a sheaf of $\mathbf{Q}$-vector spaces on the site $(Sch/k)_h$. It is called a *homotopy invariant* sheaf if for any $X \in ob(Sch/k)$ the morphism:

$$F(pr_1) : F(X) \longrightarrow F(X \times \mathbf{A}^1)$$

is an isomorphism.

Denote by HI_k the full subcategory in the category $\mathbf{Q} - vect((Sch/k)_h)$ of sheaves of $\mathbf{Q}$-vector spaces on $(Sch/k)_h$ which is generated by homotopy invariant sheaves.

Pretheorem 0.0.6 *The category HI_k is an abelien category.*

Examples:

1. Any locally constant sheaf of $\mathbf{Q}$-vector spaces in the etale topology on Sch/k is in fact a homotopy invariant h-sheaf.

2. The *h-sheaf* associated with the presheaf

$$U \longrightarrow gr_i K^j(U) \otimes \mathbf{Q}$$

   is a homotopy invariant sheaf. We will denote it by $\underline{K}_i^j$.

3. The h-sheaf representable by an Abelien variety is a homotopy invariant sheaf.

For any sheaf of $\mathbf{Q}$-vector spaces F on $(Sch/S)_h$ we define its *Suslin complex* $C_*(F)$ to be the complex of sheaves which terms are the internal Hom-objects $\underline{Hom}(\mathbf{Q}(\mathbf{A}^n_k), F)$ and the differential is the alternated sum of the morphisms defined be the standard co-simplicial structure on $(\mathbf{A}^n_k)$.

Denote by $\underline{h}_i(F)$ the sheaves $\underline{H}^{-i}(C_*(F))$.

Pretheorem 0.0.7 *The functor*

$$\underline{h}_0 : \mathbf{Q} - vect((Sch/k)_h) \longrightarrow \mathbf{Q} - vect((Sch/k)_h)$$

take values in the category HI and is left adjoint to the natural embedding $HI \longrightarrow \mathbf{Q} - vect((Sch/k)_h)$. In particular it is right exact. The functors $\underline{h}_i$ are the left derived functors of the functor $\underline{h}_0$.

Homotopy invariant sheaves seem to posses a lot of nice properties. For example the following pretheorems take place.

Pretheorem 0.0.8 *Let $X \in ob(Sch/k)$ be a regular scheme. Then for any homotopy invariant h-sheaf F of $\mathbf{Q}$-vector spaces and any $i \geq 0$ one has canonical isomorphsm:*

$$H^i_h(X, F) = H^i_{Zar}(X, F), i \geq 0.$$

Pretheorem 0.0.9 *Let X be an object of Sch/k and F be a homotopy invariant sheaf. Then for any $i \geq 0$ the canonical morphisms*

$$H^i_h(X, F) \longrightarrow H^i_h(X \times \mathbf{A}^1, F)$$

are isomorphisms.

The role homotopy invariant sheaves play in our theory is clear from the following pretheorem.

Pretheorem 0.0.10 *The canonical projection*

$$p : D^- \longrightarrow DM_k$$

has a right adjoint which we denote by $MC : DM_k \longrightarrow D^-$. This functor is a full embedding and its image consists of the classes of complexes of sheaves with the homotopy invariant cohomologies.

In particular for any sheaf F of $\mathbf{Q}$-vector spaces the object $MC(p(F))$ is canonically isomorphic to the Suslin complex $C_*(F)$ of F.

This pretheorem implies in particular that for any X in Sch/k the Suslin complex $C_*(\mathbf{Q}(X))$ is a *motivic complex* of X.

Note, that though the category HI_k has nice enough properties, its derived category has almost nothing in common with the category DM_k. In fact I would not be very surprised if for an algebraically closed field k the category HI_k were of Ext-dimension one (but it is not even a hypothesis!).

One can easily see that our pretheorems imply, that the embedding $MC : DM_k \longrightarrow D^-$ let us to define a t-structure on DM_k which is induced by the standard t-structure on D^-. We will call it the *homotopy t-structure* on DM_k. The corresponding abelien category is obviously the category HI_k.

I want also to make some remarks on the tensor structure on HI_k. One can define the tensor product of the homotopy invariant sheaves setting:

$$F \overset{h}{\otimes} G = \underline{h}_0(F \otimes G).$$

This tensor product is highly non exact. Moreover it is easy to show that the category HI_k does not have sufficiently many flat objects with respect to $\overset{h}{\otimes}$. Nevertheless we can define something like Tor-functors just setting:

$$Tor^n(F, G) = \underline{h}_n(F \otimes G).$$

The pretheorems above imply in particular that there are canonical isomorphisms:

$$\mathbf{G}_m^{\overset{h}{\otimes} n} = \underline{K}_n^n = \underline{K}_M^n \otimes \mathbf{Q}.$$

4.The geometrical filtration and the duality(conjectures). It follows easily from the definition of the category DM_k (and the cohomological dimension theorem for h-topology) that for any object X in DM_k^{ft} and any object Y in DM_k there is defined the internal Hom-object $\underline{Hom}_{DM_k}(X, Y)$.

Conjecture 0.0.11 *For any pair of objects X, Y in DM_k^{ft} the object $\underline{Hom}_{DM_k}(X, Y)$ belongs to the category DM_k^{ft}.*

Conjecture 0.0.12 *The Tate object $\mathbf{Q}(1)$ is quasi-invertible, i.e. for any Y in DM_k the canonical morphism*

$$Y \longrightarrow \underline{Hom}_{DM_k}(\mathbf{Q}(1), Y(1))$$

is an isomorphism.

Denote by $DM_k[\mathbf{Q}(-1)]$ (resp. $DM_k^{ft}[\mathbf{Q}(-1)]$) the category obtained from DM_k (resp. from DM_k^{ft}) by the formal adding of the object $\mathbf{Q}(-1)$ which is inverse to $\mathbf{Q}(1)$.

For any object X in the category $DM_k^{ft}[\mathbf{Q}(-1)]$ let X^* be the object $\underline{Hom}_{DM_k^{ft}[\mathbf{Q}(-1)]}(X, \mathbf{Q})$.

Conjecture 0.0.13 *Let X be an object of Sch/k of the dimension n. Then for any $m \geq n$ the canonical morphism*

$$M(X) \longrightarrow \underline{Hom}_{DM_k}(\underline{Hom}_{DM_k}(X, \mathbf{Q}(m)), \mathbf{Q}(m))$$

is an isomorphism.

One can easily see, that this conjecture imply the following form of the *Verdier duality*:

Conjecture 0.0.14 *For any object X in the category $DM_k^{ft}[\mathbf{Q}(-1)]$ one has:*

$$(X^*)^* \cong X.$$

In fact I even think that the following hypothesis should be true.

Hypothesis 0.0.15 *The category $DM_k^{ft}[\mathbf{Q}(-1)]$ as a tensor additive category is a ridgit tensor category.*

Basically it means just that we should have canonical isomorphisms:

$$X^* \otimes Y^* \cong (X \otimes Y)^*.$$

I am going now to define the *geometrical filtration* on the category DM_k^{ft} as follows. By the definition a filtration G on a category $\mathcal{C}$ is a family of functors

$$G_{\leq n} : \mathcal{C} \longrightarrow \mathcal{C}$$

and natural transformations $G_{\leq n} \longrightarrow G_{\leq n-1}$ such, that these data satisfy some obvious conditions. In particular the functors $G_{\leq n}$ are to be projectors.

For any object X of the category DM_k^{ft} we set:

$$G_{\leq n}(X) = \underline{Hom}_{DM_k}(\underline{Hom}_{DM_k}(X, \mathbf{Q}(n)), \mathbf{Q}(n)).$$

Conjecture 0.0.16 *The family of functors $G_{\leq n}$ defined above is indeed a filtration on the category DM_k^{ft}.*

Theorem 0.0.17 *The category $G_{\leq 0}(DM_k^{ft})$ is canonically equivalent to the derived category of the category of Artin motives, i.e. of the category of finite dimensional $Gal(\bar{k}/k)$-representations with rational coefficients.*

Pretheorem 0.0.18 *The category $G_{\leq 1}(DM_k^{ft})$ is canonically equivalent to the derived category of the category of Deligne 1-motives over k.*

Conjecture 0.0.19 *The category $G_{\leq n}(DM_k^{ft})$ coincide with the full triangulated subcategory of the category DM_k^{ft} generated by objects of the form $M(X)$ where X is an object of Sch/k of the dimension $\leq n$.*

As far as I understand it is not a problem to extend this filtration to the category DM_k.

Note that we already have two filtrations on the category DM_k, namely the geometrical filtration $(G_{\leq n})$ and the canonical filtration $(\tau^h_{\leq n})$ associated with the homotopy t-structure on DM_k.

To finish with the duality conjectures, let me formulate the following one, which seems to be a correct version of the Poincare duality in our context.

Conjecture 0.0.20 *Let $X \in ob(Sch/k)$ be a smooth proper scheme of the dimension n. Then the characteristic class of the diagonal*

$$M(X) \otimes M(X) \longrightarrow \mathbf{Q}(n)[2n]$$

induces the isomorphism:

$$M(X) \longrightarrow \underline{Hom}_{DM_k}(M(X), \mathbf{Q}(n))[2n].$$

In terms of the category $DM_k^{ft}[\mathbf{Q}(-1)]$ it means that there is a canonical isomorphism:

$$M(X)^* \cong M(X)(-n)[-2n].$$

5.The motivic t-structure and the weight filtration (hypothesises). Let me define now a pair of full subcategories $(DM_k^{ft})^{\leq 0}$ and $(DM_k^{ft})^{\geq 0}$ in the category DM_k^{ft} which is, hypothetically, a t-structure. We will call this would be t-structure the *motivic t-structure* on DM_k^{ft}.

To define our pair of subcategories we set:

1. An object Y of DM_k^{ft} belongs to the subcategory $(DM_k^{ft})^{\leq 0}$ if and only if for any affine scheme $X \in ob(Sch/k)$ and any $m > dim(X)$ one has:
$$Hom_{DM_k}(Y, M(X)[-m]) = 0.$$
2. An object Y of DM_k^{ft} belongs to the subcategory $(DM_k^{ft})^{\geq 0}$ if and only if for any $X \in ob((DM_k^{ft})^{\leq 0})$ one has:
$$Hom_{DM_k}(X[1], Y) = 0.$$

Hypothesis 0.0.21 *1. The pair $((DM_k^{ft})^{\leq 0}, (DM_k^{ft})^{\geq 0})$ is a t-structure on the category DM_k^{ft}.*

2. Denote by
$$MM_k^{ft} = (DM_k^{ft})^{\leq 0} \cap (DM_k^{ft})^{\geq 0}$$
the abelien category which corresponds to this t-structure. Then there is a canonical equivalence:
$$D^b(MM_k^{ft}) = DM_k^{ft}.$$

Let $M^i : DM_k^{ft} \longrightarrow MM_k^{ft}$ be the family of cohomological functors which corresponds to our t-structure.

Our definition of the subcategories $(DM_k^{ft})^{\leq 0}$,$(DM_k^{ft})^{\geq 0}$ is based on the remark, that according to the common feelings, for an affine $X \in ob(Sch/k)$ of the dimension n one should have
$$M_i(X) = 0 \text{ for } i > n.$$

It is easy to see that this hypothesis implies your vanishing conjectures. Moreover, modulo the duality conjectures from the previous section, the vanishing conjectures are equivalent to the fact, that
$$\mathbf{Q} \in ob((DM_k^{ft})^{\leq 0}).$$

At the present moment I do not know if the inverse is true, i.e. if the vanishing conjectures imply our hypothesis.

Note that the duality conjectures imply that MM_k^{ft} is a rigid tensor abelien category.

Let us consider now an embedding $i : k \longrightarrow \mathbf{C}$ of our base field in the field of complex numbers. Any such embedding let us to define an exact functor of singular homologies:

$$H_*^{sing} : DM_k^{ft} \longrightarrow D^b(\mathbf{Q} - vect)^{ft}$$

where $D^b(\mathbf{Q} - vect)^{ft}$ is the derived category of bounded complexes of finite dimensional $\mathbf{Q}$-vector spaces. Let $H_i^{sing} : DM_k^{ft} \longrightarrow \mathbf{Q} - vect$ be the corresponding family of cohomological functors.

Conjecture 0.0.22 *Let $X \in ob((DM_k^{ft})^{\geq 0})$ (resp. $X \in ob((DM_k^{ft})^{\leq 0})$). Then $H_i^{sing}(X) = 0$ for $i < 0$ (resp. for $i > 0$).*

Together with the previous hypothesises it would imply the following two major results:

Hypothesis 0.0.23

1. *The functor H_*^{ft} is conservative, i.e. a morphism $f : X \longrightarrow Y$ in DM_k^{ft} is an isomorphism if and only if $H_*^{sing}(f)$ is an isomorphism.*

2. *The functor*

$$H_0^{sing} : MM_k^{ft}[\mathbf{Q}(-1)] \longrightarrow \mathbf{Q} - vect$$

is a fiber functor.

(Note that to prove these results we do not need any subtle properties of the singular homologies. We have only to know that it is indeed a homological theory in our sense and that there is the Kunnet formula.)

Let us consider now the category DM_k. The theorem (0.0.5) implies easily that all our hypothesises except for those connected with the duality are still to be true for this bigger category. In particular, DM_k should be equivalent to the derived category of an abelien category MM_k of effective infinite dimensional motives over k. The category MM_k^{ft} should be the full subcategory of objects of finite type in MM_k and any object of this last category should be a direct limit of objects of MM_k^{ft} (but not an inductive direct limit any more). In particular we got a (hypothetical) third filtration $\tau^M_{\leq n}$ on DM_k which is the canonical filtration associated with the motivic t-structure on DM_k.

I am going now to "define" the fourth (and the last in this letter) filtration on this category - the weight filtration. The definition is based on the properties one would expect from this filtration.

Hypothesis 0.0.24 *1. There exists a family of exact functors*

$$W_{\leq n} : DM_k^{ft} \longrightarrow DM_k^{ft}$$

such, that for any smooth projective $X \in ob(Sch/k)$ one has a canonical isomorphism $W_{\leq n}(M(X)) \cong \tau^M_{\leq n}(M(X))$.

2. The family of functors $(W_{\leq n})$ is a filtration on the category DM_k^{ft}.

Note, that the theorem (0.0.5(4)) implies that if the functors $W_{\leq n}$ exist there are defined uniquely up to a canonical isomorphism. Note also that we should be able to extend this filtration to the category DM_k.

So, finally, we should have four filtrations on the category DM_k. Let me list them here.

Homotopy canonical filtration $(\tau^h_{\leq n})$. The functors $\tau^h_{\leq n}$ are not exact and do not preserve DM_k^{ft}. This is the only filtration which we can really construct. The associated functors h_i take values in the abelien category HI_k of homotopy invariant sheaves over k.

Geometrical filtration $(G_{\leq n})$. The functors $G_{\leq n}$ are exact and preserve DM_k^{ft}. It seems to me to be quite clear how to show that this is indeed a filtration.

Motivic canonical filtration $(\tau^M_{\leq n})$. The functors $\tau^M_{\leq n}$ are not exact, but do preserve DM_k^{ft}. The problem of the construction of this thing seems to me to be quite mysterious at the present moment.

Weight filtration $(W_{\leq n})$. The functors $W_{\leq n}$ are exact and preserve DM_k^{ft}. It seems to me that there should be an alternative description of this filtration which should make it possible to construct it without references to the motivic t-structure. This filtration defines the functors W_n

which take values in the category $W_n(DM_k)$ which should be equivalent to the derived category of the category of pure effective motives of the weight n. According to the common feelings this category should be semisimple and therefore the categories $W_n(DM_k)$ are to be abelien categories. In seems to me that the category $\oplus W_n(DM_k^{ft})$ should be equivalent to the category of the Grothendieck's effective pure motives modulo numerical (as well as homological) equivalence.

6.K-motives. The following hypothesis was actually the main reason for me to write this letter.

Hypothesis 0.0.25 *Consider the sheaves*

$$\underline{K}_i^j = gr_i \underline{K}^j \otimes \mathbf{Q}$$

as objects of the category DM_k. Then one has:

$$\underline{K}_i^j[-n(i,j)] \in MM_k$$

where:

$$n(i,j) = \begin{cases} 2i - j & for \quad i = j \\ 2i - j - 1 & for \quad j > i \end{cases}$$

Since

$$\underline{K}_i^j = h_{2i-j}(\mathbf{Q}(i))$$

it can be reformulated in the form:

$$h_q(\mathbf{Q}(p)) \in MM_k[m(p,q)],$$

where

$$m(p,q) = \begin{cases} q & for \quad q = p \\ q - 1 & for \quad q < p \end{cases}$$

This hypothesis means in particular that there exist some quite canonical motives which correspond to the sheaves $\underline{K}_i^j$. One can easily see that it is true for $p = 0, 1$. Let us consider the first nontrivial case $p = 2$. Then our definitions and the vanishing conjecture in the weight two imply that there is an exact triangle in DM_k of the following form:

$$\mathbf{Q}(2) \longrightarrow \underline{K}_2^2[-2] \longrightarrow \underline{K}_2^3 \longrightarrow \mathbf{Q}(2)[1].$$

Now, the rigidity conjecture for indecomposable K^3 means that the sheaf $\underline{K}_2^3$ is in fact a *locally constant sheaf*. It belongs, therefore to MM_k. Since $\mathbf{Q}(2) \in MM_k$ it implies that $\underline{K}_2^2 \in MM_k[-2]$.

Let us suppose now that $p = 3$. Then there are the following pair of exact triangles:

$$\mathbf{Q}(3) \longrightarrow \underline{K}_3^3[-3] \longrightarrow cone \longrightarrow \mathbf{Q}(3)[1]$$

$$\underline{K}_3^5 \longrightarrow cone \longrightarrow \underline{K}_3^4[-1] \longrightarrow \underline{K}_3^5[1].$$

By the rigidity we have $K_3^5 \in MM_k$. Therefore we have only to show, that $\underline{K}_3^4 \in MM_k[1]$. The standard weight arguments (together with the fact that $K_3^4 = 0$ for number fields) show, that there exists a surjection of sheaves of the form:

$$\oplus_\alpha \underline{Alb}(X_\alpha) \longrightarrow \underline{K}_3^4$$

where $\underline{Alb}(X_\alpha)$ are the sheaves representable by the Albanese varieties of some curves X_α over $Spec(\mathbf{Q})$. The result we need follows now from the fact that any quotient sheaf of such a form is indeed an object of the category $MM_k[1]$.

For $p = 4$ the situation is somehow less clear and I am not sure that I can deduce the hypothesis in this case just from the standard properties of motives (the main problem is indeed to show, that $\underline{K}_4^5 \in MM_k[2]$).

Anyway, one would expect that for any $j > i$ one has

$$\underline{K}_i^j[1 + j - 2i] \in G_{\leq 1+j-2i} MM_k$$

and the object $\underline{K}_i^i[-i] \in MM_k$ is has a filtration which factors are the Tate object $\mathbf{Q}(i)$ and the objects $\underline{K}_i^j[1 + j - 2i]$ for $j < i$.

It seems to be quite natural to ask the following questions:

1. How to describe (at least hypothetically) the objects $\underline{K}_i^{2i-2}$ for $i > 2$ which are supposed to be just infinite dimensional 1-motives over $Spec(\mathbf{Q})$?

2. How to describe (at least hypothetically) the Hodge and l-adic realizations of the motives which correspond to the sheaves $\underline{K}_i^j$?

It seems to be quite possible that your conjectures on the regulators actually provide the answers. There is a form of this hypothesis which can actually be formulated in the very precise form which does not require any of the conjectures or hypothesis above.

Let us consider the sheaf $\underline{K}^i_i$, i.e. the sheaf of Milnor's K-group of the degree i. One can ask then the following question. What are the singular homologies of this sheaf? Though this question seems to be a strange one it makes sens. Moreover if our hypothesis is true, then the corresponding homologies should be nonzero only in the dimension i. If $i = 2$ it should be an extension of the one dimensional $\mathbf{Q}$-vector space (in the weight 4 if we will also consider the Hodge structures) by means of the infinite dimensional $\mathbf{Q}$-vector space (in the weight 0) which is to be isomorphic to the $K^3_{ind}(\bar{\mathbf{Q}}) \otimes \mathbf{Q}$.

The point is that one can actually construct a simplicial scheme which homologies are the homologies of $\underline{K}^i_i$ quite explicitly. Just for the simplicity let us consider the case $i = 2$. The sheaf $\underline{K}^2_2$ is the quotient sheaf of the sheaf freely generated by $\mathbf{A}^2 - \{xy = 0\}$ with respect to the subsheaf which is a quotient sheaf of the sheaf freely generated by several copies of $\mathbf{A}^2 - \{xy = 0\}$ and $\mathbf{A}^1 - \{0, 1\}$. It is easy to see, that there is actually a simplicial scheme $\mathcal{X}$ of the simplicial dimension 2 such, that $\underline{K}^2_2 = \underline{H}_1(\mathcal{X})$. One may construct now quite formally (and quite explicitly!) a simplicial scheme $\mathcal{X}'$ such, that

$$\underline{H}_i(\mathcal{X}') = \begin{cases} \underline{H}_1(\mathcal{X}) = \underline{K}^2_2 & \text{for} \qquad i = 1 \\ 0 & \text{otherwise} \end{cases}$$

The singular homologies of $\mathcal{X}'$ are the groups we are interested in (shifted by one). The schemes of k-dimensional simplexes of $\mathcal{X}'$ will be of the form $\coprod X_{k,i}$ where $X_{k,i}$ are finite dimensional schemes.

7.Concluding remarks. While I was writing this letter it was more and more clear for me that it is absolutely crucial for the developing of a coherent theory of motives to consider infinite dimensional motives. Besides the K-motives from the previous section there should be another infinite dimensional motives which play important role in the theory. The easiest example is " the motive of singular cohomologies".

Let us consider an embedding $i : k \longrightarrow \mathbf{C}$ and the corresponding functors of singular cohomologies.

There should exist something like the representability theorem, which states that any reasonable exact functor $DM_k^{ft} \longrightarrow D^b(\mathbf{Q} - vect)^{op}$ is representable by an object of DM_k. In particular there should be an object which represents to the singular cohomologies. The corresponding motives should be connected with the "regular representation" of the motivic Galois group.

Another example is provided by the following observation. Let us consider for a while the category of Tate motives over k. It should be equivalent to the category of finite dimensional representations of a graded (pro-) Lie algebra, which Sasha Goncharov denotes by $L(k)$. Following his approach to the construction of motivic complexes we may consider the category of the representations of the ideal $L_{\geq 2}(k)$ generated by the elements of the weight 1 (or > 2 in our notations). It is quite natural to ask what is the analog of this category for all (not necessarily Tate) motives? It seems to me that this category should be "the category of motives modulo algebraic equivalence". To have a good theory here we obviously must consider not only the forgetful functor from $L(k)$-representations to $L_{\geq 2}$-representations, but also the adjoint functor of the induced representations. In general it takes values in the category of infinite dimensional motives.

I've tried for a while to collect problems which admit a formulation in classical terms such, that their solutions follow from the standard properties of mixed motives. So far I only have the following examples (1)-(4). Can you add something to this list?

1. Let X be a a smooth projective 2-dimensional variety over $\mathbf{C}$ such, that the following conditions hold:

   (a) $H^1(X, \mathbf{Q}) = 0$

   (b) $p_g(X) = 0$

   Then the conjecture of S.Bloch implies that $A_0(X) \otimes \mathbf{Q} = \mathbf{Q}$. It follows easily from the motivic description of A_0, the Hodge conjecture for divisors and the hypothesises (0.0.23(1)).

2. The fact that homological equivalence coincide with the numerical equivalence should follow from any reasonable theory of motives. I think that in particular the hypothesises formulated in this letter should imply it.

3. It seems to me that the following thing should be true (and even not very hard to prove, at least for $i = 2j$):

Let X be a smooth projective variety over $\mathbf{C}$. Then one has:

$$L_jH_i(X) \otimes \mathbf{Q} \cong H_i^{sing}(\underline{Hom}_{DM^{ft}_{\mathbf{C}}}(\mathbf{Q}(j), M(X))).$$

Here $L_jH_i(X)$ are Lawson homologies, i.e. the $(i-2j)$-dimensional homotopy groups of the Chow variety of j-cycles on X.

Then the conjecture (0.0.11) will imply that these groups (and, in particular, the groups $L_jH_{2j}(X)$ of j-cycles on X modulo algebraic equivalence) have finite rank.

So, what do you think about all this nonsense?
Do vstrechi,
Vladimir. 12.6.92.

P.S. There is one thing which I did not mentioned at all in the letter - the theory of motives with integral coefficients.

It seems to be very nontrivial problem to formulate any reasonable conjectures here (especialy in the case of p-torsion in characteristic p where the theory should allow us, in particular, to deal with the Drinfeld modules and which should *not* be the special case). Lately I start to believe that it is not completely impossible to write down such conjectures, but all ideas I have are too raw to present them even in the form of hypothesises.