Polylogarithmic Extensions on Mixed Shimura varieties. Part I: Construction and basic properties

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Introduction

In [B1], § 7, elements in the motivic cohomology of a cyclotomic field

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\mu_d)), \mathbb{Q}(k))$$

were constructed. Their images under the Deligne regulator were calculated and shown to give the "correct" $\mathbb{Q}$–structure on

$$H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}(\mu_d))_{\mathbb{R}}, \mathbb{R}(k))\,,$$

i.e., the one predicted by Beilinson's conjectures.
The latter group can be interpreted as

$$\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(k))$$

in a certain category of variations of $\mathbb{R}$–Hodge structure on $\mathrm{Spec}(\mathbb{Q}(\mu_d))(\mathbb{C})$.

As Beilinson observed in [B2], these elements, for $d \geq 2$ and $k \geq 1$, can be interpolated by a one–extension $\mathcal{P}$ of pro–variations of Hodge structure on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.
More precisely, $\mathcal{P}$ is fully described by its period matrix, whose inverse is given by the pro–matrix multivalued function

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2\pi i}\mathrm{Li}_1 & 1 & 0 & 0 & \ldots \\ -\left(\frac{1}{2\pi i}\right)^2 \mathrm{Li}_2 & -\frac{1}{2\pi i}\log & 1 & 0 & \ldots \\ \left(\frac{1}{2\pi i}\right)^3 \mathrm{Li}_3 & \frac{1}{2!}\left(\frac{1}{2\pi i}\log\right)^2 & -\frac{1}{2\pi i}\log & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\mathcal{P}$ should be viewed as a projective system of elements

$$\mathcal{P}^{(n)} \in \mathrm{Ext}^1_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}(\mathbb{R}(0), \mathfrak{a}^{(n)})\,,$$

where $\mathfrak{a}^{(n)}$ is a variation of Tate–Hodge structure whose graded objects are $\mathbb{R}(1), \mathbb{R}(2), \ldots, \mathbb{R}(n)$.
Over roots of unity unequal to one, $\mathfrak{a}^{(n)}$ splits canonically into the direct sum of its weight–graded parts. So $\mathcal{P}$ gives a collection of one–extensions of $\mathbb{R}(0)$ by $\mathbb{R}(k), k \geq 1$, for any such root.

If all d–th primitive roots are taken together, then these one–extensions can be interpreted as an element of

$$\prod_{k\geq 1} H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}(\mu_d))_{\mathbb{R}}, \mathbb{R}(k))\,.$$

Finally, the Galois conjugates of this element generate the same $\mathbb{Q}$–structure on

$$\prod_{k\geq 1} H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}(\mu_d))_{\mathbb{R}}, \mathbb{R}(k))$$

as does the Deligne regulator.

The essential new information provided by this viewpoint is the action of the fundamental group of $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$. In fact, as Beilinson pointed out, $\mathcal{P}$ satisfies a rigidity principle ([B2], 2.1): it is uniquely determined by the one–extension of pro–local systems underlying it.
Analogous statements are true in the l–adic setting where the elements in Galois cohomology constructed by Deligne and Soulé turn out to be specializations of a one–extension of pro–l–adic mixed sheaves on $\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$, which is uniquely characterized by the underlying "topological" extension on $\mathbb{P}^1_{\overline{\mathbb{Q}}}\setminus\{0,1,\infty\}$.

So if one is prepared to accept the existence of a formalism of mixed motivic sheaves, meaning in particular that a faithful tensor functor to the category of perverse sheaves exists, then essentially "the same proofs" should show that the two versions of $\mathcal{P}$ come from one and the same motivic object.

In particular, their specializations at spectra of cyclotomic fields should come from the same collection of elements in motivic cohomology.
This is precisely what is needed to complete the proof of the Tamagawa number conjecture for Tate motives ([BK], Theorem 6.1).

When the author first read [B2] some time ago, it was suggested that he consider the possibility of a similar interpolation process for the elements in the motivic cohomology of a CM–elliptic curve constructed in [De].
The answer was again provided by Beilinson, in a lecture given in June 1991 at the MPI Bonn:
there is a one–extension of pro–variations of Hodge structure on any punctured elliptic curve over $\mathbb{C}$, whose values at torsion points are "closely connected" to

the Deligne regulators of Deninger's elements if the curve has complex multiplication.

A few months later, the author studied parts of Pink's thesis ([P]), learning about the concepts of mixed Shimura data and varieties and the canonical construction of mixed sheaves from algebraic representations of the groups underlying the Shimura data.
He then realized that what he had learned provided the long sought–after setting for a satisfactory treatment of polylogarithms. Both the classical and the elliptic polylogarithm are extensions of pro–sheaves on an object, that can be seen as the complement of one mixed Shimura variety in another.
The pro–sheaves in both cases arise via the canonical construction from certain pro–algebraic representations, which are formed in a completely analogous manner.
Finally, and maybe most convincingly, the splitting of these sheaves over torsion points can be seen to follow almost trivially from the semisimplicity of the representation category of a reductive group.

The material contained in this and in forthcoming work ([W4], [W5]) is organized as follows:

Part I: Construction and basic properties

§ 1 Definition of polylogarithms
§ 2 Rigidity
§ 3 Interrelation between polylogarithms associated to different unipotent extensions
§ 4 The small polylogarithmic extension
§ 5 Norm compatibility
§ 6 Values at Levi sections

Part II: The classical polylogarithm

§ 1 The Shimura data $(P_0, \mathfrak{X}_0)$
§ 2 The topological extension underlying *pol*
§ 3 The Hodge version of *pol*
§ 4 The l–adic version of *pol*
§ 5 Remarks on the Tamagawa number conjecture for Tate motives

We give a description of the content of part I:

Let $[\pi] : M^K \to M^L$ denote the projection of a mixed Shimura variety to the underlying pure Shimura variety. In [W3], we defined and studied the logarithmic pro–sheaf $\mathcal{L}og$ on M^K. Using the results of [W2] and [W3], it is possible to calculate $[\pi]_* \mathcal{L}og$ (Proposition 1.1). If $\widetilde{M}^K \overset{j}{\hookrightarrow} M^K$ is the complement in M^K of a Shimura variety associated to smaller Shimura data, then the mixed formalism, in particular purity for smooth sheaves, enables one to calculate $\mathcal{H}^q([\pi] \circ j)_*(j^* \mathcal{L}og)$ as well (Theorem 1.3). Typically, these higher direct images will vanish up to some "large" degree q_0. This observation allows us to define the polylogarithmic extension associated to the situation

$$\begin{array}{ccc} \widetilde{M}^K & \overset{j}{\hookrightarrow} & M^K \\ & \searrow & \downarrow [\pi] \\ & & M^L \end{array} :$$

it is the universal q_0–extension by $j^* \mathcal{L}og$ of a sheaf coming from M^L. The name "polylogarithm" will be justified in part II.

Let us note that one might occasionally (part IV, ...) find it necessary, in order to generate "interesting" extensions, to remove from M^K a finite union of sub–Shimura varieties or even subvarieties of more general type, e.g. divisors associated to mixed Shimura varieties, which are torus torsors over other Shimura varieties etc.

The techniques used for these more general situations don't differ dramatically from those used here, and we chose to postpone their discussion until they are really needed.

The rigidity principle (Theorem 2.1) says that the polylogarithm is uniquely determined by the underlying extension of topological sheaves. It turns out to be a most useful device, both for the further development of the general theory and the explicit construction of polylogarithms in special situations (parts II and III).

In § 4, we generalize Beilinson's and Levin's definition of the small elliptic polylogarithm ([BL], 1.3.13) to the general case. We prove the analogue of [BLp], Remark 2.5.5: it is possible to recover the large from the small polylogarithm (Theorem 4.3).

§§ 3 and 5 explore the interrelation between polylogarithms associated to different unipotent extensions and different levels, respectively.

We conclude with another basic property of polylogarithms, "values at Levi sections" (§ 6). While the constructions of § 1 could be performed in a much more general context (compare [BLp], §§ 1–2), the splitting principle 6.1 is true only if the fibres of the morphism in question are of a very specific homogeneous nature. At least in the examples of parts II and III, the restrictions of the polylogarithm to pure Shimura varieties via Levi embeddings turn out to deserve our full attention, and we consider these extensions as one of the main justifications for the study of polylogarithms. We have no general statement for the subgroups of the relevant Ext–groups generated by them, yet admit that we find the following question very tantalizing:

is the polylogarithmic construction a way to generate interesting extensions on *pure* Shimura varieties?

This article is a revised version of §§ 6 and 7 of my doctoral thesis ([W1]). I thank C. Deninger for allowing me enough time and leisure to develop the ideas put forward here. I am obliged to U. Jannsen and T. Scholl for useful conversations, and to A. Beilinson for supplying me with copies of [BLpp] and [BLp]. Readers familiar with the preliminary versions of [BL] will note that many of the general constructions and results of this article are modelled after those of Beilinson and Levin in the elliptic case, and I don't find it difficult to admit that without their work the present article would not have been possible. Finally, it is a pleasure to thank G. Weckermann for her friendly and patient collaboration while typing my manuscript.

§ 1 Definition of polylogarithms

For a survey of the results of [P] relevant for us, see [W3], § 1. We use the notation of [P]. We let

$P/\mathbb{Q}$ be a connected algebraic group,

$W := R_u(P)$ its unipotent radical,

$G := P/W\ ,\ \pi : P \longrightarrow\!\!\!\!\rightarrow G$,

$U \leq W$ a normal subgroup of P,

$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ the Deligne torus, $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ the weight,

$\mathfrak{X}$ a homogeneous space under $P(\mathbb{R})\cdot U(\mathbb{C})$,

$h : \mathfrak{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ a $P(\mathbb{R})\cdot U(\mathbb{C})$–equivariant map with finite fibres.

Write h_x for $h(x)$.

Let $V := W/U\ ,\ \pi_m : P \longrightarrow\!\!\!\!\rightarrow P/U$.

Definition: ([P], Definition 2.1.)

$(P, \mathfrak{X})$ is called *mixed Shimura data* if the following holds for some (hence all) $x \in \mathfrak{X}$:

i) $\pi_m \circ h_x : \mathbb{S}_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

ii) $\pi \circ h_x \circ w : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter of the center $Z(G)_{\mathbb{R}}$ of $G_{\mathbb{R}}$.

iii) $\mathrm{Ad}_P \circ h_x$ induces on $\mathrm{Lie}\, P$ a mixed graded–polarizable $\mathbb{Q}$–Hodge structure ($\mathbb{Q}$–MHS) of type

$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}\,.$$

iv) the weight filtration on $\mathrm{Lie}\, P$ is given by

$$W_n(\mathrm{Lie}\, P) = \begin{cases} 0 & , \quad n \leq -3 \\ \mathrm{Lie}\, U & , \quad n = -2 \\ \mathrm{Lie}\, W & , \quad n = -1 \\ \mathrm{Lie}\, P & , \quad n \geq 0 \end{cases}\,.$$

v) $\mathrm{int}\,(\pi(h_x(\sqrt{-1})))$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.

vi) $G^{\mathrm{ad}}_{\mathbb{R}}$ has no nontrivial factors of compact type, that are defined over $\mathbb{Q}$.

vii) $Z(G)$ acts on U and on V through a torus, that is an almost direct product of a $\mathbb{Q}$–split torus with a torus of compact type defined over $\mathbb{Q}$.

Because of weight reasons, the algebraic group V is abelian, and U is contained in $Z(W)$.
If $W = 1$ then $(P, \mathfrak{X})$ is called pure.

As in [W3] we shall restrict ourselves to those mixed Shimura data satisfying

vii)' $Z(G)^0$ is an almost direct product of a $\mathbb{Q}$–split torus with a torus of compact type defined over $\mathbb{Q}$.

This condition implies that any real cocharacter of $Z(G)$ is defined over $\mathbb{Q}$.

Again, because of weight reasons, $\pi : P \longrightarrow G$ is injective on $Z(P)$, so $Z(P)^0$ is a torus of the same type.

$E = E(P, \mathfrak{X})$ denotes the reflex field of $(P, \mathfrak{X})$, and for an open compact subgroup K of $P(\mathbb{A}_f)$, we let $M^K(P, \mathfrak{X})$ denote the Shimura variety of level K. It is a normal quasi–projective variety over E. The number field E is given together with fixed embeddings $\overline{\sigma_0} : \overline{E(P, \mathfrak{X})} \hookrightarrow \mathbb{C}$ and $\sigma_0 := \overline{\sigma_0}\,|_{E(P,\mathfrak{X})}$. The set of complex points of $M^K(P, \mathfrak{X})$ is $P(\mathbb{Q})\backslash(\mathfrak{X} \times (P(\mathbb{A}_f)/K))$.

For $K \le P(\mathbb{A}_f)$ neat, open and compact, there are functors ([W3], §§ 2 and 4)

$$\begin{aligned} \mu_{K,\infty,\sigma_0} : \mathrm{Rep}_{\mathbb{Q}}(P) &\longrightarrow [\pi]_{\mathbb{C}}\text{–}U\mathrm{Var}_{\mathbb{Q}}(M^K(P,\mathfrak{X})_{\mathbb{C}}) \quad \text{and} \\ \mu_{K,l} : \mathrm{Rep}_{\mathbb{Q}_l}(P) &\longrightarrow [\pi]\text{–}U\mathrm{Et}^l_{\mathbb{Q}_l}(M^K(P,\mathfrak{X}))\,. \end{aligned}$$

The categories on the right hand sides are defined to be the full subcategories of those objects of the category $\mathrm{Var}_{\mathbb{Q}}(M^K(P, \mathfrak{X})_{\mathbb{C}})$ of graded–polarizable admissible variations of $\mathbb{Q}$–MHS or the category of lisse l–adic sheaves $\mathrm{Et}^l_{\mathbb{Q}_l}(M^K(P, \mathfrak{X}))$ respectively, that admit a filtration, whose graded objects come from $M^L(G, \mathcal{H})$, where $L := \pi(K)$ and $(G, \mathcal{H}) := (P, \mathfrak{X})/W$.

It is expected ([W3], Conjecture 4.2) that in fact $\mu_{K,l}$ lands in the full subcategory $[\pi]$–$U\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(M^K(P, \mathfrak{X}))$ of objects of $[\pi]$–$U\mathrm{Et}^l_{\mathbb{Q}_l}(M^K(P, \mathfrak{X}))$, which are mixed in the sense of [D1], VI. It will however not be necessary for us to assume this. Fix a section $i : (G, \mathcal{H}) \to (P, \mathfrak{X})$ of π. It defines an action of P on the completion $\hat{\mathfrak{U}}(\mathrm{Lie}\, W)$ of the universal envelope of $\mathrm{Lie}\, W$ with respect to the augmentation ideal a: W acts by multiplication, and $i(G)$ acts by conjugation. If K is of the shape $K^W \rtimes i(L)$, then the pro–object $\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$ coincides with the logarithmic sheaf $\mathcal{L}og(i, K)$ of [W3], §§ 2 and 4 ([W3], Theorems 2.1 and 4.4). In particular, $\mu_{K,l}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$ is mixed, and in both settings, the higher direct images under $[\pi]$ of $\mathcal{L}og(i, K)$ can be calculated using cohomology of W ([W3], Theorems 2.3 and 4.7.)

In what follows, we treat the Hodge and l–adic version in parallel. Since this will reduce the amount of notation, we use the conventions of [W2], § 4. So whenever an area of paper is divided by a vertical bar:

$$\Big|$$

(usually)

$$\Big|$$

or

$$\Big|$$

(rarely)

the text on the left of it will concern the Hodge–theoretic setting, while the text on the right will deal with the l–adic setting. This understood, we let

$$A := \mathbb{C}, \quad \left| \begin{array}{l} A := \text{a number field}, \\ l := \text{a fixed prime number}, \end{array} \right.$$

$\varphi : X \longrightarrow Y$ a morphism of type (S) between schemes over A, that is, a smooth morphism with geometrically connected fibres between smooth, separated schemes of finite type over A, φ being compactifiable in such a way that X is the complement of a relative divisor with normal crossings in a smooth, projective Y–scheme,

$$\begin{array}{l} \overline{X} := X(\mathbb{C}), \\ \overline{Y} := Y(\mathbb{C}) \text{ as topological spaces}, \\ \overline{\varphi} : \overline{X} \to \overline{Y}. \end{array} \left| \begin{array}{l} \overline{X} := X \otimes_A \overline{A}, \\ \overline{Y} := Y \otimes_A \overline{A}, \\ \overline{\varphi} : \overline{X} \to \overline{Y}. \end{array} \right.$$

$\mathrm{Sh}^s(Y)$	$:=$	$\mathrm{Var}_{\mathbb{Q}}(Y)$,	$\mathrm{Sh}^s(Y)$	$:=$	$\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(Y)$,
$\mathrm{Sh}^s_{\varphi}(X)$	$:=$	φ–$U\mathrm{Var}_{\mathbb{Q}}(X)$,	$\mathrm{Sh}^s_{\varphi}(X)$	$:=$	φ–$U\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X)$,
$\mathrm{Sh}^s(\overline{Y})$	$:=$	the category of local systems of $\mathbb{Q}$–vector spaces on $\overline{Y}$,	$\mathrm{Sh}^s(\overline{Y})$	$:=$	the category of lisse constructible $\mathbb{Q}_l$–sheaves on $\overline{Y}$,
$\mathrm{Sh}^s_{\overline{\varphi}}(\overline{X})$	$:=$	the category of $\overline{\varphi}$–unipotent local systems of $\mathbb{Q}$–vector spaces on $\overline{X}$.	$\mathrm{Sh}^s_{\overline{\varphi}}(\overline{X})$	$:=$	the category of $\overline{\varphi}$–unipotent lisse constructible $\mathbb{Q}_l$–sheaves on $\overline{X}$.

Each of these categories is naturally contained in one of the following:

$\mathrm{Sh}(Y)$	$:=$	$MHM_F(Y)$,	$\mathrm{Sh}(Y)$	$:=$	$\mathrm{Perv}^m_F(Y)$,
$\mathrm{Sh}(X)$	$:=$	$MHM_F(X)$,	$\mathrm{Sh}(X)$	$:=$	$\mathrm{Perv}^m_F(X)$,
$\mathrm{Sh}(\overline{Y})$	$:=$	$\mathrm{Perv}_F(\overline{Y})$,	$\mathrm{Sh}(\overline{Y})$	$:=$	$\mathrm{Perv}_F(\overline{Y})$,
$\mathrm{Sh}(\overline{X})$	$:=$	$\mathrm{Perv}_F(\overline{X})$.	$\mathrm{Sh}(\overline{X})$	$:=$	$\mathrm{Perv}_F(\overline{X})$.

Here, MHM_F denotes the category of algebraic mixed F–Hodge modules ([S], § 4), and Perv^m_F is the category of mixed perverse $\mathbb{Q}_l$–sheaves (see [W2], § 4). Perv_F denotes the category of perverse sheaves on the topological space underlying a complex manifold ([BBD], 2.1) or on a smooth scheme over an algebraically closed field of characteristic zero ([BBD], 2.2).

Now let $\pi : (P, \mathfrak{X}) \to (G, \mathcal{H})$ be as before, and let i be a fixed splitting.
Let $W' \lneqq W$ be a closed subgroup, stable under conjugation by $i(G)$ but not necessarily normal in W.
By [P], Proposition 2.17.a), there are mixed Shimura data $(P', \mathfrak{X}')$ and a morphism

$$\pi' : (P', \mathfrak{X}') \longrightarrow (G, \mathcal{H})$$

covering

$$\pi' : P' := W' \rtimes i(G) \overset{k}{\hookrightarrow} P \overset{\pi}{\twoheadrightarrow} G$$

and inducing an isomorphism of $(P', \mathfrak{X}')/W'$ and $(G, \mathcal{H})$.
$(P', \mathfrak{X}')$ and π' are unique up to isomorphism.
Futhermore, by [P], Proposition 2.17.b), there is a unique morphism

$$k : (P', \mathfrak{X}') \longrightarrow (P, \mathfrak{X})$$

covering the immersion k of P' into P.
It follows from the proof of [P], Proposition 2.17.b) that k is an embedding.

Also, the morphism i factors uniquely through $(P', \mathfrak{X}')$, giving an embedding

$$(G, \mathcal{H}) \stackrel{i'}{\hookrightarrow} (P', \mathfrak{X}')$$

of Shimura data. Let $h^{-1,-1} := \dim U$, and $h^{0,-1} := \frac{1}{2} \dim V$.
So $d := h^{-1,-1} + h^{0,-1}$ is the relative dimension of $[\pi]$, and $N := h^{-1,-1} + 2h^{0,-1}$ is the dimension of W.
Similarly, $h'^{-1,-1} := \dim U'$ etc., and $h''^{-1,-1} := h^{-1,-1} - h'^{-1,-1}$ etc.
So $[k]$ is of codimension d''.

Let $K \leq P(\mathbb{A}_f)$ be neat, open and compact and of the shape $K = K^W \rtimes L$. Set $K' := k^{-1}(K)$.
We have the following commutative diagram:

$$\begin{array}{ccccc}
M^{K'}(P', \mathfrak{X}') & \xrightarrow{[k]_{K',K}} & M^K(P, \mathfrak{X}) & \xleftarrow{j_K} & \widetilde{M}^K(P, \mathfrak{X}) \\
& {\scriptstyle [\pi']_{K',L}} \searrow & \downarrow {\scriptstyle [\pi]_{K,L}} & \swarrow {\scriptstyle \widetilde{[\pi]}_{K,L}} & \\
& & M^L(G, \mathcal{H}) & &
\end{array}$$

Here,

$$j := j_K : \widetilde{M}^K(P, \mathfrak{X}) := M^K(P, \mathfrak{X}) - [k](M^{K'}(P', \mathfrak{X}')) \hookrightarrow M^K(P, \mathfrak{X})$$

is the open immersion complementary to $[k]$, and $\widetilde{[\pi]} := \widetilde{[\pi]}_{K,L} := [\pi]_{K,L} \circ j_K$.
By purity, which is a formal consequence of relative duality ([S], (4.3.5); sheafified version of [SGA4,III], Exp. XVIII, Théorème 3.2.5) and the usual adjointness property of the pairs $[k]_!$, $[k]^!$ and $[k]^*$, $[k]_*$, for any $\mathbf{V} \in \mathrm{Sh}^s(M^K(P, \mathfrak{X}))$ we have canonically

$$[k]^! \mathbf{V} = [k]^* \mathbf{V}(-d'')[-2d''],$$

hence an exact triangle

$$\begin{array}{ccc} [k]_*[k]^*\mathbf{V}(-d'')[-2d''] & \longrightarrow & \mathbf{V} \\ {\scriptstyle \text{shift by } [1]} \nwarrow & & \swarrow \\ & j_*j^*\mathbf{V} & \end{array} \qquad (*)$$

in $D^b(\mathrm{Sh}(M^K(P,\mathfrak{X})))$.
Using the exact triangle $[\pi]_*(*)$ for $\mathbf{V} = \mathcal{L}og(i,K)(d)$, it is possible to calculate $\widetilde{[\pi]}_* j^*\mathcal{L}og(i,K)(d)$:
thanks to [W2], Corollary 4.4, we know $[\pi]_*\mathcal{L}og(i,K)(d)$.

Proposition 1.1: There is a canonical isomorphism

$$[\pi]_*\mathcal{L}og(i,K)(d) \stackrel{\sim}{\longrightarrow} \mathbb{Q}_{(l)}(0)[-N+d]\,.^\dagger$$

Remark: This should be regarded as a statement on the projective system of higher direct images under $[\pi]$ of the noetherian quotients of $\mathcal{L}og(i,K)(d)$. See the remark following 1.2.

Proof of Proposition 1.1: There is an exact sequence

$$0 \longrightarrow \mathrm{Lie}\,U \longrightarrow \mathrm{Lie}\,W \longrightarrow \mathrm{Lie}\,V \longrightarrow 0\,.$$

By [W2], Corollary 4.4 and [W3], Theorems 2.3 and 4.7, we have to calculate $\mu_{L,-}(\Lambda^N(\mathrm{Lie}\,W)^\vee)$, and by [W3], Theorem 1.3 we are reduced to one of the following cases:

1. $[\pi]$ is an abelian scheme. Here, $N = 2d$.
   Then there are canonical isomorphisms

$$\mu_{L,-}((\mathrm{Lie}\,W)^\vee) \stackrel{\sim}{\longrightarrow} \mathcal{H}^{-d+1}[\pi]_*\mathbb{Q}_{(l)}(0)$$

   and

$$\Lambda^N\mathcal{H}^{-d+1}[\pi]_*\mathbb{Q}_{(l)}(0) \stackrel{\sim}{\longrightarrow} \mathcal{H}^{-d+N}[\pi]_*\mathbb{Q}_{(l)}(0) \stackrel{\sim}{\longrightarrow} \mathbb{Q}_{(l)}(-d)$$

   given by the cup product.

† The subscript (l) takes the two values *blank* and l, depending on whether we are in the Hodge theoretic or the l–adic setting.

2. $[\pi]$ is a product of copies of $\mathbb{G}_m$. Here, $N = d$.
   Then there is a canonical isomorphism

$$\mu_{L,-}((\operatorname{Lie} W)^\vee) \xrightarrow{\sim} \mathcal{H}^{-d+1}[\pi]_* \mathbb{Q}_{(l)}(0) \xrightarrow{\sim} (\mathbb{Q}_{(l)}(-1))^d$$

   given by the map "residue at 0". q.e.d.

A similar formula holds for $[\pi']_*[k]^*\mathcal{L}og(i,K)(d')[-2d'']$. Before stating it, note that due to our conventions,

$$[k]^*\mathcal{L}og(i,K) = ([k]^s)^*\mathcal{L}og(i,K)[d'']$$

where $([k]^s)^*$ is the inverse image in the category of smooth sheaves:
on M^K, we identify a smooth sheaf $\mathbf{V}$ with the complex of sheaves, concentrated in degree $-\dim(M^K)$, the cohomology object $\mathcal{H}^{-\dim(M^K)}$ being equal to $\mathbf{V}$.

On $M^{K'}$, the same rule applies with $-\dim(M^{K'})$ instead of $-\dim(M^K)$.

Proposition 1.2: There is a canonical isomorphism

$$[\pi']_*([k]^*\mathcal{L}og(i,K)(d')[-2d'']) \xrightarrow{\sim} \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\operatorname{Lie} W))[-N' + d' - d''] \, .$$

Proof: This is due to [W2], Theorem 4.3 and Corollary 1.13 and the canonical isomorphism

$$\mu_{L,-}(\Lambda^{N'}(\operatorname{Lie} W')^\vee) \xrightarrow{\sim} \mathbb{Q}_{(l)}(-d')$$

of the proof of 1.1. q.e.d.

As before, such a statement on direct images of pro–sheaves should be interpreted appropriately: here, the projective systems of cohomology objects sitting in the wrong degrees are ML–zero, while the projective system in the highest possible degree coincides with the system

$$(\mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\operatorname{Lie} W)/\mathfrak{a}^n))_{n\in\mathbb{N}} \, .$$

Theorem 1.3: Let $m := N' - d' + d'' - 1 = h^{0,-1} + h''^{-1,-1} - 1$.

a) $\mathcal{H}^q\widetilde{[\pi]}_* j^*\mathcal{L}og(i,K)(d) = \begin{cases} 0 \text{ for } N - d \neq q < m \\ \mathbb{Q}_{(l)}(0) \text{ for } N - d = q < m \end{cases}$.

   In particular, if $h''^{-1,-1} \leq 1$, then the second possibility does not occur.

b) $W_{-1}(\mathcal{H}^m\widetilde{[\pi]}_* j^* \mathcal{L}og(i,K)(d)) = \mu_{L,-}(b(W',i))$, where we set

$$b(W',i) := W_{-1}(H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}\, W)))\,.$$

More precisely, there is a canonical morphism of projective systems

$$(\mathcal{H}^m\widetilde{[\pi]}_* j^* \mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W)/\mathfrak{a}^n)(d))_{n\in\mathbb{N}} \longrightarrow (\mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}\, W)/\mathfrak{a}^n))_{n\in\mathbb{N}}\,.$$

The weight ≤ -1–parts of the projective systems of kernels and cokernels are ML–zero.

Proof: We apply $[\pi]_*$ to the exact triangle $(*)$, distinguishing three cases:

1. $h''^{-1,-1} = 0$:
   so $m = h^{0,-1} - 1 = N - d - 1$, and 1.1 and 1.2 yield the following:

   $$\mathcal{H}^q\widetilde{[\pi]}_* = 0 \text{ for } q \notin \{m, m+1\}.$$

   Furthermore, there is an exact sequence

   $$0 \to \mathcal{H}^m\widetilde{[\pi]}_* \to \mu_{L,-} H_0 \to \mathbb{Q}_{(l)}(0) \to \mathcal{H}^{m+1}\widetilde{[\pi]}_* \to 0\,,$$

   where we used appropriate abbreviations.

2. $h''^{-1,-1} = 1$:
   so $m = N - d$. We get

   $$\mathcal{H}^q\widetilde{[\pi]}_* = 0 \text{ for } q \neq m$$

   and an exact sequence

   $$0 \to \mathbb{Q}_{(l)}(0) \to \mathcal{H}^m\widetilde{[\pi]}_* \to \mu_{L,-} H_0 \to 0\,.$$

3. $h''^{-1,-1} > 1$:
   we get

   $$\mathcal{H}^q\widetilde{[\pi]}_* = \begin{cases} 0, & q \notin \{N-d, m\} \\ \mathbb{Q}_{(l)}(0), & q = N-d \\ \mu_{L,-} H_0, & q = m \end{cases}\,.$$

q.e.d.

As the proof shows, we could give more precise statements on the higher direct images of $j^*\mathcal{L}og(i,K)(d)$. However, for our purposes the result in 1.3 will do.

The polylogarithmic extension will be defined as a certain extension of pro–sheaves. To make sure that the naive conception of such an extension as a projective system of extensions is correct, we state the following

Lemma 1.4: Let $SS(B)$ be the category of spectral sequences in an abelian category B admitting countable products.
Let $q_0 \in \mathbb{N}$, $(S_n)_{n\in\mathbb{N}}$ a countable projective system in $SS(B)$,

$$S_n = (E_{2,n}^{p,q} \Rightarrow E_n^{p+q})\,.^\dagger$$

Assume that for all n, $E_{2,n}^{p,q} = 0$ for $p \le 0$ or $q \le 0$. For any $q \in \{q_0, q_0+1\}$ and any $0 \le l < q_0$, assume that the projective system

$$(E_{2,n}^{q-l,l})_{n\in\mathbb{N}}$$

is ML–zero.
Then the projective limit of the edge homomorphisms

$$E_n^{q_0} \longrightarrow E_{2,n}^{0,q_0}$$

is an isomorphism.
Proof: The condition for q_0+1 implies

$$\varprojlim_{n\in\mathbb{N}} E_{2,n}^{2,q_0-1} = \varprojlim_{n\in\mathbb{N}} E_{3,n}^{2,q_0-2} = \ldots = \varprojlim_{n\in\mathbb{N}} E_{q_0+1,n}^{q_0+1,0} = 0\,,$$

hence $\varprojlim E_{2,n}^{0,q_0} = \varprojlim E_{\infty,n}^{0,q_0}$.
The condition for q_0 implies that the systems

$$(E_{\infty,n}^{q_0-l,l})_{n\in\mathbb{N}}$$

are ML–zero as well, for all $0 \le l < q_0$.
Hence the projective limit of the homomorphisms

$$E_n^{q_0} \longrightarrow E_{\infty,n}^{0,q_0}$$

remains surjective and has trivial kernel. q.e.d.

[†] By definition, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non–negative integers.

We are now able to state the main result so far:

Theorem 1.5: Let $q_0 := N' + 2d'' - 1 = N + h^{''-1,-1} - 1$, and assume that $\mathbf{V} \in \mathrm{Sh}^s(M^L(G,\mathcal{H}))$ has weights ≤ -1.

a)
$$\mathrm{Ext}^q_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}((\widetilde{[\pi]}^s)^*\mathbf{V}, j^*\mathcal{L}og(i,K)(d)) = 0$$
for $q < q_0$.

b) There are canonical isomorphisms

$$\begin{aligned} & \mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}((\widetilde{[\pi]}^s)^*\mathbf{V}, j^*\mathcal{L}og(i,K)(d)) \\ \xrightarrow{\sim} \; & \mathrm{Hom}_{\mathrm{Sh}(M^L(G,\mathcal{H}))}(\mathbf{V}, \mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^*\mathcal{L}og(i,K)(d)) \\ \xrightarrow{\sim} \; & \mathrm{Hom}_{\mathrm{Sh}(M^L(G,\mathcal{H}))}(\mathbf{V}, \mu_{L,-}(b(W',i)))\,. \end{aligned}$$

More precisely, the first isomorphism is the projective limit of the edge homomorphisms

$$\begin{aligned} & \mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}((\widetilde{[\pi]}^s)^*\mathbf{V}, j^*\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\,W)/\mathfrak{a}^n)(d)) \\ \longrightarrow \; & \mathrm{Hom}_{\mathrm{Sh}(M^L(G,\mathcal{H}))}(\mathbf{V}, \mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^*\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\,W)/\mathfrak{a}^n)(d)) \end{aligned}$$

in the Leray spectral sequence for $\widetilde{[\pi]}$.

The second isomorphism is induced by the isomorphism of 1.3.b).

Proof: Note that as usual $(\widetilde{[\pi]}^s)^*\mathbf{V} = [\pi]^*\mathbf{V}[d]$.
Our central technical tool will be the Leray spectral sequence for $\widetilde{[\pi]}$. It exists because $\mathcal{H}^q\widetilde{[\pi]}_*$ and $\mathcal{H}^q\widetilde{[\pi]}^*$ are defined not only as cohomological functors but as cohomology objects of functors $\widetilde{[\pi]}_*$ and $\widetilde{[\pi]}^*$ defined on the level of derived categories. $\widetilde{[\pi]}_*$ and $\widetilde{[\pi]}^*$ are adjoint, and although they do not in general appear as right or left derived functors, one may employ the theory of exact couples ([Hu], VIII, § 6) to construct the Leray spectral sequence.
We have to analyze its values at the $j^*\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\,W)/\mathfrak{a}^n)(d), n \in \mathbb{N}$. In order to see that the hypotheses of Lemma 1.4 are met, we have to show that

$$(\mathrm{Ext}^{q-l}_{M^L}(\mathbf{V}, \mathcal{H}^{l-d}\widetilde{[\pi]}_* j^*\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}\,W)/\mathfrak{a}^n)(d)))_{n\in\mathbb{N}}$$

is ML-zero for any $q \leq q_0 + 1$, $0 \leq l \leq q$, $l < q_0$.
This follows from Theorem 1.3 and the next proposition. q.e.d.

Proposition 1.6: Let X be a smooth variety over k, and assume $\mathbf{V} \in \mathrm{Sh}^s(X)$ is of weights ≤ -1. Then $\mathrm{Ext}^q_{\mathrm{Sh}(X)}(\mathbf{V}, \mathbb{Q}_{(l)}(0)) = 0$ for any q.

Proof: $\mathrm{Ext}^q_{\mathrm{Sh}(X)}(\mathbf{V}, \mathbb{Q}_{(l)}(0)) = \mathrm{Ext}^q_{\mathrm{Sh}(X)}(\mathbb{Q}_{(l)}(0), \mathbf{V}^\vee)$, and we apply the Leray spectral sequence for

$$a : X \longrightarrow \mathrm{Spec}(k)\,.$$

Since a is smooth, any $\mathcal{H}^q a_*(\mathbf{V}^\vee)$ has weights ≥ 1 as follows from

[S], 4.5.2. Note that as a Hodge module, $\mathbf{V}^\vee$ has weights $\geq \dim X + 1$ ([S], Theorem 3.27), hence the same is true for the complex $a_*(\mathbf{V}^\vee)$. So the Hodge structure $\mathcal{H}^q a_*(\mathbf{V}^\vee)$ has weights $\geq \dim X + q + 1$, for $q = -\dim X, \ldots, \dim X$.	[D1], Théorème 6.1.2, generic base change ([SGA4 1/2], Th. finitude, Théorème 1.9) and [D1], Corollaire 3.3.5.

Hence $\mathrm{Ext}^p_{\mathrm{Sh}(\mathrm{Spec}(k))}(\mathbb{Q}_{(l)}(0), \mathcal{H}^q a_*(\mathbf{V}^\vee)) = 0$ for any p, q. q.e.d.

Definition: The *polylogarithmic extension*

$$\mathcal{P}ol(W', i, K)$$

is the universal q_0–extension in

$$\mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}(j^* \mu_{K,-} \mathrm{res}^P_G(b(W', i)), j^* \mathcal{L}og(i, K)(d))\,,$$

with $b(W', i) = W_{-1}(H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}\, W)))$ as in 1.3.b), corresponding to

$$\mathrm{id} \in \mathrm{Hom}_{\mathrm{Sh}(M^L(G,\mathcal{H}))}(\mu_{L,-}(b(W', i)), \mu_{L,-}(b(W', i)))$$

under the isomorphism in 1.5.b).

More precisely, if we have projective systems $(\mathbf{V}_m)_{m \in \mathbb{N}}$ and $(\mathbf{W}_n)_{n \in \mathbb{N}}$ of mixed sheaves on X, such that the respective projective systems of quotients of weight $\geq w$ become constant for any integer w, we denote by

$$\mathrm{Ext}^q_{\mathrm{Sh}(X)}(\varprojlim_m \mathbf{V}_m, \varprojlim_n \mathbf{W}_n)$$

the vector space

$$\varprojlim_n (\varinjlim_m \mathrm{Ext}^q_{\mathrm{Sh}(X)}(\mathbf{V}_m, \mathbf{W}_n)).$$

In the case of $\mathcal{P}ol(W', i, K)$, observe that by the semisimplicity of $\mathrm{Rep}_F(G)$, $b(W', i)$ is canonically the product of its weight–graded objects:

$$b(W', i) = \prod_{m \le -1} \mathrm{Gr}_m^W (b(W', i)) \, .$$

So $\mathcal{P}ol(W', i, K)$ is an element of

$$\prod_{m \le -1} \lim_{\substack{\longleftarrow \\ n \in \mathbb{N}}} \mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))} (b_m(W', i), j^* \mu_{K,-} (\hat{\mathfrak{U}}(\mathrm{Lie}\, W)/\mathfrak{a}^n)(d)) \, ,$$

where we let

$$b_m(W', i) := j^* \mu_{K,-} \mathrm{res}^P_G (\mathrm{Gr}^W_m b(W', i)) \, .$$

§ 2 Rigidity

The first basic property of polylogarithms will turn out to be an extremely powerful tool. It will allow us (Corollary 2.2) to conclude that there is a version of the polylogarithmic extension in the category of smooth mixed systems of sheaves (see [W2], § 2 or [W3], § 6) if $q_0 = 1$ and also (Theorem 2.3) that in some cases $\mathcal{P}ol(W', i, K)$ can be represented by an extension of smooth sheaves. Furthermore, it indicates how to actually construct polylogarithms; at least under the hypotheses of 2.3.a), which are met in the examples of parts II and III, it will be comparatively easy to construct the extension of topological sheaves underlying $\mathcal{P}ol(W', i, K)$. Theorem 2.1 then predicts that there is exactly one way to equip it with a mixed structure, and that this will be the polylogarithmic extension.

Theorem 2.1: (Rigidity principle.)
$\mathcal{P}ol(W', i, K)$ is uniquely determined by the underlying extension of topological sheaves. More precisely, for any $m \le -1$, the natural map

$$\begin{aligned} &\mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))} (b_m(W', i), j^* \mathcal{L}og(i, K)(d)) \\ \longrightarrow \; &\mathrm{Ext}^{q_0}_{\overline{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}} (b_m(W', i), j^* (\mathcal{L}og(i, K)/W_{m-1} \mathcal{L}og(i, K))(d)) \end{aligned}$$

is injective.

Proof: The first isomorphism of 1.5.b) is part of a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{q_0}_{\widetilde{M}^K} & \longrightarrow & \mathrm{Ext}^{q_0}_{\overline{\widetilde{M}^K}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{M^L} & \longrightarrow & \mathrm{Hom}_{\overline{M^L}} \end{array}$$

where we have omitted the arguments.

The lower horizontal map is injective as the forgetful functor is faithful.

More precisely, it identifies $\mathrm{Ext}^{q_0}_{\widetilde{M}^K}$ with

$$\mathrm{Hom}_{\mathrm{Spec}(k)}(\mathbb{Q}_{(l)}(0), \mathrm{Hom}_{\overline{M^L}}) \, .$$

It remains to show the following:

let $\mathbf{V} \in \mathrm{Sh}^s(M^L)$ be pure of weight $m \leq -1$.

Then the homomorphism

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathbb{Q}_{(l)}(0), \mathrm{Hom}_{\overline{M^L}}(\mathbf{V}, \mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^* \mathcal{L}og(d))) \\ \longrightarrow \; & \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathbb{Q}_{(l)}(0), \mathrm{Hom}_{\overline{M^L}}(\mathbf{V}, \mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^* (\mathcal{L}og/W_{m-1})(d))) \end{aligned}$$

is injective.

Since we may replace $\mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^* \mathcal{L}og(d)$ by its W_{-1}–part, $\mathbf{V}$ being of weights smaller or equal to -1, we may apply 1.3.b) and replace $\mathcal{H}^{q_0-d}\widetilde{[\pi]}_* j^* \mathcal{L}og(d)$ by $\mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie} W))$.

Composing with the boundary homomorphism of the exact triangle $[\pi]_*(*)$, applied to $\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}W)/W_{m-1})(d)$, we have to show that the map

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathbb{Q}_{(l)}(0), \mathrm{Hom}_{\overline{M^L}}(\mathbf{V}, \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W)))) \\ \longrightarrow \; & \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathbb{Q}_{(l)}(0), \mathrm{Hom}_{\overline{M^L}}(\mathbf{V}, \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W)/W_{m-1}))) \end{aligned}$$

is injective.

But this homomorphism equals

$$\begin{aligned} & \mathrm{Hom}_{M^L}(\mathbf{V}, \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W))) \\ \longrightarrow \; & \mathrm{Hom}_{M^L}(\mathbf{V}, \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W)/W_{m-1})) \, . \end{aligned}$$

Now observe that

$$\mu_{L,-} H_0(W', W_{m-1})$$

surjects onto

$$\mathcal{L} := \ker(\mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W)) \to \mu_{L,-} H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W)/W_{m-1}))$$

since H_0 is right exact. So $\mathcal{L}$ is of weights $\leq m-1$.
But the kernel of our homomorphism equals

$$\mathrm{Hom}_{M^L}(\mathbf{V}, \mathcal{L}),$$

which therefore is trivial. q.e.d.

Remark: As was pointed out in the proof of Theorem 1.5, the Leray spectral sequence exists because the functors $\mathcal{H}^q\widetilde{[\pi]}_*$ and $\mathcal{H}^q\widetilde{[\pi]}^*$ are cohomology objects of adjoint functors $\widetilde{[\pi]}_*$ and $\widetilde{[\pi]}^*$ on the level of derived categories. It is precisely this point that prevents us from defining polylogarithmic extensions in the context of mixed systems of sheaves (see [W2], § 2 or [W3], § 6 for the definition of smooth objects in this category). While it is conceivable that one may define $\mathcal{H}^q\widetilde{[\pi]}_*$ and $\mathcal{H}^q\widetilde{[\pi]}^*$ "componentwise" without to great an effort, it would require a lot more work to show that they are actually induced by functors $\widetilde{[\pi]}_*$ and $\widetilde{[\pi]}^*$ of derived categories.
However, if $q_0 = 1$, we may think of $\mathcal{P}ol(W', i, K)$ as a collection of framed pro–sheaves. The rigidity principle allows us to show that the Hodge and l–adic versions defined so far are in fact components of a mixed system of smooth sheaves on $\widetilde{M}^K(P, \mathfrak{X})$:

Corollary 2.2: Let $q_0 = 1$. Then there is a unique one–extension $\mathcal{P}ol(W', i, K)$ of $j^*\mu_{K,MS}\operatorname{res}^P_G(b(W', i))$ by $j^*\mathcal{L}og(i, K)(d)$ in $\widetilde{[\pi]}$–$UMS^s_{\mathbb{Q}}(\widetilde{M}^K(P, \mathfrak{X}))$ (see [W2], § 3 or [W3], § 6), whose (∞, σ_0)–component is the Hodge theoretic polylogarithm and whose underlying pro–vector bundle over $\mathbb{C}$ carries the canonical algebraic structure of [D2], II, Théorème 5.9. Its l–component is the l–adic polylogarithm.
Proof: Since $\mathcal{P}ol_{\infty,\sigma_0}$ is admissible, its Hodge filtration is a filtration by pro–subbundles, that are algebraic with respect to the canonical algebraic structure on the flat pro–bundle $\mathrm{For}(\mathcal{P}ol_{\infty,\sigma_0})$ underlying $\mathcal{P}ol_{\infty,\sigma_0}$ ([Ka], Proposition 1.11.3), as is the weight filtration. On the other hand, note that since $q_0 = 1$, we must have $h''^{-1,-1} \leq 1$. So we may copy the proof of Theorem 1.5 in the category of regular holonomic $\mathcal{D}$–modules ([Bo], V–VIII) on $\widetilde{M}^K(P, \mathfrak{X})$: note that by [W3], § 3, the flat bifiltered pro–vector bundle underlying $j^*\mathcal{L}og(i, K)_{\infty,\sigma_0}$ descends to $E(P, \mathfrak{X})$. The calculations of 1.1 to 1.3 run through: in the proofs, replace [W3], Theorems 2.3 and 4.7 by [W3], Theorem 3.5.

We arrive at a universal one–extension $\mathcal{P}ol_{RH}$ in

$$\mathrm{Ext}^1_{[\widetilde{\pi}]-U\mathrm{VB}(\widetilde{M}^K)}(j^*\mathrm{For}(\mu_{K,DR}\,\mathrm{res}^P_G(b(W',i))), j^*\mathrm{For}(\mathcal{L}og(i,K)_{DR}(d)))\,,$$

where $[\widetilde{\pi}]$–UVB$(\widetilde{M}^K)$ is the category of $[\widetilde{\pi}]$–unipotent flat vector bundles on $\widetilde{M}^K$, whose connection is regular at infinity. $\mathcal{P}ol_{RH}$ defines an $E(P,\mathfrak{X})$–structure on For$(\mathcal{P}ol_{\infty,\sigma_0})$. We need to show that the weight and Hodge filtrations of $\mathcal{P}ol_{\infty,\sigma_0}$ descend to $E(P,\mathfrak{X})$. In order to apply [W2], Lemma 2.10, we must know that they are fixed under any automorphism τ of $\mathbb{C}$ over $E(P,\mathfrak{X})$. Now the rigidity principle shows that the *extension class* defined by $\mathcal{P}ol_{\infty,\sigma_0}$ is fixed under any automorphism. The claim follows: by Theorem 1.5.a), the pro–variation $\mathcal{P}ol_{\infty,\sigma_0}$ admits no non–trivial automorphisms, that induce the identity on both $j^*\mu_{K,\infty,\sigma_0}\mathrm{res}^P_G(b(W',i))$ and $j^*\mathcal{L}og(i,K)(d)$.
So we have defined a bifiltered flat pro–vector bundle $\mathcal{P}ol_{DR}$ on $\widetilde{M}^K$. The Hodge components $\mathcal{P}ol_{\infty,\sigma}$ for arbitrary embeddings of $E(P,\mathfrak{X})$ into $\mathbb{C}$ are defined by considering Shimura data conjugate to the given ones ([W3], §§ 5 and 6). Again, the rigidity principle assures that $\mathcal{P}ol_{DR}$ does not depend on the choice of σ and hence that the data fit together to define a mixed system of $[\widetilde{\pi}]$–unipotent smooth sheaves. q.e.d.

One might wonder whether it is possible to find an element of an Ext–group in a category of smooth sheaves mapping to $\mathcal{P}ol(W',i,K)$. The answer in most cases is provided by the following

Theorem 2.3:

a) Assume that the codimension d'' of $[k]$ is one and that the fibres of $[\widetilde{\pi}]$ are unipotent $K(\pi,1,\leq q_0)$s (see [W2], § 4).
Then $\mathcal{P}ol(W',i,K)$ comes from a unique extension in

$$\mathrm{Ext}^{q_0}_{\mathrm{Sh}^s_{[\widetilde{\pi}]}(\widetilde{M}^K(P,\mathfrak{X}))}(j^*\mu_{K,-}\mathrm{res}^P_G(b(W',i)), j^*\mathcal{L}og(i,K)(d))\,,$$

which as in 2.1 is uniquely determined by the underlying extension of smooth relatively unipotent topological sheaves.

b) Assume that $d''\geq 2$.
Then it is impossible to represent $\mathcal{P}ol(W',i,K)$ by an extension of smooth sheaves.

Proof: a) From [W3], Theorem 1.3, it can be deduced that we are in the situation of [W2], Theorem 4.3. So 1.3 remains correct if we replace $\mathcal{H}^q\widetilde{[\pi]}_*$ by $R^{q+d}\widetilde{[\pi]}_*^s$.

Similarly, 1.5 remains true if we replace $\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))$ by $\mathrm{Sh}^s_{\widetilde{[\pi]}}(\widetilde{M}^K(P,\mathfrak{X}))$. In the proof, replace the Leray spectral sequence for $\widetilde{[\pi]}$ by the Hochschild–Serre spectral sequence for the category of smooth unipotent sheaves. Note that it is not necessary to employ an analogue of 1.6 since $h''^{-1,-1} \le d'' \le 1$, so the second possibility in 1.3.a) does not occur.

So we may define a smooth version $\mathcal{P}ol^s(W', i, K)$ and prove a rigidity principle as in 2.1.

In order to show that $\mathcal{P}ol^s$ maps to $\mathcal{P}ol$, we have to convince ourselves that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^{q_0}_{\mathrm{Sh}^s_{\widetilde{[\pi]}}(\widetilde{M}^K)} & \longrightarrow & \mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K)} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Sh}^s(M^L)} & = & \mathrm{Hom}_{\mathrm{Sh}(M^L)} \end{array}$$

commutes.

Here, the vertical maps are given by the edge homomorphisms of the Leray and Hochschild–Serre spectral sequences.

Observe that in general this would require more than what was actually proved in [W2], Theorem 4.3.

Namely, recalling how the edge homomorphisms are defined, we would have to know that the *complexes*

$$R\widetilde{[\pi]}_*^s \quad \text{and} \quad \widetilde{[\pi]}_*[-d]$$

are quasi–isomorphic in degrees $\le q_0$.

While one may expect this to be true in our situation, for the time being we have to think of a different proof: by 2.1, $\mathcal{P}ol$ is uniquely determined by the underlying extension of topological sheaves. Therefore, is suffices to prove that the topological version of the above diagram commutes. This can be checked, by [W2], Lemma 1.7, in the respective ind–categories. Now the central point is that $\overline{\widetilde{[\pi]}}_*$ has a natural extension to the derived category of ind–constructible topological sheaves.

Since the extension of $R\overline{\widetilde{[\pi]}}_*^s$ is a derived functor, we get a transformation

$$R\overline{\widetilde{[\pi]}}_*^s \longrightarrow \overline{\widetilde{[\pi]}}_*[-d]$$

which by our hypothesis is a quasi–isomorphism in degree $\le q_0$, when evaluated on smooth relatively unipotent topological sheaves.

b) Assume that $\mathcal{P}ol$ is represented by an extension $\mathcal{E}$ consisting of smooth sheaves. Then since the codimension of the complement of $\widetilde{M}^K$ in M^K is greater than one, $\mathcal{E}$ can be extended to the whole of M^K, thereby yielding a q_0–extension of

$\mu_{K,-}\mathrm{res}^P_G(b(W',i))$ by $\mathcal{L}og(i,K)(d)$. Namely, because of the codimension condition, $\mathcal{H}^0\overline{j}_*$ induces an equivalence of categories between smooth topological sheaves on $\widetilde{M}^K$ and smooth topological sheaves on M^K, the inverse being given by $\overline{j}^*$. Both functors are exact, and analogous statements hold on the level of smooth mixed sheaves.

But by 1.1 and the Leray spectral sequence for $[\pi]$, we have for any
$\mathbf{V} \in \mathrm{Sh}^s(M^L)$:

$$\begin{aligned}
&\mathrm{Ext}^q_{M^K}(([\pi]^s)^*\mathbf{V}, \mathcal{L}og(i,K)(d)) = 0 \text{ for } q \neq N\,,\\
&\mathrm{Ext}^N_{M^K}(([\pi]^s)^*\mathbf{V}, \mathcal{L}og(i,K)(d)) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{M^L}(\mathbf{V}, \mathbb{Q}_{(l)}(0))\,.
\end{aligned}$$

So even if q_0 equals N, the Ext–group is trivial if the weights of $\mathbf{V}$ are ≤ -1.

q.e.d.

Remarks: a) Theorem 2.3.a) will be of great help when we describe explicitly polylogarithms for special Shimura varieties.

It will be a relatively easy matter to write down an extension of smooth relatively unipotent topological sheaves, which is a candidate for For($\mathcal{P}ol$). 2.3.a) asserts that as soon as we manage to equip it with a mixed structure, this will necessarily be the polylogarithmic extension.

b) As long as there is no satisfactory formalism of mixed systems of sheaves available, we may use an analogue of 2.3.a) as a preliminary definition of the "mixed systems" version of $\mathcal{P}ol(W',i,K)$ if $d''=1$ and if the fibres of $\widetilde{[\pi]}$ are unipotent $K(\pi,1,\le q_0)$s. For the proof of the "smooth mixed systems" version of 1.3, use the fact that the diagrams

$$\begin{array}{ccc}
[\widetilde{\pi}]\text{–}UMS^s_{\mathbb{Q}}(\widetilde{M}^K) & \xrightarrow{(\infty,\sigma_0)} & [\widetilde{\pi}]_{\mathbb{C}}\text{–}U\mathrm{Var}_{\mathbb{Q}}(\widetilde{M}^K_{\mathbb{C}}) \\
R^q[\widetilde{\pi}]^s_* \downarrow & & \downarrow R^q([\widetilde{\pi}]_{\mathbb{C}})^s_* \\
MS^s_{\mathbb{Q}}(M^L) & \xrightarrow{(\infty,\sigma_0)} & \mathrm{Var}_{\mathbb{Q}}(M^L_{\mathbb{C}})
\end{array} \quad ,$$

$$\begin{array}{ccc}
[\widetilde{\pi}]\text{–}UMS^s_{\mathbb{Q}}(\widetilde{M}^K) & \xrightarrow{(l)} & [\widetilde{\pi}]\text{–}U\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(\widetilde{M}^K) \\
R^q[\widetilde{\pi}]^s_* \downarrow & & \downarrow R^q[\widetilde{\pi}]^s_* \\
MS^s_{\mathbb{Q}}(M^L) & \xrightarrow{(l)} & \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(M^L)
\end{array} \quad \text{and}$$

$$\begin{array}{ccc}
[\widetilde{\pi}]\text{–}UMS^s_{\mathbb{Q}}(\widetilde{M}^K) & \xrightarrow{\mathrm{For}\circ(DR)} & [\widetilde{\pi}]\text{–}U\mathrm{VB}(\widetilde{M}^K) \\
R^q[\widetilde{\pi}]^s_* \downarrow & & \downarrow R^q[\widetilde{\pi}]^s_* \\
MS^s_{\mathbb{Q}}(M^L) & \xrightarrow{\mathrm{For}\circ(DR)} & \mathrm{VB}(M^L)
\end{array}$$

commute: [W2], Corollaries 3.2.i) and 3.4.i) and Theorem 3.6.b) take care of the first two diagrams. For the third diagram, we have to define $R^q[\widetilde{\pi}]^s_*$ – note that unless $\widetilde{M}^K$ has $E(P,\mathfrak{X})$–rational points, the categories on the right won't be neutral Tannakian: this can be done in two equivalent manners, either by defining $R^q[\widetilde{\pi}]^s_*$ to be the restriction of the functor $\mathcal{H}^{q-d}[\widetilde{\pi}]_*$ on the category of regular holonomic $\mathcal{D}$–modules or by applying Galois descent. In any case, it can be checked over $\mathbb{C}$ that the natural transformation

$$\mathrm{For}\circ(DR)\circ R^q[\widetilde{\pi}]^s_* \longrightarrow R^q[\widetilde{\pi}]^s_*\circ\mathrm{For}\circ(DR)$$

is an isomorphism. There, it follows from [W2], Theorem 3.6.b).

c) Let us note that in the situation of b) or 2.2 where we managed to define a "mixed systems" version of the polylogarithmic extension, the results of §§ 3 and 4 together with the splitting principle 6.1 will carry over without difficulty. Furthermore, the morphisms used to construct the norm map in § 5 can be defined in the context of smooth mixed systems, and hence the remaining results of §§ 5 and 6 are also true.

§ 3 Interrelation between polylogarithms associated to different unipotent extensions

In case W' is nonzero, we are going to identify $\mathcal{P}ol(W', i, K)$ with the cup–product of a polylogarithmic extension associated to certain quotient Shimura data and an extension arising via the canonical construction.
In terms of our goal of generating "interesting extensions", Theorem 3.1 and the splitting principle we shall establish in § 6 tell us that one should either start with $W' = 0$ or remove more than just one sub–Shimura variety. Namely, by the reductiveness of G, extensions obtained via the canonical construction of extensions on the level of representations will always split along embeddings of pure Shimura varieties given by Levi sections.
This observation is in fact what we would like readers of § 3 to keep in mind, and we advise them to omit the proof of 3.1 at first reading.

So assume W' is nonzero.
Let $0 \neq W_0 \leq W'$ be normal in P.
Set $W_1' := W'/W_0$, $W_1 := W/W_0$,

$$\begin{gathered}(P_1', \mathfrak{X}_1') := (P', \mathfrak{X}')/W_0\,,\ (P_1, \mathfrak{X}_1) := (P, \mathfrak{X})/W_0\,,\\ \varphi' : (P', \mathfrak{X}') \to (P_1', \mathfrak{X}_1')\,,\ \varphi : (P, \mathfrak{X}) \to (P_1, \mathfrak{X}_1)\,,\\ \pi_1' : (P_1', \mathfrak{X}_1') \to (G, \mathcal{H})\,,\ \pi_1 : (P_1, \mathfrak{X}_1) \to (G, \mathcal{H})\,.\end{gathered}$$

The existence of π_1' and π_1 is guaranteed by [P], Proposition 2.9. The sections

$$i_1' := \varphi' \circ i' : G \hookrightarrow P_1' \quad \text{and} \quad i_1 := \varphi \circ i : G \hookrightarrow P_1$$

are covered by embeddings of Shimura data

$$i_1' : (G, \mathcal{H}) \hookrightarrow (P_1', \mathfrak{X}_1') \quad \text{and} \quad i_1 : (G, \mathcal{H}) \hookrightarrow (P_1, \mathfrak{X}_1)$$

as is the immersion $k_1 : P_1' \hookrightarrow P_1$.
By [P], Proposition 2.17, we have a commutative diagram

$$\begin{array}{ccc} (P', \mathfrak{X}') & \xrightarrow{\ k\ } & (P, \mathfrak{X}) \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (P_1', \mathfrak{X}_1') & \xrightarrow{\ k_1\ } & (P_1, \mathfrak{X}_1) \end{array}$$

which by another application of the same proposition is cartesian.

As before, let $K \leq P(\mathbb{A}_f)$ be neat, open and compact and of the shape $K = K^W \rtimes L$.

Set $K_1 := \varphi(K), K_1' := k_1^{-1}(K_1)$. We have

$$K' = K_1' \times_{K_1} K\,.$$

By [P], Lemma 3.11 and the remark preceding its proof, we get a cartesian diagram

$$\begin{array}{ccccc}
M^{K'}(P', \mathfrak{X}') & \xrightarrow{[k]_{K',K}} & M^K(P, \mathfrak{X}) & \xleftarrow{j_K} & \widetilde{M}^K(P, \mathfrak{X}) \\
{\scriptstyle [\varphi']_{K',K_1'}} \downarrow & & \downarrow {\scriptstyle [\varphi]_{K,K_1}} & & \downarrow {\scriptstyle \widetilde{[\varphi]}_{K,K_1}} \\
M_1^{K'}(P_1', \mathfrak{X}_1') & \xrightarrow{[k_1]_{K_1',K_1}} & M^{K_1}(P_1, \mathfrak{X}_1) & \xleftarrow{j_{K_1}} & \widetilde{M}^{K_1}(P_1, \mathfrak{X}_1)
\end{array}$$

of $M^L(G, \mathcal{H})$–schemes.

We have universal extensions

$$\begin{aligned}
\mathcal{P}ol(W') &\in \operatorname{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}(j_K^* \mu_{K,-} \mathrm{res}^P_G(b(W', i)), j_K^* \mathcal{L}og(i, K)(d))\,, \\
\mathcal{P}ol(W_1') &\in \operatorname{Ext}^{q_{1,0}}_{\mathrm{Sh}(\widetilde{M}^{K_1}(P_1,\mathfrak{X}_1))}(j_{K_1}^* \mu_{K_1,-}(\mathrm{res}^{P_1}_G(b(W_1', i_1))), j_{K_1}^* \mathcal{L}og(i_1, K_1)(d_1))\,, \\
\mathcal{P}_\emptyset &\in \operatorname{Ext}^{N_0}_{\mathrm{Rep}_{\mathbb{Q}_{(l)}}(P)}(\mathrm{res}^P_{P_1}(\hat{\mathfrak{U}}(\mathrm{Lie} W_1)), \hat{\mathfrak{U}}(\mathrm{Lie} W) \hat{\otimes}_{\mathbb{Q}_{(l)}} \Lambda^{N_0}(\mathrm{Lie} W_0))\,.
\end{aligned}$$

We have to explain what we mean by the last Ext–group. This time, both arguments are pro–objects and we define this group to be

$$\varprojlim_{m \in \mathbb{N}} \left(\varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^{N_0}_{\mathrm{Rep}_{\mathbb{Q}_{(l)}}(P)}(\mathrm{res}^P_{P_1}(\hat{\mathfrak{U}}(\mathrm{Lie} W_1)/\mathfrak{a}_1^n), (\hat{\mathfrak{U}}(\mathrm{Lie} W)/\mathfrak{a}^m) \otimes_{\mathbb{Q}_{(l)}} \Lambda^{N_0}(\mathrm{Lie} W_0)) \right)\,.$$

Here, $N_0 := N - N_1 = \dim W_0$, and we take the adjoint representation of P on $\mathrm{Lie} W_0$. So the induced representation on $\Lambda^{N_0}(\mathrm{Lie} W_0)$ factors through G. We saw in the proof of 1.1 that $\mu_{K,-}(\Lambda^{N_0}(\mathrm{Lie} W_0))$ is canonically isomorphic to $\mathbb{Q}_{(l)}(d_0)$, where $d_0 := d - d_1$.

Still, it remains to define the element $P_\emptyset$ of this group.

We apply 1.4 to the direct limit over n of the Hochschild–Serre spectral sequence for

$$1 \longrightarrow W_0 \longrightarrow P \longrightarrow P_1 \longrightarrow 1\,,$$

applied to $\mathrm{res}^{P}_{P_1}(\hat{\mathfrak{U}}(\mathrm{Lie}W_1)/\mathfrak{a}_1^n), n \in \mathbb{N}$.

By [W2], Corollary 1.13, we have

$$H^q(W_0, \hat{\mathfrak{U}}(\mathrm{Lie}W)) \hat{\otimes}_{\mathbb{Q}_{(l)}} \Lambda^{N_0}(\mathrm{Lie}W_0) = \begin{cases} 0 & \text{for } q \neq N_0 \\ \hat{\mathfrak{U}}(\mathrm{Lie}W_1) & \text{for } q = N_0 \end{cases} .$$

As usual, this is a statement on the projective systems

$$(H^q(W_0, \hat{\mathfrak{U}}(\mathrm{Lie}W)/\mathfrak{a}^m) \otimes_{\mathbb{Q}_{(l)}} \Lambda^{N_0}(\mathrm{Lie}W_0))_{m\in\mathbb{N}} ,$$

which shows that the hypothesis of 1.4 is met.

Therefore, we get an isomorphism of the above Ext–group with

$$\lim_{\substack{\longleftarrow \\ m\in\mathbb{N}}} \left(\lim_{\substack{\longrightarrow \\ n\in\mathbb{N}}} \mathrm{Hom}_{\mathrm{Rep}_{\mathbb{Q}_{(l)}}(P_1)}(\hat{\mathfrak{U}}(\mathrm{Lie}W_1)/\mathfrak{a}_1^n, \hat{\mathfrak{U}}(\mathrm{Lie}W_1)/\mathfrak{a}_1^m) \right) ,$$

and we define $\mathcal{P}_\emptyset$ to be the extension corresponding to the identity.

By the same argument, replacing the Hochschild–Serre spectral sequence by the Leray spectral sequence and using [W3], Theorem 4.3 and the remark following [W3], Corollary 1.4, one sees that there is a universal N_0–extension in

$$\mathrm{Ext}^{N_0}_{\mathrm{Sh}(M^K(\mathcal{P},\mathfrak{X}))}([\varphi]^*\mathcal{L}og(i_1, K_1)(d_1), \mathcal{L}og(i, K)(d))$$

which is uniquely determined by its underlying extension of topological sheaves. Because of the same reason as in the proof of 2.3.a), it is true that via the isomorphism of [W3], Theorems 2.1 and 4.4, this is precisely the extension $\mu_{K,-}(\mathcal{P}_\emptyset)(d_1)$.

Note also that

$$b(W', i) = W_{-1}(H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}W))) = W_{-1}(H_0(W_1', \hat{\mathfrak{U}}(\mathrm{Lie}W_1))) = b(W_1', i_1)$$

and that

$$[\varphi]^* \circ \mu_{K_1,-} = \mu_{K,-} \circ \mathrm{res}^P_{P_1} .$$

Theorem 3.1: Up to a sign, $\mathcal{P}ol(W')$ is the cup–product of

$$\widetilde{[\varphi]}^* \mathcal{P}ol(W_1') \quad \text{and} \quad j_K^*(\mu_{K,-}(\mathcal{P}_\emptyset)(d_1)) .$$

Proof: Observe that because $[\varphi]$ is of type (S), $[\varphi]_*$ satisfies base change.

We have to show the following:

1. given sheaves $\mathbf{V}^L$ on M^L, $\widetilde{\mathbf{V}}^{K_1}$ on $\widetilde{M}^{K_1}$ and a smooth $\widetilde{[\varphi]}$–unipotent sheaf $\widetilde{\mathbf{V}}^K$ on $\widetilde{M}^K$, the diagram

$$
\begin{array}{ccc}
\mathrm{Ext}^q_{\widetilde{M}^{K_1}}(\widetilde{[\pi_1]}^*\mathbf{V}^L, \widetilde{\mathbf{V}}^{K_1}) \times \mathrm{Ext}^{N_0-d_0}_{\widetilde{M}^K}(\widetilde{[\varphi]}^*\widetilde{\mathbf{V}}^{K_1}, \widetilde{\mathbf{V}}^K) & \longrightarrow & \mathrm{Ext}^{q+N_0-d_0}_{\widetilde{M}^K}(\widetilde{[\pi]}^*\mathbf{V}^L, \widetilde{\mathbf{V}}^K) \\
(\alpha, \beta) & \longmapsto & \beta \cup \widetilde{[\varphi]}^*\alpha \\
\downarrow \text{edge} \times \text{edge} & & \downarrow \text{edge} \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q\widetilde{[\pi_1]}_*\widetilde{\mathbf{V}}^{K_1}) \times \mathrm{Hom}_{\widetilde{M}^{K_1}}(\widetilde{\mathbf{V}}^{K_1}, \mathcal{H}^{N_0-d_0}\widetilde{[\varphi]}_*\widetilde{\mathbf{V}}^K) & & \\
(h, \widetilde{g}) \longmapsto \mathcal{H}^q\widetilde{[\pi_1]}_*(\widetilde{g}) \circ h & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+N_0-d_0}\widetilde{[\pi]}_*\widetilde{\mathbf{V}}^K) \\
\downarrow & \swarrow \text{edge} & \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q\widetilde{[\pi_1]}_*\mathcal{H}^{N_0-d_0}\widetilde{[\varphi]}_*\widetilde{\mathbf{V}}^K) & &
\end{array}
$$

commutes up to a sign.

Here, the morphism

$$\mathcal{H}^{q+N_0-d_0}\widetilde{[\pi]}_*\widetilde{\mathbf{V}}^K \longrightarrow \mathcal{H}^q\widetilde{[\pi_1]}_*\mathcal{H}^{N_0-d_0}\widetilde{[\varphi]}_*\widetilde{\mathbf{V}}^K$$

is an edge homomorphism in the Grothendieck spectral sequence belonging to the formula

$$\widetilde{[\pi]}_* = \widetilde{[\pi_1]}_* \circ \widetilde{[\pi]}_* \,.$$

Observe that $\mathcal{H}^p\widetilde{[\varphi]}_*\widetilde{\mathbf{V}}^K$ vanishes for $p > N_0 - d_0$ because $\widetilde{\mathbf{V}}^K$ was assumed to be $\widetilde{[\varphi]}$–unipotent and we may apply [W2], Theorem 4.3.

2. assume that the sheaves $\widetilde{\mathbf{V}}^{K_1}$ and $\widetilde{\mathbf{V}}^K$ of 1. come from sheaves $\mathbf{V}^{K_1}$ and $\mathbf{V}^K$ on M^{K_1} and M^K:

$$\widetilde{\mathbf{V}}^{K_1} = j_{K_1}^*\mathbf{V}^{K_1}\,,\ \widetilde{\mathbf{V}}^K = j_K^*\mathbf{V}^K\,,$$

and assume that $\mathbf{V}^K$ is smooth and $[\varphi]$–unipotent. Then the diagrams

$$
\begin{array}{ccc}
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* \widetilde{\mathbf{V}}^{K_1}) \times \mathrm{Hom}_{M^{K_1}}(\mathbf{V}^{K_1}, \mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) & \searrow & \\
\Big\downarrow \text{boundary} \times \text{id} & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* \widetilde{\mathbf{V}}^{K_1}) \times \mathrm{Hom}_{\widetilde{M}^{K_1}}(\widetilde{\mathbf{V}}^{K_1}, \mathcal{H}^{N_0-d_0}[\widetilde{\varphi}]_* \widetilde{\mathbf{V}}^K) \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathbf{V}^{K_1}) \times \mathrm{Hom}_{M^{K_1}}(\mathbf{V}^{K_1}, \mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) & & \Big\downarrow \quad (h, \widetilde{g}) \mapsto \mathcal{H}^q[\widetilde{\pi_1}]_*(\widetilde{g}) \circ h \\
\Big\downarrow \quad (h, g) \mapsto \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)(g) \circ h & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* \mathcal{H}^{N_0-d_0}[\widetilde{\varphi}]_* \widetilde{\mathbf{V}}^K) \\
 & & \Big\| \\
 & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* j_{K_1}^* \mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) \\
 & \swarrow \text{boundary} & \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) & &
\end{array}
$$

and

$$
\begin{array}{ccc}
 & \swarrow \text{edge} & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+N_0-d_0}[\widetilde{\pi}]_* \widetilde{\mathbf{V}}^K) \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* \mathcal{H}^{N_0-d_0}[\widetilde{\varphi}]_* \widetilde{\mathbf{V}}^K) & & \Big\downarrow \text{boundary} \\
\Big\| & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^! \mathbf{V}^K) \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^q[\widetilde{\pi_1}]_* j_{K_1}^* \mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) & & \Big\downarrow \text{edge} \\
\Big\downarrow \text{boundary} & & \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1-d''}[\pi_1']_* \mathcal{H}^{N_0-d_0}[\varphi']_*([k]^! \mathbf{V}^K[d''])) \\
 & \swarrow \tau_* & \\
\mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K) & &
\end{array}
$$

commute up to a sign.

Here, the isomorphism τ_* comes from the isomorphism

$$\tau : \mathcal{H}^{N_0-d_0}[\varphi']_*([k]^!\mathbf{V}^K[d'']) \xrightarrow{\sim} ([k_1]^!\mathcal{H}^{N_0-d_0}[\varphi]_*\mathbf{V}^K)[d''] \,,$$

which is the isomorphism of the cohomology objects of largest degree induced by the isomorphism of functors

$$\tau : [\varphi']_*[k]^! \xrightarrow{\sim} [k_1]^![\varphi]_* \,.$$

Note that $\mathbf{V}^K$ is smooth, so $[k]^!\mathbf{V}^K$ is concentrated in one degree because of purity.

Since $\mathbf{V}^K$ is $[\varphi]$–unipotent, the sheaf $[k]^!\mathbf{V}^K[d'']$ is $[\varphi']$–unipotent, and $\mathcal{H}^q[\varphi']_*([k]^!\mathbf{V}^K[d''])$ vanishes for $q > N_0 - d_0$.

The boundary homomorphisms all come from the exact triangles

$$\begin{array}{ccc} [k]_*[k]^! & \longrightarrow & \mathrm{id} \\ {\scriptstyle \text{shift by } [1]} \nwarrow & & \swarrow \\ & (j_K)_*j_K^* & \end{array} \quad \text{and} \quad \begin{array}{ccc} [k_1]_*[k_1]^! & \longrightarrow & \mathrm{id} \\ {\scriptstyle \text{shift by } [1]} \nwarrow & & \swarrow \\ & (j_{K_1})_*j_{K_1}^* & \end{array} \,.$$

3. for a smooth sheaf $\mathbf{V}^K$ on M^K with smooth higher direct images $\mathcal{H}^q[\varphi]_*\mathbf{V}^K$, the diagram

$$\begin{array}{ccc} [\varphi']_*[k]^!\mathbf{V}^K & \xrightarrow[\sim]{\tau} & [k_1]^![\varphi]_*\mathbf{V}^K \\ {\scriptstyle \text{purity}} \wr\downarrow & & \wr\downarrow {\scriptstyle \text{purity}} \\ [\varphi']_*[k]^*\mathbf{V}^K(-d'')[-2d''] & \xleftarrow{\sim} & [k_1]^*[\varphi]_*\mathbf{V}^K(-d'')[-2d''] \end{array}$$

commutes.

Here, the lower horizontal arrow is the base change isomorphism.

Let us first see why 1.–3. solve our problem:

take a smooth $[\varphi]$–unipotent sheaf $\mathbf{V}^K$ on M^K, a sheaf $\mathbf{V}^{K_1}$ on M^{K_1} and $g \in \mathrm{Hom}_{M^{K_1}}(\mathbf{V}^{K_1}, \mathcal{H}^{N_0-d_0}[\varphi]_*\mathbf{V}^K)$. So g induces a map

$$\widetilde{g} : \widetilde{\mathbf{V}}^{K_1} := j_{K_1}^*\mathbf{V}^{K_1} \longrightarrow \mathcal{H}^{N_0-d_0}\widetilde{[\varphi]}_*\widetilde{\mathbf{V}}^K \,,$$

where $\widetilde{\mathbf{V}}^K := j_K^*\mathbf{V}^K$.

Let $\mathbf{V}^L \in \mathrm{Sh}(M^L)$ and $f \in \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^!)\mathbf{V}^K)$. Assume that we have elements

$$\alpha \in \mathrm{Ext}^q_{\widetilde{M}^{K_1}}(\widetilde{[\pi_1]}^*\mathbf{V}^L, \widetilde{\mathbf{V}}^{K_1}) \,,$$

where $\widetilde{\mathbf{V}}^{K_1} := j_{K_1}^* \mathbf{V}^{K_1}$, and

$$\beta \in \operatorname{Ext}^{N_0 - d_0}_{\widetilde{M}^K}(\widetilde{[\varphi]}^* \widetilde{\mathbf{V}}^{K_1}, \widetilde{\mathbf{V}}^K)$$

satisfying the following:

a) under the homomorphism

$$(\mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)(g))_* \circ \text{boundary} \circ \text{edge} :$$

$$\operatorname{Ext}^q_{\widetilde{M}^{K_1}}(\widetilde{[\pi_1]}^* \mathbf{V}^L, \widetilde{\mathbf{V}}^{K_1}) \to \operatorname{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K)\,,$$

α maps to the morphism composed of

$$f : \mathbf{V}^L \longrightarrow \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^!)\mathbf{V}^K,$$

the homomorphism

$$\text{edge} : \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^!)\mathbf{V}^K \to \mathcal{H}^{q+1-d''}[\pi_1']_* \mathcal{H}^{N_0-d_0}[\varphi']_*([k]^! \mathbf{V}^K[d''])$$

and the isomorphism

$$\mathcal{H}^{q+1-d''}[\pi_1']_*(\tau) :$$

$$\mathcal{H}^{q+1-d''}[\pi_1']_* \mathcal{H}^{N_0-d_0}[\varphi']_*([k]^! \mathbf{V}^K[d'']) \xrightarrow{\sim} \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K\,.$$

b) under the homomorphism

$$\text{edge} : \operatorname{Ext}^{N_0-d_0}_{\widetilde{M}^K}(\widetilde{[\varphi]}^* \widetilde{\mathbf{V}}^{K_1}, \widetilde{\mathbf{V}}^K) \to \operatorname{Hom}_{\widetilde{M}^{K_1}}(\widetilde{\mathbf{V}}^{K_1}, \mathcal{H}^{N_0-d_0} \widetilde{[\varphi]}_* \widetilde{\mathbf{V}}_K)\,,$$

β maps to $\tilde{g}$.

It follows from the diagram in 1. and the first diagram in 2. that

$$\beta \cup \widetilde{[\varphi]}^* \alpha \in \operatorname{Ext}^{q+N_0-d_0}_{\widetilde{M}^K}(\widetilde{[\pi]}^* \mathbf{V}^L, \widetilde{\mathbf{V}}^K)$$

maps, up to a sign, to the same element in

$$\operatorname{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1}([\pi_1']_*[k_1]^!)\mathcal{H}^{N_0-d_0}[\varphi]_* \mathbf{V}^K)$$

under the homomorphism boundary $\circ$ edge $\circ$ edge as the one described in a). By the second diagram in 2., the homomorphism

$$\text{boundary} \circ \text{edge} :$$

$$\operatorname{Ext}^{q+N_0-d_0}_{\widetilde{M}^K}(\widetilde{[\pi]}^* \mathbf{V}^L, \widetilde{\mathbf{V}}^K) \longrightarrow \operatorname{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^!)\mathbf{V}^K)$$

maps $\beta \cup \widetilde{[\varphi]}^* \alpha$ to a morphism, whose image under

$$\begin{aligned} &\text{edge} : \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1+N_0-d_0}([\pi']_*[k]^!)\mathbf{V}^K) \\ &\longrightarrow \ \mathrm{Hom}_{M^L}(\mathbf{V}^L, \mathcal{H}^{q+1-d''}[\pi'_1]_* \mathcal{H}^{N_0-d_0}[\varphi']_*([k]^! \mathbf{V}^K[d''])) \end{aligned}$$

coincides, up to a sign, with that of f.

More specifically, we let $\mathbf{V}^K$ run through the finite–dimensional quotients of $\mathcal{L}og(i,K)(d)$ and g through the natural projections of large enough quotients of $\mathcal{L}og(i_1,K_1)(d_1)$. Then the claim follows from the definition of $\mathcal{P}ol(W')$: namely, the vital ingredients for that definition (1.5.b)) were edge and boundary homomorphisms. Note that because of diagram 3., our identification of $b(W',i)$ and $b(W'_1,i_1)$, which after applying $\mu_{L,-}$ yields $[\pi'_1]_*$ of the base change isomorphism, corresponds via 1.2 to the isomorphism $\mathcal{H}^{q+1-d''}[\pi'_1]_*(\tau)$ above, for suitable q.

Now for the proof of 1.–3.:

1. is a statement on the level of cohomology objects which follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\widetilde{M}^{K_1}}(\widetilde{[\pi_1]}^* C^L, \widetilde{C}^{K_1}) \times \mathrm{Hom}_{\widetilde{M}^K}(\widetilde{[\varphi]}^* \widetilde{C}^{K_1}, \widetilde{C}^K) & \longrightarrow & \mathrm{Hom}_{\widetilde{M}^K}(\widetilde{[\pi]}^* C^L, \widetilde{C}^K) \\
\widetilde{[\pi_1]}_* \times \widetilde{[\varphi]}_* \Big\downarrow \wr \qquad (\alpha,\beta) & \longmapsto & \beta \circ \widetilde{[\varphi]}^*(\alpha) \qquad \\
\mathrm{Hom}_{M^L}(C^L, \widetilde{[\pi_1]}_* \widetilde{C}^{K_1}) \times \mathrm{Hom}_{\widetilde{M}^{K_1}}(\widetilde{C}^{K_1}, \widetilde{[\varphi]}_* \widetilde{C}^K) & & \wr \Big\downarrow \widetilde{[\pi]}_* \\
(h,\widetilde{g}) \mapsto \widetilde{[\pi_1]}_*(\widetilde{g}) \circ h \ \Big\downarrow & & \mathrm{Hom}_{M^L}(C^L, \widetilde{[\pi]}_* \widetilde{C}^K) \\
\mathrm{Hom}_{M^L}(C^L, \widetilde{[\pi_1]}_* \widetilde{[\varphi]}_* \widetilde{C}^K) & \longleftarrow &
\end{array}$$

which in turn follows from the projection formula, applied to $\widetilde{[\varphi]}$: if $\sigma : Y \to X$ is a morphism of varieties over k, then for objects C^1_X, C^2_X of $D^b(\mathrm{Sh}(X))$ and C_Y of $D^b(\mathrm{Sh}(Y))$, the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_X(C_X^1, C_X^2) \times \mathrm{Hom}_Y(\sigma^* C_X^2, C_Y) & \xrightarrow{\mathrm{id} \times \sigma_*} & \mathrm{Hom}_X(C_X^1, C_X^2) \times \mathrm{Hom}_X(C_X^2, \sigma_* C_Y) \\
\downarrow{\scriptstyle \sigma^* \times \mathrm{id}} & & \downarrow{\scriptstyle \circ} \\
\mathrm{Hom}_Y(\sigma^* C_X^1, \sigma^* C_X^2) \times \mathrm{Hom}_Y(\sigma^* C_X^2, C_Y) & & \mathrm{Hom}_X(C_X^1, \sigma_* C_Y) \\
\downarrow{\scriptstyle \circ} & \nearrow{\scriptstyle \sigma_*} & \\
\mathrm{Hom}_Y(\sigma^* C_X^1, C_Y) & &
\end{array}
$$

commutes.

This can be checked e.g. on topological level.

2. Again, the claims follow from statements on the level of complexes. For the first, observe that it suffices to prove the claim for $C^L := \widetilde{[\pi_1]}_* \widetilde{C}^{K_1}$ and the universal morphism id. The claim then follows from the fact that

$$
\begin{array}{ccc}
[k_1]_*[k_1]^! & \longrightarrow & \mathrm{id} \\
{\scriptstyle \text{shift by } [1]} \nwarrow & & \swarrow \\
& (j_{K_1})_* j_{K_1}^* &
\end{array}
$$

is an exact triangle of *functors*.

Similarly, the second follows from the commutativity of

$$
\begin{array}{ccc}
\widetilde{[\pi_1]}_* \widetilde{[\varphi]}_* j_K^* & \xleftarrow{=} & \widetilde{[\pi]}_* j_K^* \\
\| \downarrow & & \downarrow \\
\widetilde{[\pi_1]}_* j_{K_1}^* [\varphi]_* & & [\pi']_* [k]^! [1] \\
\downarrow & & \downarrow \| \\
[\pi_1']_* [k_1]^! [\varphi]_* [1] & \underset{\sim}{\xleftarrow{\tau}} & [\pi_1']_* [\varphi']_* [k]^! [1]
\end{array}
$$

which is $[\pi_1]_*$ applied to

$$
\begin{array}{ccc}
(j_{K_1})_* \widetilde{[\varphi]}_* j_K^* & \xleftarrow{=} & [\varphi]_* (j_K)_* j_K^* \\
\| \downarrow & & \downarrow \\
(j_{K_1})_* j_{K_1}^* [\varphi]_* & & [\varphi]_* [k]_* [k]^! [1] \\
\downarrow & & \downarrow \| \\
[k_1]_* [k_1]^! [\varphi]_* [1] & \underset{\sim}{\xleftarrow{\tau}} & [k_1]_* [\varphi']_* [k]^! [1] \, .
\end{array}
$$

The commutativity of this diagram holds since it comes from a cartesian situation.

3. Purity for smooth sheaves is nothing but the isomorphisms

$$\begin{aligned}\gamma_{M^K} : [k]^! &\xrightarrow{\sim} \mathbb{D}_{M^{K'}}[k]^*\mathbb{D}_{M^K}\,, \\ \gamma_{M^{K_1}} : [k_1]^! &\xrightarrow{\sim} \mathbb{D}_{M^{K'_1}}[k_1]^*\mathbb{D}_{M^{K_1}}\end{aligned}$$

written out.

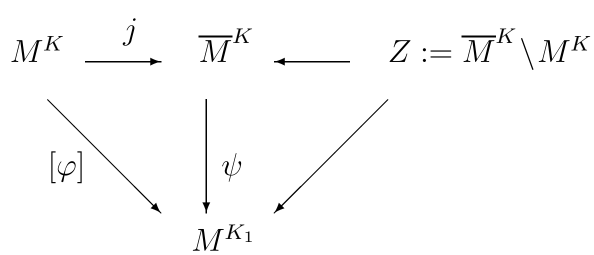
τ comes from an isomorphism of functors and transforms into

$$[\varphi']_*\mathbb{D}_{M^{K'}}[k]^*\mathbb{D}_{M^K} \xrightarrow{\sim} \mathbb{D}_{M^{K'_1}}[k_1]^*\mathbb{D}_{M^{K_1}}[\varphi]_*\,.$$

If $[\varphi]$ is proper, then $[\varphi]_* = [\varphi]_!$, hence $\mathbb{D}_{M^{K_1}}[\varphi]_* = [\varphi]_*\mathbb{D}_{M^K}$ and similarly for $[\varphi']_*$, and τ is the dual of the usual base change isomorphism

$$[k_1]^*[\varphi]_! \xrightarrow{\sim} [\varphi']_![k]^*\,.$$

Here, we haven't used anything but the properness of $[\varphi]$ and the fact that $[\varphi], [k], [\varphi']$ and $[k_1]$ constitute a cartesian diagram.
In the general case, we apply this argument to a compactification of $[\varphi]$ as in [W3], Corollary 1.4:



Base change by

$$[k_1] : M^{K'_1} \hookrightarrow M^{K_1}$$

gives a similar diagram for $[\varphi']$.
Now the base change isomorphism

$$[k_1]^*[\varphi]_* \xrightarrow{\sim} [\varphi']_*[k_1]^*$$

equals the composition of the base change isomorphism for ψ, applied to j_*:

$$[k_1]^*[\varphi]_* = [k_1]^*\psi_*j_* \xrightarrow{\sim} \psi'_*l^*j_*\,,$$

where l denotes the immersion of $\overline{M}^{K'}$ into $\overline{M}^{K}$,
and ψ'_* applied to the base change morphism

$$l^* j_* \longrightarrow (j')_* [k]^* ,$$

which is an isomorphism.
It remains to observe that the isomorphism

$$\overline{\tau} : \psi'_* l^! \xrightarrow{\sim} [k_1]^! \psi_* ,$$

applied to j_*, yields an isomorphism

$$[\varphi']_* [k]^! = \psi'_* l^! j_* \xrightarrow{\sim} [k_1]^! \psi_* j_* = [k_1]^! [\varphi]_* ,$$

which coincides with τ. q.e.d.

§ 4 The small polylogarithmic extension

By [BL], Corollary 1.3.4, our definition in § 1 coincides with what Beilinson and Levin call the large polylogarithmic extension ([BL], 1.3.5) in the elliptic case. There, they also define the small elliptic polylogarithm. The aim of this paragraph is to extend this definition to the general case and to prove (Theorem 4.3) that it is possible to recover the large from the small polylogarithm $pol(W', i, K)$ (compare [BLp], Remark 2.5.5). This means in particular that all extensions occurring in $\mathcal{P}ol(W', i, K)$ already turn up in $pol(W', i, K)$.

We keep the notation of the previous paragraphs.

Lemma 4.1: There is a canonical multiplicative isomorphism of G–modules

$$\operatorname{res}^G_P \hat{\mathfrak{U}}(\operatorname{Lie} W) \xrightarrow{\sim} \prod_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1} ,$$

which is compatible with change of the group G.
Here, $\mathfrak{a}$ denotes the augmentation ideal of $\hat{\mathfrak{U}}(\operatorname{Lie} W)$.
Proof: If $W = U$, then we have $\mathfrak{a}^k = W_{-2k}(\hat{\mathfrak{U}}(\operatorname{Lie} W))$. On the other hand, if the commutator morphism

$$[\ ,\ ] : V \otimes_{\mathbb{Q}} V \longrightarrow U$$

is surjective, we have $\mathfrak{a}^k = W_{-k}(\hat{\mathfrak{U}}(\mathrm{Lie}\,W))$.

In both cases, there is a unique isomorphism as in the claim because of weight reasons and because G is reductive. In general, *choose* a G–complement $\widetilde{U}$ of $\mathrm{im}\,([\ ,\ ])$ in U and let $\widetilde{W} \trianglelefteq W$ be defined by $\mathrm{Lie}\,\widetilde{W} := \mathrm{Lie}\,V \oplus \mathrm{im}\,([\ ,\ ])$ to get an isomorphism

$$\psi : \hat{\mathfrak{U}}(\mathrm{Lie}\,\widetilde{U})\hat{\otimes}_{\mathbb{Q}}\hat{\mathfrak{U}}(\mathrm{Lie}\,\widetilde{W}) \xrightarrow{\ \sim\ } \hat{\mathfrak{U}}(\mathrm{Lie}\,W)\,.$$

Here we view $\mathrm{Lie}\,V$ as a sub–G–module of $\mathrm{Lie}\,W$, as we may because of weight reasons.The map $\mathrm{res}^G_P\psi^{-1}$ induces a multiplicative isomorphism

$$\begin{array}{rcl} \mathrm{res}^G_P\,\hat{\mathfrak{U}}(\mathrm{Lie}\,W) & \xrightarrow{\ \sim\ } & \prod_{l\geq 0} \mathfrak{a}^l_{\widetilde{U}}/\mathfrak{a}^{l+1}_{\widetilde{U}} \hat{\otimes}_{\mathbb{Q}} \prod_{m\geq 0} \mathfrak{a}^m_{\widetilde{W}}/\mathfrak{a}^{m+1}_{\widetilde{W}} \\ & \xrightarrow{\ \sim\ } & \prod_{n\geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1}\,, \end{array}$$

which is easily seen to be independent of the choice of $\widetilde{U}$. q.e.d.

So in particular, the morphism

$$\mathrm{mult} : \hat{\mathfrak{U}}(\mathrm{Lie}\,W)\hat{\otimes}_{\mathbb{Q}}\mathrm{res}^P_G\mathrm{res}^G_P\,\hat{\mathfrak{U}}(\mathrm{Lie}\,W) \longrightarrow \hat{\mathfrak{U}}(\mathrm{Lie}\,W)$$

of P–modules induces morphisms

$$\mathrm{mult}_n : \hat{\mathfrak{U}}(\mathrm{Lie}\,W)\hat{\otimes}_{\mathbb{Q}}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes n} \longrightarrow \hat{\mathfrak{U}}(\mathrm{Lie}\,W)\,,$$

which in turn yield morphisms

$$H_0(W', \mathrm{mult}_n) : b(W', i)\hat{\otimes}_{\mathbb{Q}}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes n} \longrightarrow b(W', i)\,,$$

where as before $b(W', i) = W_{-1}(H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}\,W)))$.

Observe that $b(W', i)$ is a quotient of $H_0(W', \mathfrak{a})$. We get a descending filtration $F^{\cdot}b(W', i)$ by the images of the $H_0(W', \mathfrak{a}^m)$. On the other hand, the natural epimorphism

$$\mathrm{res}^G_P\,\hat{\mathfrak{U}}(\mathrm{Lie}\,W) \longrightarrow\!\!\!\!\rightarrow H_0(W', \hat{\mathfrak{U}}(\mathrm{Lie}\,W))$$

identifies $b(W', i)$ with a quotient of $\mathrm{res}^G_P\mathfrak{a}$, which by Lemma 4.1 is equal to the product $\prod_{n\geq 1} \mathfrak{a}^n/\mathfrak{a}^{n+1}$. The filtration step $F^m b(W', i)$ is precisely the image of $\prod_{n\geq m} \mathfrak{a}^n/\mathfrak{a}^{n+1}$.

Corollary 4.2: The projection

$$F^m b(W',i) \longrightarrow\!\!\!\!\rightarrow \mathrm{Gr}_F^m b(W',i)$$

has a canonical right inverse.

Proof: The image of $\mathfrak{a}^m/\mathfrak{a}^{m+1}$ in F^m is a complement of F^{m+1}: take a decomposition $W = \widetilde{U} \times \widetilde{W}$ as in the proof of 4.1, such that

$$W' = (\widetilde{U} \cap W') \times (\widetilde{W} \cap W').$$

q.e.d.

In particular, there is a monomorphism $\mathrm{Gr}_F^1 b(W',i) \hookrightarrow b(W',i)$.
Define $b^n(W',i)$ as $\mu_{L,-}\mathrm{Gr}_F^n b(W',i)$.

Definition: The *small polylogarithmic extension*

$$pol(W',i,K)$$

is the q_0–extension in

$$\mathrm{Ext}^{q_0}_{\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))}(\widetilde{[\pi]}^* b^1(W',i), j^*\mathcal{L}og(i,K)(d))$$

corresponding to the inclusion $b^1(W',i) \hookrightarrow \mu_{L,-}(b(W',i))$ under the isomorphism in 1.5.b).

We now describe how to recover $\mathcal{P}ol(W',i,K)$ from $pol(W',i,K)$. Consider the morphism of G–modules

$$\mathrm{Gr}_F^1 b(W',i) \otimes_{\mathbb{Q}} (\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)} \hookrightarrow b(W',i) \otimes_{\mathbb{Q}} (\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)} \xrightarrow{H_0(W',\mathrm{mult}_{n-1})} b(W',i)\,.$$

It is easily seen to map epimorphically to the direct summand $\mathrm{Gr}_F^n b(W',i)$ of $F^n b(W',i)$. Any right inverse ψ_{n-1} of this epimorphism induces a map

$$\begin{aligned}
\varphi_n :\ & \mathrm{Ext}^{q_0}(\widetilde{[\pi]}^* b^1(W',i), j^*\mathcal{L}og(i,K)(d)) \\
\longrightarrow\ & \mathrm{Ext}^{q_0}(\widetilde{[\pi]}^* b^n(W',i), j^*\mathcal{L}og(i,K)(d) \otimes_{\mathbb{Q}} \mu_{K,-}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)}) \\
\xrightarrow{(\mathrm{mult}_{n-1})_*}\ & \mathrm{Ext}^{q_0}(\widetilde{[\pi]}^* b^n(W',i), j^*\mathcal{L}og(i,K)(d))\,.
\end{aligned}$$

Here, the Ext groups are formed in the category $\mathrm{Sh}(\widetilde{M}^K(P,\mathfrak{X}))$, and the first map is the composition of ψ_{n-1}^* with the map induced by tensoring with $\mu_{K,-}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)}$. The product over all n is a map

$$\begin{aligned}
\varphi :\ & \mathrm{Ext}^{q_0}(\widetilde{[\pi]}^* b^1(W',i), j^*\mathcal{L}og(i,K)(d)) \\
\longrightarrow\ & \mathrm{Ext}^{q_0}(j^*\mu_{K,-}\mathrm{res}^P_G(b(W',i)), j^*\mathcal{L}og(i,K)(d))\,,
\end{aligned}$$

and we have:

Theorem 4.3: For any choice of the ψ_{n-1}, the morphism φ maps $pol(W', i, K)$ to $\mathcal{P}ol(W', i, K)$.

Proof: By the projection formula, it is possible to calculate the higher direct images under the morphism $\widetilde{[\pi]}$ of $j^*\mathcal{L}og(i,K)(d) \otimes_{\mathbb{Q}} \mu_{K,-}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)}$ from those of $j^*\mathcal{L}og(i,K)(d)$. Furthermore, the maps φ_n are seen to correspond to the maps

$$\begin{aligned} & \operatorname{Hom}(b^1(W',i), \mu_{L,-}(b(W',i))) \\ \longrightarrow\ & \operatorname{Hom}(b^n(W',i), \mu_{L,-}(b(W',i)) \otimes_{\mathbb{Q}} \mu_{L,-}(\mathfrak{a}/\mathfrak{a}^2)^{\otimes(n-1)}) \\ \xrightarrow{H_0(W',\mathrm{mult}_{n-1})_*}\ & \operatorname{Hom}(b^n(W',i), \mu_{L,-}(b(W',i)))\,. \end{aligned}$$

The product over all n clearly maps the inclusion $b^1(W',i) \hookrightarrow \mu_{L,-}(b(W',i))$ to the identity. q.e.d.

§ 5 Norm compatibility

The subject of this paragraph is the study of the interrelation of polylogarithmic extensions of different level K. The first result is quite immediate:

Proposition 5.1:

a) Let $K_2 = K^W \rtimes L_2 \le P(\mathbb{A}_f)$ be neat, open and compact, $L_1 \le L_2$ open, $K_1 := K^W \rtimes L_1$. We have a cartesian diagram

$$\begin{array}{ccc} \widetilde{M}^{K_1}(P,\mathfrak{X}) & \xrightarrow{\widetilde{[\cdot 1]}} & \widetilde{M}^{K_2}(P,\mathfrak{X}) \\ \widetilde{[\pi]}\big\downarrow & & \big\downarrow\widetilde{[\pi]} \\ M^{L_1}(G,\mathcal{H}) & \xrightarrow{[\cdot 1]} & M^{L_2}(G,\mathcal{H})\,. \end{array}$$

Via the canonical isomorphism

$$\mathcal{L}og(i,K_1) \xrightarrow{\sim} [\cdot 1]^*\mathcal{L}og(i,K_2),$$

$\mathcal{P}ol(W',i,K_1)$ is the inverse image under $\widetilde{[\cdot 1]}$ of $\mathcal{P}ol(W',i,K_2)$.

b) Let $L \le G(\mathbb{A}_f)$ be neat, open and compact, $K_1^W \le K_2^W \le W(\mathbb{A}_f)$ two open compact subgroups stable under conjugation by $i(L)$. For $j = 1,2$, define $K_j := K_j^W \rtimes L$ and $K_j^{W'} := K_j^W \cap W'(\mathbb{A}_f)$. If we have

$$[K_2^{W'} : K_1^{W'}] = [K_2^W : K_1^W],$$

then the diagram

$$\begin{array}{ccc} M^{K_1'}(P',\mathfrak{X}') & \xrightarrow{[\cdot 1]} & M^{K_2'}(P',\mathfrak{X}') \\ {\scriptstyle [k]}\downarrow & & \downarrow{\scriptstyle [k]} \\ M^{K_1}(P,\mathfrak{X}) & \xrightarrow{[\cdot 1]} & M^{K_2}(P,\mathfrak{X}) \end{array}$$

is cartesian, we have a well defined map

$$\widetilde{[\cdot 1]} : \widetilde{M}^{K_1}(P,\mathfrak{X}) \longrightarrow \widetilde{M}^{K_2}(P,\mathfrak{X}),$$

and $\mathcal{P}ol(W', i, K_1)$ is the inverse image under $\widetilde{[\cdot 1]}$ of $\mathcal{P}ol(W', i, K_2)$.

Proof: left to the reader. q.e.d.

It remains to study the situation complementary to that of 5.1.b). Namely, let $L, K_1^W \leq K_2^W$ be as before, but assume now that

$$K_1^{W'} = K_2^{W'} .$$

We have a diagram

$$\begin{array}{ccccccc} M^{K'}(P',\mathfrak{X}') & \xrightarrow{[k]} & M^{K_1}(P,\mathfrak{X}) & \xleftarrow{j_{K_1}} & \widetilde{M}^{K_1}(P,\mathfrak{X}) & \xleftarrow{j} & [\cdot 1]^{-1}(\widetilde{M}^{K_2}(P,\mathfrak{X})) \\ \| & & {\scriptstyle [\cdot 1]}\downarrow & & & & \downarrow{\scriptstyle \widetilde{[\cdot 1]}} \\ M^{K'}(P',\mathfrak{X}') & \xrightarrow{[k]} & M^{K_2}(P,\mathfrak{X}) & & \xleftarrow{j_{K_2}} & & \widetilde{M}^{K_2}(P,\mathfrak{X}) \\ & {\scriptstyle [\pi']}\searrow & {\scriptstyle [\pi]}\downarrow & & \swarrow{\scriptstyle \widetilde{[\pi]}} & & \\ & & M^L(G,\mathcal{H}) & & & & \end{array}$$

Here, $K' := k^{-1}(K_1) = k^{-1}(K_2)$.
Again, we have the canonical isomorphism

$$\mathcal{L}og(i, K_1) \xrightarrow{\sim} [\cdot 1]^*\mathcal{L}og(i, K_2) .$$

For any two sheaves $\mathbf{V}_1$ and $\mathbf{V}_2$ on $\widetilde{M}^{K_2}(P,\mathfrak{X})$, there is a functorial homomorphism

$$N_{K_1,K_2} : \operatorname{Ext}^q_{\mathrm{Sh}([\cdot 1]^{-1}(\widetilde{M}^{K_2}))}(\widetilde{[\cdot 1]}^*\mathbf{V}_1, \widetilde{[\cdot 1]}^*\mathbf{V}_2) \to \operatorname{Ext}^q_{\mathrm{Sh}(\widetilde{M}^{K_2})}(\mathbf{V}_1,\mathbf{V}_2)$$

given as follows:
as $\widetilde{[\cdot 1]}$ is finite and étale, $\widetilde{[\cdot 1]}_* = \mathcal{H}^0\widetilde{[\cdot 1]}_*$ is exact, and

$$\widetilde{[\cdot 1]}_* = \widetilde{[\cdot 1]}_! \quad \text{and} \quad \widetilde{[\cdot 1]}^* = \widetilde{[\cdot 1]}^! .$$

We have natural transformations

$$\alpha : \widetilde{[\cdot 1]}_*\widetilde{[\cdot 1]}^* \to \mathrm{id} \quad \text{and} \quad \beta : \mathrm{id} \to \widetilde{[\cdot 1]}_*\widetilde{[\cdot 1]}^* .$$

N_{K_1,K_2} is defined to be the composite of

$$\widetilde{[\cdot 1]}_* : \operatorname{Ext}^q_{\mathrm{Sh}([\cdot 1]^{-1}(\widetilde{M}^{K_2}))}(\widetilde{[\cdot 1]}^*\mathbf{V}_1, [\cdot 1]^*\mathbf{V}_2) \to \operatorname{Ext}^q_{\mathrm{Sh}(\widetilde{M}^{K_2})}(\widetilde{[\cdot 1]}_*\widetilde{[\cdot 1]}^*\mathbf{V}_1, \widetilde{[\cdot 1]}_*\widetilde{[\cdot 1]}^*\mathbf{V}_2)$$

and $(\alpha_*)_{\mathbf{V}_2} \circ \beta^*_{\mathbf{V}_1}$.

Theorem 5.2: (Norm compatibility.)
$j^*\mathcal{P}ol(W', i, K_1)$ is mapped to $\mathcal{P}ol(W', i, K_2)$ under N_{K_1,K_2}.
Proof: Let

$$k' : Z := M^{K'} \times_{M^{K_2}} M^{K_1} \hookrightarrow M^{K_1}$$

be the closed immersion complementary to $j_{K_1} \circ j$. The scheme Z is equipped with a finite étale map

$$[\cdot 1] : Z \longrightarrow M^{K'},$$

which has a section.
By arguments similar to the ones used in the construction of $\mathcal{P}ol$, for any $\mathbf{V} \in \mathrm{Sh}^s(M^L)$ of weights ≤ -1 we get an isomorphism

$$\begin{aligned} & \operatorname{Ext}^{q_0}_{[\cdot 1]^{-1}(\widetilde{M}^{K_2})}((\widetilde{[\pi]}^s \circ \widetilde{[\cdot 1]}^s)^*\mathbf{V}, j^*j^*_{K_1}\mathcal{L}og(i, K_1)(d)) \\ \stackrel{\sim}{\longrightarrow}\ & \operatorname{Hom}_{M^L}(\mathbf{V}, \mathcal{H}^{q_0-d}(\widetilde{[\pi]} \circ \widetilde{[\cdot 1]})_*j^*j^*_{K_1}\mathcal{L}og(i, K_1)(d)) \\ \stackrel{\sim}{\longrightarrow}\ & \operatorname{Hom}_{M^L}(\mathbf{V}, \mathcal{H}^{q_0-d+1}([\pi'] \circ [\cdot 1])_*((k')^*\mathcal{L}og(i, K_1)(d')[-2d''])) . \end{aligned}$$

Under this isomorphism, N_{K_1,K_2} corresponds to the functor $\mathcal{H}^{q_0-d+1}[\pi']_*$ applied to the morphism

$$[\cdot 1]_*(k')^*\mathcal{L}og(i, K_1)(d') = [\cdot 1]_*[\cdot 1]^*[k]^*\mathcal{L}og(i, K_2)(d') \to [k]^*\mathcal{L}og(i, K_2)(d')$$

coming from the natural transformation

$$[\cdot 1]_! [\cdot 1]^! \longrightarrow \mathrm{id}\,.$$

Observe that since $j^* \mathcal{P}ol(W', i, K_1)$ can be extended to $\widetilde{M}^{K_1}$, the composition of the above isomorphism, for $\mathbf{V} = \mu_{L,-}(b(W', i))$, with the projection to

$$\mathrm{Hom}_{M^L}(\mu_{L,-}(b(W', i)), \mathcal{H}^{q_0 - d + 1}([\pi'] \circ r)_* (\widetilde{k}^* \mathcal{L}og(i, K_1)(d')[-2d'']))$$

maps $j^* \mathcal{P}ol(W', i, K_1)$ to zero. Here, r and $\widetilde{k}$ are the morphisms

$$\begin{array}{ccc} Z - M^{K'} & \stackrel{\widetilde{k}}{\hookrightarrow} & M^{K_1} \\ r \downarrow & & [\cdot 1] \downarrow \\ M^{K'} & \stackrel{[k]}{\hookrightarrow} & M^{K_2}\,. \end{array}$$

It follows that its image under N_{K_1, K_2} is equal to its projection to the component $M^{K'}$, which by construction is the identity. q.e.d.

§ 6 Values at Levi sections

The reader may have noted that the construction of the polylogarithmic extension can be carried out in a much more general context than that of Shimura varieties. In fact, this is what is done in [BLp], §§ 1–2, for relative curves, which are unipotent $K(\pi, 1)$s. In this paragraph, we shall study the restriction of the polylogarithm to closed pure sub–Shimura varieties given by Levi sections. The essential ingredient, which we feel makes the polylogarithmic extension interesting, will be the splitting principle for the logarithmic sheaf or, in fact, for any sheaf arising via the canonical construction: its restriction to such subvarieties splits into the direct product of its weight–graded parts.

So the restriction of $\mathcal{P}ol$ can be considered to be a collection of extensions of proper sheaves, i.e., there is no longer any need to talk about projective limits. At least in the examples of parts II and III, these restricted extensions turn out to be very interesting indeed. So one may ask whether polylogarithms can be used more generally to construct non–trivial extensions of sheaves on *pure* Shimura varieties. They are components of a "mixed system" version if this is true for $\mathcal{P}ol$. A similar remark holds for the property of being of geometric origin. Trivial as these observations appear, the reader should note that it seems

reasonable to expect the Hodge– and l–adic versions of $\mathcal{P}ol$ to fit together to form an extension of mixed systems (compare Corollary 2.2 and Remark b) after Theorem 2.3). It should be true that they are realizations of one and the same motivic object. This is not at all clear a priori for the Levi restrictions of $\mathcal{P}ol$. Already in the simplest case of all, the classical polylogarithm (compare part II), this is up to date the most elegant way of getting any information about the l–adic regulators of the elements in the K–theory of cyclotomic fields defined by Beilinson in [B1], § 7.
We consider this to be one of the most important observations in [B2] and certainly one of our main motivations to study and generalize polylogarithms.

The need to look at subvarieties over which the logarithmic sheaf splits forced us to restrict our attention to Shimura varieties, which appear to be particularly well suited for that type of considerations. Note however that results similar to those of this article hold e.g. for any fibre of $[\pi]$.

So let $K = K^W \rtimes L\,,\ K' := k^{-1}(K)$, and the diagram

$$\begin{array}{ccccc} M^{K'}(P',\mathfrak{X}') & \xrightarrow{[k]_{K',K}} & M^K(P,\mathfrak{X}) & \xleftarrow{j_K} & \widetilde{M}^K(P,\mathfrak{X}) \\ & {\scriptstyle [\pi']_{K',L}} \searrow & \downarrow {\scriptstyle [\pi]_{K,L}} & \swarrow {\scriptstyle \widetilde{[\pi]}_{K,L}} & \\ & & M^L(G,\mathcal{H}) & & \end{array}$$

as in the previous paragraph. For $v \in W(\mathbb{Q}) - K^W W'(\mathbb{A}_f)$, let

$$i_v : (G,\mathcal{H}) \longrightarrow (P,\mathfrak{X})$$

be the splitting covering

$$i_v := \mathrm{int}(v) \circ i : G \longrightarrow P,$$

and

$$[i_v] : M^{L_v}(G,\mathcal{H}) \hookrightarrow M^K(P,\mathfrak{X})$$

the embedding on the level of varieties, where $L_v := i_v^{-1}(K) \le L$. Because v is supposed not to belong to $K^W W'(\mathbb{A}_f)$, the set

$$[i_v](M^{L_v}(\mathbb{C})) = P(\mathbb{Q})\backslash\Big(P(\mathbb{Q})(i(\mathcal{H}) \times (i(G)(\mathbb{A}_f)v^{-1}K/K))\Big)$$

is disjoint from

$$[k](M^{K'}(\mathbb{C})) = P(\mathbb{Q})\backslash\Big(P(\mathbb{Q})(k(\mathfrak{X}') \times ((P'(\mathbb{A}_f))K/K))\Big),$$

so $[i_v]$ factors through $\widetilde{M}^K$:

$$\begin{array}{ccccc} M^{L_v}(G,\mathcal{H}) & \xrightarrow{[i_v]} & \widetilde{M}^K(P,\mathfrak{X}) & \xrightarrow{j_K} & M^K(P,\mathfrak{X}) \\ & {\scriptstyle [\cdot 1]_{L_v,L}} \searrow & \downarrow {\scriptstyle [\widetilde{\pi}]_{K,L}} & \swarrow {\scriptstyle [\pi]_{K,L}} & \\ & & M^L(G,\mathcal{H}) & & \end{array}$$

Proposition 6.1: (Splitting principle.)
$[i_v]^*\mathcal{L}og(i,K)$ splits canonically into the direct product of its weight–graded parts. More precisely, for any $n \le 0$, the injection

$$[i_v]^*\mathrm{Gr}_n^W(\mathcal{L}og(i,K)) \hookrightarrow [i_v]^*(\mathcal{L}og(i,K)/W_{n-1}(\mathcal{L}og(i,K)))$$

has a unique left inverse.
Proof: There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_{(l)}}(P) & \xrightarrow{i_v^*} & \mathrm{Rep}_{\mathbb{Q}_{(l)}}(G) \\ {\scriptstyle \mu_{K,-}}\downarrow & & \downarrow{\scriptstyle \mu_{L_v,-}} \\ \mathrm{Sh}^s_{[\pi]}(M^K) & \xrightarrow{[i_v]^*} & \mathrm{Sh}^s(M^{L_v})\,, \end{array}$$

and $\mathrm{Rep}_{\mathbb{Q}_{(l)}}(G)$ is semisimple ([DM], Proposition 2.23). By [W3], Theorem 2.1 and 4.4, the pro-sheaf $\mathcal{L}og(i,K)$ is contained in the image of $\mu_{K,-}$. q.e.d.

Hence $[i_v]^*\mathcal{P}ol(W',i,K)$ is an element of

$$\prod_{m\le -1}\prod_{n\le 0}\mathrm{Ext}^{q_0}_{\mathrm{Sh}(M^{L_v}(G,\mathcal{H}))}(\mu_{L_v,-}\mathrm{Gr}_m^W b(W',i), \mu_{L_v,-}(\mathrm{res}_P^G\mathrm{Gr}_n^W\hat{\mathfrak{U}}(\mathrm{Lie}W))(d)),$$

where as before $q_0 = N + h''^{-1,-1} - 1$. Because of Theorem 3.1 and the semisimplicity of $\mathrm{Rep}_{\mathbb{Q}_{(l)}}(G)$, the extension is zero if $W' \neq 0$.
So assume $W' = 0$. Theorem 5.1.a) has a rather immediate consequence for $[i_v]^*\mathcal{P}ol$, which we don't write down explicitly.
The following is no less immediate, but we find it worth to be noted.

Let $L \leq G(\mathbb{A}_f)$ be neat, open and compact, $K_1^W \leq K_2^W \leq W(\mathbb{A}_f)$ two open compact subgroups stable under conjugation by $i(L)$, $v \in W(\mathbb{Q}) - K_2^W$, $K_j := K_j^W \rtimes L$, $j = 1, 2$, and $L_{2,v} := i_v^{-1}(K_2) \leq L$. We have a diagram

$$
\begin{array}{ccccccc}
M^L(G,\mathcal{H}) & \xrightarrow{[i]} & M^{K_1}(P,\mathfrak{X}) & \xleftarrow{j_{K_1}} & \widetilde{M}^{K_1}(P,\mathfrak{X}) & \xleftarrow{\bigcup_l [i_{v_l}]} & \bigcup_{l=1}^{r} M^{L_{1,v_l}}(G,\mathcal{H}) \\
\| & & \downarrow{\scriptstyle [\cdot 1]} & & & & \downarrow{\scriptstyle \bigcup_l [\cdot 1]} \\
M^L(G,\mathcal{H}) & \xrightarrow{[i]} & M^{K_2}(P,\mathfrak{X}) & \xleftarrow{j_{K_2}} & \widetilde{M}^{K_2}(P,\mathfrak{X}) & \xleftarrow{[i_v]} & M^{L_{2,v}}(G,\mathcal{H}) \\
& \searrow{\scriptstyle =} & \downarrow{\scriptstyle [\pi]} & & & \swarrow{\scriptstyle [\cdot 1]} & \\
& & M^L(G,\mathcal{H}) & & & &
\end{array}
$$

Here, $v_1, \ldots, v_r \in W(\mathbb{Q})$ are chosen such that

$$K_2 v \cdot i(G)(\mathbb{A}_f) = \bigcup_{l=1}^{r} K_1 v_l \cdot i(G)(\mathbb{A}_f)$$

is a disjoint union. This is possible: since W is unipotent, we can find elements $v_1, \ldots, v_{r'} \in W(\mathbb{Q})$ constituting a set of representatives of $K_1 \backslash K_2 v = K_1^W \backslash K_2^W v$. Some of them may define the same class $K_1 v_l \cdot i(G)(\mathbb{A}_f)$.
Define L_{1,v_l} to be $i_{v_l}^{-1}(K_1) \leq L_{2,v}$.
We claim that the upper right part of the diagram is cartesian. This can be checked on the level of $\mathbb{C}$–valued points.
We need to show the equality

$$
\begin{aligned}
& P(\mathbb{Q}) \backslash \Big(P(\mathbb{Q})(i(\mathcal{H}) \times (i(G)(\mathbb{A}_f) v^{-1} K_2 / K_1)) \Big) \\
= & \bigcup_{l=1}^{r} P(\mathbb{Q}) \backslash \Big(P(\mathbb{Q})(i(\mathcal{H}) \times (i(G)(\mathbb{A}_f) v_l^{-1} K_1 / K_1)) \Big) ,
\end{aligned}
$$

the only non–trivial point being that the union is disjoint. This follows from the next claim:
let $x_1, x_2 \in i(\mathcal{H}), p \in P(\mathbb{Q})$ such that $x_2 = p x_1$. Then $p \in i(G)(\mathbb{Q})$:
since $P(\mathbb{Q}) = W(\mathbb{Q}) \rtimes G(\mathbb{Q})$, we may assume $p \in W(\mathbb{Q})$. We have to show that $p = 1$. But this follows from the bijection between $W(\mathbb{Q})$ and the set of Levi decompositions of P defined over $\mathbb{Q}$, which we recalled in [W3], § 1.

Let N be the norm belonging to the finite étale map

$$\bigcup_l [\cdot 1] : \bigcup_{l=1}^r M^{L_{1,v_l}}(G,\mathcal{H}) \longrightarrow M^{L_{2,v}}(G,\mathcal{H}).$$

Theorem 6.2: (Norm compatibility.)
$(\bigcup_l [i_{v_l}])^* \mathcal{P}ol(0,i,K_1)$ is mapped to $[i_v]^* \mathcal{P}ol(0,i,K_2)$ under N. Here, the identifications of

$$[\cdot 1]^* \mu_{L_{2,v,-}}(\mathrm{Gr}_m^W b(W',i)) \quad \text{and} \quad \mu_{L_{1,v_l,-}}(\mathrm{Gr}_m^W b(W',i))$$

and of

$$[\cdot 1]^* \mu_{L_{2,v,-}}(\mathrm{res}_P^G(\mathrm{Gr}_n^W \hat{\mathfrak{U}}(\mathrm{Lie} W))) \quad \text{and} \quad \mu_{L_{1,v_l,-}}(\mathrm{res}_P^G(\mathrm{Gr}_n^W \hat{\mathfrak{U}}(\mathrm{Lie} W)))$$

are the natural ones given by the canonical construction.
Proof: Theorem 5.2. q.e.d.

Remark: If we have the equality

$$G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot L,$$

then all pure sub–Shimura varieties of $M^K(P,\mathfrak{X})$ associated to $(G,\mathcal{H})$ are of the shape $[i_v](M^{L_v}(G,\mathcal{H}))$. Else, we have to consider Levi sections of the more general shape

$$[i_v] \circ [\cdot g_f],$$

for $g_f \in G(\mathbb{A}_f)$. Of course, the results of this paragraph continue to hold for these more general morphisms.

Index of Notations

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