Polylogarithmic Extensions on Mixed Shimura varieties.

Part II:

The classical polylogarithm

Jörg Wildeshaus
Mathematisches Institut
der Universität Münster
Einsteinstr. 62
D–48149 Münster*

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*current address: Dept. of Math. Sciences, South Road, GB–Durham DH1 3LE
e–mail: wildesh@math.uni-muenster.de

Introduction

In this work, we carry out explicitly the program developed in [W4] for the most elementary example, i.e., the classical polylogarithm. As shown in [W4], Theorem 4.3, the large polylogarithmic extension *Pol* is determined by the small extension *pol*, and we attempt to describe the latter in its Hodge and l–adic incarnations.

None of the main results is new:

the description of the extension of local systems underlying *pol* (Theorem 2.2) already appears in [B2], 2.1, while the Hodge version (Theorem 3.5) was described in [B2], 3.1 and [BD2], § 1. The l–adic polylogarithm (§ 4) was constructed, at least stalkwise, in [B2], 3.3. The splitting principle ([W4], Proposition 6.1) for the classical polylogarithm as well as values at Levi sections, i.e., spectra of cyclotomic fields (Theorems 3.11 and 4.5) were discussed in [B2], § 4.

Still, we feel that it is justified to restate these results and reprove them in the context of Shimura varieties, not least because we are thus able to show how the general principles of [W4] work "in practise".

This article is a revised version of § 8 of my doctoral thesis ([W1]). I thank C. Deninger for suggesting its topic and providing me with support in every respect. I am obliged to F. Oort and J. Stienstra for their invitation to Utrecht in March 1992. In the course of my visit there I began to understand the l–adic version of the classical polylogarithm.

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§ 1 The Shimura data $(P_0, \mathfrak{X}_0)$

Throughout this article, the notation is as follows:
$(P_0, \mathfrak{X}_0)$ is as in [P], 2.24. However, we prefer the use of lower triangular matrices:
$G_0 := \mathbb{G}_{m,\mathbb{Q}}$,
$U_0 := \mathbb{G}_{a,\mathbb{Q}}$,
$P_0 := U_0 \rtimes G_0$ where we use the standard action of G_0 on U_0.
We think of P_0 as the subgroup of $GL_{2,\mathbb{Q}}$ of matrices of the shape

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} .$$

There are pure Shimura data $(G_0, \mathcal{H}_0)$ ([P], 2.8):
$\mathcal{H}_0$ is the set $\{2\pi i, -2\pi i\}$ of isomorphisms

$$\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(1) := 2\pi i \cdot \mathbb{Z} \subset \mathbb{C} ,$$

and $G_0(\mathbb{R})$ acts via the sign.
$h : \mathcal{H}_0 \to \mathrm{Hom}(\mathbb{S}, G_{0,\mathbb{R}})$ is the constant map, whose image is the norm character

$$N : \mathbb{S} \longrightarrow G_{0,\mathbb{R}} ,$$

which on $\mathbb{C}$–valued points is given by $(z_1, z_2) \longmapsto z_1 z_2$.
$(P_0, \mathfrak{X}_0)$ is the unipotent extension ([P], Proposition 2.17) of $(G_0, \mathcal{H}_0)$ by U_0.
Explicitly, $h(\mathfrak{X}_0)$ is the $P_0(\mathbb{R}) \cdot U_0(\mathbb{C})$–conjugation class of morphisms $\mathbb{S}_{\mathbb{C}} \to P_{0,\mathbb{C}}$ of the shape

$$(z_1, z_2) \longmapsto \begin{pmatrix} 1 & 0 \\ (1 - z_1 z_2) z & z_1 z_2 \end{pmatrix}$$

where $z \in \mathbb{C}$, and $\mathfrak{X}_0 = \mathcal{H}_0 \times h(\mathfrak{X}_0)$ with the diagonal action of $P_0(\mathbb{R}) \cdot U_0(\mathbb{C})$.

It is easy to check that $\mathrm{Ad}_{P_0} \circ h_x$ induces on $\mathrm{Lie} P_0$ an MHS, which is of type $\{(0,0), (-1,-1)\}$, for any $x \in \mathfrak{X}_0$.
U_0, the unipotent radical of P_0, is pure of weight -2.

Lemma 1.1: The following diffeomorphism is $P_0(\mathbb{R})\cdot U_0(\mathbb{C})$–equivariant:

$$\begin{aligned} \{+,-\}\times\mathbb{C} &\xrightarrow{\sim} \mathfrak{X}_0, \\ (\epsilon,z) &\longmapsto (\epsilon\cdot 2\pi i, (z_1,z_2)\mapsto \left(\begin{matrix} 1 & 0 \\ (1-z_1z_2)z & z_1z_2 \end{matrix}\right)). \end{aligned}$$

On the left hand side,

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \in P_0(\mathbb{R})\cdot U_0(\mathbb{C})$$

acts by sending the pair (ϵ, z) to

$$(\epsilon\cdot\mathrm{sgn}(a), az+b)\,.$$

Proof: This is a direct calculation. q.e.d.

Next we convince ourselves that the above diffeomorphism is an isomorphism of complex structures. By [P], Proposition 1.7.a), we have to show that the Hodge filtration of some faithful representation $\mathbf{V}$ of P_0 depends holomorphically on the coordinate z.

This follows from

Lemma 1.2: Let $\mathbf{V}$ be the standard representation of $P_0 \leq GL_{2,\mathbb{Q}}$, and let $x = (\epsilon, z) \in \mathfrak{X}_0$.

Then the Hodge structure on $\mathbf{V}$ induced by h_x is given as follows:

$W_{-3}(\mathbf{V}) = 0$,

$W_{-2}(\mathbf{V}) = W_{-1}(\mathbf{V}) = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle_{\mathbb{Q}}$,

$W_0(\mathbf{V}) = \mathbf{V}$,

$F^1(\mathbf{V}_{\mathbb{C}}) = 0$,

$F^0(\mathbf{V}_{\mathbb{C}}) = \langle \begin{pmatrix} 1 \\ z \end{pmatrix} \rangle_{\mathbb{C}}$,

$F^{-1}(\mathbf{V}_{\mathbb{C}}) = \mathbf{V}_{\mathbb{C}}$.

Proof: We decompose $\mathbf{V}_{\mathbb{C}}$ into eigenspaces under the action of $\mathbb{S}_{\mathbb{C}}$ given by h_x. The vector $\begin{pmatrix} 1 \\ z \end{pmatrix}$ is an eigenvector for the trivial action of $\mathbb{S}_{\mathbb{C}}$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector for the character N.

So $V_{\mathbb{C}} = H^{0,0} \oplus H^{-1,-1}$, where

$$H^{0,0} = \langle \begin{pmatrix} 1 \\ z \end{pmatrix} \rangle_{\mathbb{C}} \quad \text{and} \quad H^{-1,-1} = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle_{\mathbb{C}}\,.$$

q.e.d.

We fix the following Levi section of $\pi : P_0 \to G_0$:

$$\begin{aligned} i : G_0 &\longrightarrow P_0, \\ a &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} . \end{aligned}$$

Let $N \in \mathbb{N}$, and define an open compact subgroup

$$K_N := \begin{pmatrix} 1 & 0 \\ N \cdot \hat{\mathbb{Z}} & \hat{\mathbb{Z}}^* \end{pmatrix}$$

of $P_0(\mathbb{A}_f)$.[†]

Although it is not neat we note that we may apply the remark following [W3], Proposition 1.2 and use freely the results of [W3] and [W4]. By approximation, $P_0(\mathbb{A}_f) = P_0(\mathbb{Q}) \cdot K_N$, and it is easy to conclude:

Lemma 1.3:

$$\begin{aligned} (P_0(\mathbb{Q})^+ \cap K_N) \backslash \mathbb{C} &\longrightarrow M^{K_N}(\mathbb{C}) = P_0(\mathbb{Q}) \backslash (\{+,-\} \times \mathbb{C} \times (P_0(\mathbb{A}_f)/K_N)), \\ [z] &\longmapsto [+, z, 1] \end{aligned}$$

is an isomorphism of complex manifolds.

Here, $P_0(\mathbb{Q})^+$ denotes the subgroup of $P_0(\mathbb{Q})$ of matrices of positive determinant, so

$$P_0(\mathbb{Q})^+ \cap K_N = \begin{pmatrix} 1 & 0 \\ N \cdot \mathbb{Z} & 1 \end{pmatrix} .$$

By 1.1, $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in P_0(\mathbb{Q})^+ \cap K_N$ acts on $\mathbb{C}$ via translation by b, so the above result gives an isomorphism

$$\begin{aligned} \mathbb{G}_m(\mathbb{C}) &\xrightarrow{\sim} M^{K_N}(P_0, \mathfrak{X}_0)(\mathbb{C}), \\ t &\longmapsto \left[+, \frac{N}{2\pi i} \log t, 1 \right] . \end{aligned}$$

Note that the pure Shimura variety $M^L(G_0, \mathcal{H}_0)$ is Spec($\mathbb{Q}$), where we let $L := \pi(K_N) = \hat{\mathbb{Z}}^*$: $M^L(G_0, \mathcal{H}_0)(\mathbb{C}) = \mathbb{G}_m(\mathbb{Q}) \backslash (\mathcal{H}_0 \times (\mathbb{G}_m(\mathbb{A}_f)/L))$ consists of one point, and the reflex field ([P], 11.1) is easily seen to be $\mathbb{Q}$.

By [W3], Theorem 1.3, we know that the canonical model for $M^{K_N}(P_0, \mathfrak{X}_0)$ is $\mathbb{G}_{m,\mathbb{Q}}$.

[†] By definition, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non–negative integers.

Proposition 1.4: The isomorphism

$$\begin{aligned} \mathbb{G}_m(\mathbb{C}) &\xrightarrow{\sim} M^{K_N}(P_0,\mathfrak{X}_0)(\mathbb{C}), \\ t &\longmapsto \left[+, \frac{N}{2\pi i}\log t, 1\right] \end{aligned}$$

is the isomorphism occurring in the definition of the canonical model ([P], Definition 11.5).
In particular, it is algebraic, descends to $\mathbb{Q}$ and respects the group structure.
Proof: This is precisely the content of [P], Proposition 11.4. q.e.d.

Note that by [W3], Theorem 4.6.a) and 4.3.a), Conjecture 4.2 of [W3] holds, and hence the image of the l–adic canonical construction functor $\mu_{K_N,l}$ lands in the category of mixed sheaves.

Next we study Levi sections:
the morphism of Shimura data covering

$$i : G_0 \hookrightarrow P_0, a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

sends $\epsilon\cdot 2\pi i \in \mathcal{H}_0$ to

$$(\epsilon\cdot 2\pi i, (z_1,z_2) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z_1 z_2 \end{pmatrix}) \in \mathfrak{X}_0 .$$

For $u \in U_0(\mathbb{Q}) = \mathbb{Q}$, we have the morphism of Shimura data

$$i_u : (G_0, \mathcal{H}_0) \longrightarrow (P_0, \mathfrak{X}_0) .$$

On the level of groups, it is given by

$$i_u(a) = \mathrm{int}(u) \circ i(a) = \begin{pmatrix} 1 & 0 \\ u(1-a) & a \end{pmatrix} .$$

So

$$L_{u,N} := i_u^{-1}(K_N) = \{a \in \hat{\mathbb{Z}}^* \mid u(1-a) \in N\hat{\mathbb{Z}}\},$$

which equals $\ker(\hat{\mathbb{Z}}^* \to (\mathbb{Z}/d\mathbb{Z})^*)$. Here, $u = \frac{b}{f}$ with coprime $b, f \in \mathbb{Z}$, and

$$d := \left|\frac{f\cdot N}{gcd(b,N)}\right| .$$

We have the embedding

$$[i_u] : M^{L_{u,N}}(G_0, \mathcal{H}_0) \longrightarrow M^{K_N}(P_0, \mathfrak{X}_0) .$$

Lemma 1.5:

a) $(\mathbb{Z}/d\mathbb{Z})^* \longrightarrow M^{L_{u,N}}(G_0,\mathcal{H}_0)(\mathbb{C}) = G_0(\mathbb{Q})\backslash(\{+,-\}\times(G_0(\mathbb{A}_f)/L_{u,N}))$,
$[n] \longmapsto [+,\tilde{n}]$, where $\tilde{n}\in\hat{\mathbb{Z}}^*$ is congruent to n modulo d,
is a bijection.

b) There is a commutative diagram

$$\begin{tikzcd}
M^{L_{u,N}}(\mathbb{C}) \arrow[r] & M^{K_N}(\mathbb{C}) \\
(\mathbb{Z}/d\mathbb{Z})^* \arrow[u, "\wr"', "a)"] \arrow[d, "\exp\left(\frac{2\pi i u}{N}\cdot \_\right)\wr"'] & \\
\mu_{d,\mathbb{C}}^{\text{prim.}} \arrow[r] & \mathbb{G}_m(\mathbb{C})\,. \arrow[uu, "\wr"', "1.4"]
\end{tikzcd}$$

Here, the upper horizontal map is the morphism $[i_u](\mathbb{C})$, while the lower horizontal map is the natural inclusion of the d–th primitive roots of unity into $\mathbb{G}_m(\mathbb{C})$.

c) For $d\geq 3$, the isomorphism

$$\mu_{d,\mathbb{C}}^{\text{prim.}} \xrightarrow{\sim} M^{L_{u,N}}(\mathbb{C})$$

in b) is the isomorphism on $\mathbb{C}$–valued points from $\mathrm{Spec}(\mathbb{Q}(\mu_d))$ to $M^{L_{u,N}}$ given by the fact that the former is the canonical model of the latter, preceded by the isomorphism

$$\zeta \longmapsto \zeta^{\frac{gcd(b,N)}{b}}\,.$$

Proof: a) left to the reader.
b) Let $n\in\mathbb{Z}$.
We have to show that the pairs

$$\left((z_1,z_2)\longmapsto\begin{pmatrix}1 & 0\\ u(1-z_1z_2) & z_1z_2\end{pmatrix}\,,\,\begin{pmatrix}1 & 0\\ u(1-\tilde{n}) & \tilde{n}\end{pmatrix}\right)$$

and

$$\left((z_1,z_2)\longmapsto\begin{pmatrix}1 & 0\\ nu(1-z_1z_2) & z_1z_2\end{pmatrix}\,,\,1\right)$$

define the same element of $M^K(\mathbb{C})$.
Let $p := \begin{pmatrix}1 & 0\\ u(n-1) & 1\end{pmatrix}\in P_0(\mathbb{Q})$.

A direct calculation shows that conjugation by p transforms the first component of the first pair into the first component of the second while multiplication by p transforms the second component of the first pair into

$$\begin{pmatrix} 1 & 0 \\ u(n-\tilde{n}) & \tilde{n} \end{pmatrix}$$

which by assumption is an element of K_N.
c) This follows from the description in [P], 11.3 and 11.4. Note that $\frac{u}{N} = \frac{b}{gcd(b,N)} \cdot \frac{1}{d}$. q.e.d.

Set $W' := 0$.
So with the notations of [W4], § 1, $(P', \mathfrak{X}')$ coincides with $(G_0, \mathcal{H}_0)$ and the embedding k coincides with i. Moreover, π' is the identity on $(G_0, \mathcal{H}_0)$ and so is i'.
We have

$$h^{-1,-1} = h''^{-1,-1} = 1\,,\; h^{0,-1} = h''^{0,-1} = 0\,,$$

hence

$$d = d'' = N = N'' = 1\,.$$

By 1.5.b), $[i](M^L(G_0, \mathcal{H}_0))$ is the $\mathbb{Q}$–valued point 1 of $\mathbb{G}_m$, so the open immersion

$$j_{K_N} : \widetilde{M}^{K_N}(P_0, \mathfrak{X}_0) \hookrightarrow M^{K_N}(P_0, \mathfrak{X}_0)$$

is identified with the natural inclusion

$$j : \mathbb{G}_{m,\mathbb{Q}} \backslash \{1\} \hookrightarrow \mathbb{G}_{m,\mathbb{Q}}\,.$$

Note that the left hand side coincides with $\mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$.

The machinery described in [W4], § 1, Corollary 2.2 and § 4 yields a projective system $pol(0, i, K_N)$ of one–extensions of mixed systems of smooth sheaves on $\mathbb{P}^1_{\mathbb{Q}} \backslash \{0, 1, \infty\}$ defined by a universal property ([W4], Theorem 1.5.b)). It is uniquely determined by the underlying extension of unipotent smooth topological sheaves ([W4], Theorem 2.3.a)).
By [W4], Theorem 5.2, the polylogarithms for different $N \in \mathbb{N}$ satisfy a certain norm compatibility with respect to the finite étale maps

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}},\; t \longmapsto t^r\;,\; r \in \mathbb{N}\,.$$

In the Hodge version, this will translate into the classical distribution property for higher logarithms. Over roots of unity, or rather the spectra of cyclotomic fields, the restriction of the polylogarithm yields one–extensions of sheaves of finite rank ([W4], § 6), which are of Tate type.

Observe that the Shimura data $(P_0, \mathfrak{X}_0)$ admit a non–trivial automorphism φ_{-1} of order two, which is trivial on $(G_0, \mathcal{H}_0)$. On the level of groups, it is given by

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} .$$

The groups K_N are invariant under φ_{-1}, and under the identification of 1.4, the morphism $[\varphi_{-1}]$ corresponds to the map $t \mapsto t^{-1}$.

The isomorphism $\varphi_{-1}^* \hat{\mathfrak{U}}(\mathrm{Lie}\, U_0) \xrightarrow{\sim} \hat{\mathfrak{U}}(\mathrm{Lie}\, U_0)$ on the level of completed enveloping algebras given by multiplication by -1 on $\mathrm{Lie}\, U_0$ identifies $[\varphi_{-1}]^* \mathcal{L}og$ and $\mathcal{L}og$ as well as $[\varphi_{-1}]^* pol$ and $-pol$.

More generally, if N is any nonzero integer, we can define an automorphism φ_N given by

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ \frac{1}{N} b & a \end{pmatrix} .$$

There is a canonical isomorphism between $[\varphi_N]^* pol(0, i, K_1)$ and $N \cdot pol(0, i, K_N)$.

§ 2 The topological extension underlying pol

Fix $N \in \mathbb{N}$, and let

$$\begin{aligned} q^{\frac{1}{N}} : \mathbb{C} &\longrightarrow M^{K_N}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}), \\ z &\longmapsto \exp\left(\frac{2\pi i}{N} z\right) \end{aligned}$$

be the universal covering in the coordinates given by 1.1, 1.3 and 1.4. If we set

$$e := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Lie}\, U_0 \, ,$$

then

$$\gamma := \exp(Ne) = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in U_0(\mathbb{Q})$$

is the generator of $\pi_1(M^{K_N}(\mathbb{C}))$ corresponding to a positively oriented loop around zero. So the isomorphism

$$H_1(M^{K_N}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} \operatorname{Lie} U_0$$

induced by the natural inclusion of the fundamental group as an arithmetic subgroup in $U_0(\mathbb{Q})$ sends the generator γ to Ne.

We may identify $\hat{\mathfrak{U}}(\operatorname{Lie} U_0)$ with the ring $\mathbb{Q}[[e]]$ of power series in the variable e. Recall the pro–algebraic action of $P_0 = U_0 \rtimes G_0$ on $\hat{\mathfrak{U}}(\operatorname{Lie} U_0)$: G_0 acts by conjugation, and U_0 acts by multiplication ([W4], § 1).

Lemma 2.1: Under this action,

$$\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \in P_0(\mathbb{Q})$$

maps $e^k \in \mathbb{Q}[[e]]$ to the power series

$$a^k e^k \exp(be) = a^k \left(e^k + b e^{k+1} + \frac{1}{2!} b^2 e^{k+2} + \dots \right) .$$

Proof: $\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ acts by conjugation and respects the multiplicative structure. It maps e to ae.

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \exp(e)$ acts by multiplication by $\exp(e)$. q.e.d.

In particular, our generator γ maps e^k to $e^k \exp(Ne)$.

So if we think of $\mathbb{Q}[[e]]$ as an infinite–dimensional vector space with basis $(e^k \mid k \in \mathbb{N}_0)$, then the pro–local system on $M^{K_N}(\mathbb{C})$ underlying $\mu_{K_N,\infty}(\hat{\mathfrak{U}}(\operatorname{Lie} U_0))$, i.e., the action of γ is given by the pro–matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ N & 1 & 0 & 0 & \dots \\ \frac{1}{2!}N^2 & N & 1 & 0 & \dots \\ \frac{1}{3!}N^3 & \frac{1}{2!}N^2 & N & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

By [W3], Theorem 2.1, we may identify $\mu_{K_N,\infty}(\hat{\mathfrak{U}}(\mathrm{Lie}\, U_0))$ and the logarithmic pro–variation $\mathcal{L}og(i, K_N)$. Let $\mathrm{For}_{\mathbb{Q}}(\mathcal{L}og(i, K_N))$ be the underlying pro–local system.

In the notation of [W4], § 1, we have $b_m(0, i) = 0$ if m is odd or greater than -2. If $m \leq -2$ is even, then we have $b_m(0, i) = \mathbb{Q}_{(l)}(-\frac{1}{2}m)$. So $pol(0, i, K_N)$ is the one–extension in

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\})}(\mathbb{Q}_{(l)}(1), j^*\mathcal{L}og(i, K_N)(1))$$

corresponding to the inclusion $\mathbb{Q}_{(l)}(1) \hookrightarrow \prod_{k \geq 1} \mathbb{Q}_{(l)}(k)$ under the isomorphism in [W4], Theorem 1.5.b).

By [W4], Theorem 2.3.a), $pol(0, i, K_N)$ is uniquely determined by the underlying one–extension of unipotent local systems.
We think of it as a pro–local system sitting in an exact sequence

$$0 \longrightarrow \mathrm{For}_{\mathbb{Q}}(j^*\mathcal{L}og(i, K_N)(1)) \longrightarrow \mathrm{For}_{\mathbb{Q}}(pol) \longrightarrow \mathrm{For}_{\mathbb{Q}}(\mathbb{Q}(1)) \longrightarrow 0\,.$$

$\widetilde{\pi}_1 := \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, \sqrt[N]{\frac{1}{2}})$ is free in the two generators α_0 and α_1, where α_k is a positively oriented circle around k, for $k = 0, 1$.
So under the inclusion

$$\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} \longrightarrow \mathbb{G}_m(\mathbb{C})\,,$$

α_0 is mapped to γ, and α_1 is mapped to 1.
The action of $\widetilde{\pi}_1$ on $\mathrm{For}_{\mathbb{Q}}(j^*\mathcal{L}og(i, K_N)(1))$ is as follows:

$$\alpha_0 : 2\pi i \cdot e^k \longmapsto 2\pi i \cdot e^k \exp(Ne)\,,$$

and α_1 acts trivially.
On $\mathrm{For}_{\mathbb{Q}}(\mathbb{Q}(1))$, $\widetilde{\pi}_1$ acts trivially. Denote the canonical base vector of $\mathrm{For}_{\mathbb{Q}}(\mathbb{Q}(1))$ by $2\pi i$.

The stalk at $\sqrt[N]{\frac{1}{2}}$ of $\mathrm{For}_{\mathbb{Q}}(pol)$ is the vector space

$$E := \langle 2\pi i \rangle_{\mathbb{Q}} \oplus \langle 2\pi i \cdot e^k \mid k \in \mathbb{N}_0 \rangle_{\mathbb{Q}}\,.$$

We define the action of $\widetilde{\pi}_1$ as follows:

$$\begin{array}{rrcl} \alpha_0 : & 2\pi i & \longmapsto & 2\pi i \,, \\ & 2\pi i\cdot e^k & \longmapsto & 2\pi i\cdot e^k \exp(Ne)\,,\ k \in \mathbb{N}_0\,, \\ \alpha_1 : & 2\pi i & \longmapsto & 2\pi i + 2\pi i\cdot e\,, \\ & 2\pi i\cdot e^k & \longmapsto & 2\pi i\cdot e^k\,,\ k \in \mathbb{N}_0\,. \end{array}$$

Theorem 2.2: The above defines the pro–local system $\mathrm{For}_{\mathbb{Q}}(pol)$ underlying pol.

Proof: Call the above object $\mathbf{E}$. By definition, it is part of an exact sequence

$$0 \longrightarrow \mathrm{For}_{\mathbb{Q}}(j^*\mathcal{L}og(i, K_N)(1)) \rightarrow \mathbf{E} \longrightarrow \mathrm{For}_{\mathbb{Q}}(\mathbb{Q}(1)) \longrightarrow 0\,. \qquad (**)$$

As in [W4], § 1, we have the diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{C}) \arrow[r, hook, "{[i]}"] \arrow[dr, "\mathrm{id}"'] & \mathbb{G}_{m,\mathbb{C}} \arrow[d, "{[\pi]}"] & \mathbb{P}^1_{\mathbb{C}} \backslash \{0,1,\infty\} \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{[\pi]}"] \\ & \mathrm{Spec}(\mathbb{C}) & \end{tikzcd}$$

$[i]$ being the inclusion of 1, and an exact triangle

$$\begin{tikzcd} {[i]_*[i]^*\mathbf{V}(-1)[-2]} \arrow[rr] & & \mathbf{V} \arrow[dl] \\ & j_*j^*\mathbf{V} \arrow[ul, "\text{shift by } [1]"] & \end{tikzcd} \qquad (*)$$

for mixed, but also for topological smooth sheaves $\mathbf{V}$.

We have to look at the exact triangle $[\pi]_*(*)$ and write down explicitly the boundary homomorphism. Note that both $\mathbb{G}_{m,\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}} \backslash \{0,1,\infty\}$ are $K(\pi,1)$s.

For $\mathbf{V} \in \mathrm{Mod}_{\mathbb{Q}[\widetilde{\pi}_1]}$, $\widetilde{[\pi]}_* \mathbf{V} = R\Gamma(\widetilde{\pi}_1, \mathbf{V})$ is the complex

$$\begin{array}{rcl} \mathbf{V} & \longrightarrow & \mathbf{V} \oplus \mathbf{V} \\ v & \longmapsto & ((\alpha_0 - 1)v, (\alpha_1 - 1)v)\,. \end{array}$$

This implies that the boundary homomorphism

$$H^0(\widetilde{\pi}_1, \mathrm{For}_{\mathbb{Q}}(\mathbb{Q}(1))) \longrightarrow H^1(\widetilde{\pi}_1, \mathrm{For}_{\mathbb{Q}}(j^*\mathcal{L}og(i, K_N)(1)))$$

coming from the exact sequence $(**)$ maps the class of the cocycle $2\pi i$ to the class of the cocycle $(0, 2\pi i \cdot e)$.

It remains to observe that the boundary homomorphism of the exact triangle $[\pi]_*(*)$

$$H^1(\widetilde{\pi}_1, \mathbf{V}) \longrightarrow \frac{1}{2\pi i}\mathbf{V}\,,$$

for any $\mathbb{Q}[\pi_1]$–module $\mathbf{V}$, maps the class of a cocycle (v_1, v_2) to $\dfrac{1}{2\pi i}\,v_2$: both maps factor over H^1 of a small punctured disc around 1, so we may reduce to trivial coefficients. But then, this is just the explicit description of the residue map.

Putting everything together, we showed that $\mathbf{E}$ corresponds to the homomorphism of vector spaces

$$\mathbb{Q}(1) \longrightarrow \prod_{l \geq 1} \mathbb{Q}(l)$$

mapping $2\pi i$ to e. q.e.d.

§ 3 The Hodge version of pol

We start by giving a description of $\mu_{K_N,\infty}(\hat{\mathfrak{U}}(\operatorname{Lie} U_0)) = \mathcal{L}og(i, K_N)$:
let $\operatorname{For}^{\text{hol.}}_{M^{K_N}(\mathbb{C})}(\mathcal{L}og(i, K_N))$ denote the underlying pro–vector bundle. We want to fix a trivialization of $\operatorname{For}^{\text{hol.}}_{M^{K_N}(\mathbb{C})}(\mathcal{L}og(i, K_N))$.

Lemma 3.1: There is an isomorphism

$$\prod_{k \in \mathbb{N}_0} \mathcal{O}_{M^{K_N}(\mathbb{C})} \xrightarrow{\sim} \operatorname{For}^{\text{hol.}}_{M^{K_N}(\mathbb{C})}(\mathcal{L}og(i, K_N))$$

given by sending e_k to the section invariant under γ

$$z \longmapsto e^k \exp(ze) = e^k + ze^{k+1} + \frac{1}{2!}z^2 e^{k+2} + \ldots$$

Proof: γ acts on $\operatorname{For}_{\mathbb{Q}}(\mathcal{L}og(i, K_N)) \hat{\otimes}_{\mathbb{Q}} \mathcal{O}_{M^{K_N}(\mathbb{C})} = \operatorname{For}^{\text{hol.}}_{M^{K_N}(\mathbb{C})}(\mathcal{L}og(i, K_N))$ by sending a section $e \hat{\otimes} f$ to $\gamma(e) \hat{\otimes} f \circ \gamma^{-1}$. Hence

$$\begin{aligned} & \gamma(z \longmapsto e^k \exp(ze)) \\ = & \ (z \longmapsto e^k \exp(Ne) \exp((z-N)e)) = \\ = & \ (z \longmapsto e^k \exp(ze))\,. \end{aligned}$$

q.e.d.

In terms of the basis $(e_k \,|\, k \in \mathbb{N}_0)$, $e^k = e_k(z)\exp(-ze)$, so we may view

$$(q^{\frac{1}{N}})^{-1}\mathrm{For}_{\mathbb{Q}}(\mathcal{L}og(i, K_N))$$

as the pro–local system over $\mathbb{Q}$ sitting inside $\prod_{k\in\mathbb{N}_0}\mathcal{O}_{\mathbb{C}}$ described by the pro–matrix valued function

$$L_N : z \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -z & 1 & 0 & 0 & \dots \\ \frac{1}{2!}z^2 & -z & 1 & 0 & \dots \\ -\frac{1}{3!}z^3 & \frac{1}{2!}z^2 & -z & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

This means that for any $k \in \mathbb{N}_0$, the multivalued section $L_N(e_k)$ is rational and flat.

We need to know the weight and Hodge filtrations of $\mathcal{L}og(i, K_N)$. Denote by $\mathcal{H}^{p,q}(\mathcal{L}og(i, K_N))$ the C^∞–subbundle, on whose fibre at $(\epsilon, z) \in \mathfrak{X}_0$ the $\mathbb{C}$–valued point (z_1, z_2) of the Deligne torus acts via multiplication by ${z_1}^{-p}{z_2}^{-q}$.

As remarked in [W3], § 1, this yields the unique decomposition of the C^∞–bundle underlying $\mathcal{L}og$, which satisfies

$$\overline{\mathcal{H}^{q,p}} = \mathcal{H}^{p,q} \mathrm{mod} \bigoplus_{\substack{p'<p \\ q'<q}} \mathcal{H}^{p',q'} .$$

Proposition 3.2:

i) For $l \in \mathbb{Z}$,
$W_{2l+1}(\mathcal{L}og(i, K_N)) = W_{2l}(\mathcal{L}og(i, K_N)) = \langle e^k \,|\, k \geq -l\rangle_{\mathbb{Q}}$, and
$W_{2l}(\mathcal{L}og(i, K_N))\hat{\otimes}_{\mathbb{Q}}\mathcal{O}_{M^{K_N}(\mathbb{C})} = \langle e_k \,|\, k \geq -l\rangle_{\mathcal{O}_{M^{K_N}(\mathbb{C})}}$.

ii) For $p \in \mathbb{Z}$,
$\mathcal{F}^p(\mathcal{L}og(i, K_N)) = \langle e_k \,|\, k \leq -p\rangle_{\mathcal{O}_{M^{K_N}(\mathbb{C})}}$.

iii) $\mathcal{H}^{p,q}(\mathcal{L}og(i, K_N)) = 0$ for $p \neq q$ or $p > 0$, and
$\mathcal{H}^{p,p}(\mathcal{L}og(i, K_N))$ is of rank one, a global generator being given by e_{-p} if $p \leq 0$.

Proof: We only need to show iii).

At $z \in \mathbb{C}$, the Deligne torus acts by the cocharacter

$$(z_1, z_2) \longmapsto \begin{pmatrix} 1 & 0 \\ (1 - z_1 z_2)z & z_1 z_2 \end{pmatrix} ,$$

which by 2.1 maps e^k to

$$z \longmapsto z_1^k z_2^k e^k \exp((1 - z_1 z_2) ze)\,.$$

So $e_{-p} : z \longmapsto e^{-p} \exp(ze)$ is mapped to

$$z \longmapsto z_1^{-p} z_2^{-p} e^{-p} \exp((1 - z_1 z_2) ze) \exp(z_1 z_2 ze) = z_1^{-p} z_2^{-p} e_{-p}\,.$$

q.e.d.

Observe in particular that $\mathcal{H}^{p,p}(\mathcal{L}og(i, K_N))$ has a holomorphic structure. While this is always the case for variations of Tate–Hodge structure, we can't expect it to be true in general. See e.g. part III.

3.2 tells us that the pro–variation $\mathcal{L}og(i, K_N)$ is in fact fully described by the pro–matrix valued function L_N. Namely, the weight and Hodge filtrations are easily expressible in terms of the canonical basis $(e_k \mid k \in \mathbb{N}_0)$ of $\prod_{k \in \mathbb{N}_0} \mathcal{O}_{M^{K_N}(\mathbb{C})}$, and the rational structure is given by L_N, meaning that for any $k \in \mathbb{N}_0$, $L_N(e_k)$ is a multivalued section of the pro–local system $\mathrm{For}_{\mathbb{Q}}(\mathcal{L}og(i, K_N))$.

In terms of the parameter t of $M^{K_N}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C})$, the matrix L_N acquires the following multivalued form:

$$L_N : t \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot s \\ -\frac{N}{2\pi i}\log(t) & 1 & 0 & 0 & \cdot s \\ \frac{1}{2!}\left(-\frac{N}{2\pi i}\log(t)\right)^2 & -\frac{N}{2\pi i}\log(t) & 1 & 0 & \cdot s \\ \frac{1}{3!}\left(-\frac{N}{2\pi i}\log(t)\right)^3 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log(t)\right)^2 & -\frac{N}{2\pi i}\log(t) & 1 & \cdot s \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We aim at a similar description of $pol(0, i, K_N)$.

We have the basis of global sections $(2\pi i{\cdot}e_k \mid k \in \mathbb{N}_0)$ of $\mathrm{For}^{\mathrm{hol.}}_{M^{K_N}(\mathbb{C})}(\mathcal{L}og(i, K_N)(1))$.

Lemma 3.3: This basis can be completed to give an isomorphism

$$\mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}} \times \prod_{k \in \mathbb{N}_0} \mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}} \xrightarrow{\sim} \mathrm{For}^{\mathrm{hol.}}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}(pol)$$

by adding the global section

$$f := 2\pi i + \sum_{k=1}^{\infty} N^{k-1} \Lambda_k \cdot 2\pi i \cdot e^k\,.$$

Here, the multivalued functions Λ_k are defined as

$$\Lambda_k := \frac{1}{(-2\pi i)^k} \sum_{n=1}^{k} \frac{(-\log)^{k-n}}{(k-n)!} \mathrm{Li}_n \, .$$

Remark: In [BL], 4.8, the functions Λ_k are called *Debye polylogarithms.*

Before giving the proof of 3.3, we recall the definition of the multivalued functions Li_k, the well-known *higher logarithms* :

$$\mathrm{Li}_1(t) := -\log(1-t) = \int_0^t \frac{1}{1-s}\, ds \, ,$$

$$\mathrm{Li}_{k+1}(t) := \int_0^t \frac{\mathrm{Li}_k(s)}{s}\, ds \, , \text{ for } k \in \mathbb{N} \text{ and } t \in \mathbb{P}^1(\mathbb{C}) \backslash \{0,1,\infty\} \, .$$

We think of these as functions on the universal cover $\widetilde{\mathfrak{X}}^0$ of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$: let

$$p : \widetilde{\mathfrak{X}}^0 \twoheadrightarrow \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$$

be the covering morphism. It factors over

$$q^{\frac{1}{N}} \,|_{\mathbb{C}\backslash N\mathbb{Z}} : \mathbb{C}\backslash N\mathbb{Z} \twoheadrightarrow \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\} \, .$$

Let $\widetilde{x} \in \widetilde{\mathfrak{X}}^0$ be a base point mapping to $\frac{1}{2\pi i}\log\left(\frac{1}{2}\right)$ under

$$\widetilde{\mathfrak{X}}^0 \twoheadrightarrow \mathbb{C}\backslash N\mathbb{Z} \, .$$

Here, we take $\log\left(\frac{1}{2}\right) \in \mathbb{R}$. So $\widetilde{x}$ maps to $\sqrt[N]{\frac{1}{2}}$ under p. By convention,

$$\log : \widetilde{\mathfrak{X}}^0 \longrightarrow \mathbb{C} \quad \text{and} \quad \mathrm{Li}_k : \widetilde{\mathfrak{X}}^0 \longrightarrow \mathbb{C} \, , \ k \in \mathbb{N}$$

are those branches of the respective functions taking the values

$$\log \circ p \quad \text{and} \quad \mathrm{Li}_k(p) \, , \ k \in \mathbb{N}$$

near $\widetilde{x}$.

Here, we let Li_k be given by the usual power series expression.

$\widetilde{\pi}_1$ acts on $\mathfrak{X}^0$ from the left. Its induced action on the multivalued functions $\log, \mathrm{Li}_k$ and Λ_k is given by the following

Proposition 3.4:

a) $\alpha_0(\log) = \log - 2\pi i,\ \alpha_1(\log) = \log.$

b) $\alpha_0(\mathrm{Li}_k) = \mathrm{Li}_k, \qquad \alpha_1(\mathrm{Li}_k) = \mathrm{Li}_k + 2\pi i \cdot \dfrac{\log^{k-1}}{(k-1)!}$ for all $k \in \mathbb{N}$.

c) $\alpha_0(\Lambda_k) = \displaystyle\sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \Lambda_{k-j}$ for all $k \in \mathbb{N}$,
$\alpha_1(\Lambda_1) = \Lambda_1 - 1,$
$\alpha_1(\Lambda_k) = \Lambda_k$ for all $k \geq 2$.

Proof: Recall that $\gamma \in \widetilde{\pi}_1$ acts on $\Gamma(\widetilde{\mathfrak{X}}^0, \mathcal{O}_{\widetilde{\mathfrak{X}}^0})$ by $g \longmapsto g \circ \gamma^{-1}$.
a) is well known, as is b) for $k = 1$.
We use induction on k:
if $\alpha_0(\mathrm{Li}_k) = \mathrm{Li}_k$, then we have

$$\mathrm{Li}_{k+1}(t) - \alpha_0(\mathrm{Li}_{k+1})(t) = \int_{\frac{1}{2}}^{t} \frac{\mathrm{Li}_k(s)}{s}\, ds - \int_{\frac{1}{2}}^{t} \int_{\alpha_0^{-1}} \frac{\mathrm{Li}_k(s)}{s}\, ds\,.$$

This difference is zero since $s \mapsto \dfrac{\mathrm{Li}_k(s)}{s}$ is defined in $|s| < 1$, hence has trivial monodromy around zero.
If the claim for α_1 is proven for k, it is true for $k+1$ up to a constant as one sees by differentiation.
But the constant is zero: form the limit $t \to 1$!
c) follows from a) and b) by a direct calculation:

$$\begin{aligned}
\alpha_0(\Lambda_k) &= \frac{1}{(-2\pi i)^k} \sum_{i=1}^{k} \frac{(-1)^{k-i} \cdot (\log - 2\pi i)^{k-i}}{(k-i)!} \mathrm{Li}_i \\
&= \frac{1}{(-2\pi i)^k} \sum_{i=1}^{k} \sum_{j=0}^{k-i} \frac{1}{j!(k-i-j)!} (-\log)^{k-i-j} (2\pi i)^j \mathrm{Li}_i \\
&= \sum_{j=0}^{k-1} \frac{1}{j!} (-1)^j \frac{1}{(-2\pi i)^{k-j}} \sum_{i=1}^{k-j} \frac{(-\log)^{k-i-j}}{(k-i-j)!} \mathrm{Li}_i \\
&= \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \Lambda_{k-j}\,, \\
\alpha_1(\Lambda_k) &= \frac{1}{(-2\pi i)^k} \sum_{i=1}^{k} \frac{(-\log)^{k-i}}{(k-i)!} \left(\mathrm{Li}_i + 2\pi i \cdot \frac{\log^{i-1}}{(i-1)!} \right) \\
&= \Lambda_k - \frac{1}{(-2\pi i)^{k-1}} \cdot \log^{k-1} \sum_{i=1}^{k} \frac{(-1)^{k-i}}{(k-i)!(i-1)!}\,.
\end{aligned}$$

The sum is nonzero only for $k = 1$, in which case it takes the value 1. q.e.d.

Proof of Lemma 3.3:

$$\begin{aligned}
\alpha_0(f) &= \alpha_0(2\pi i) + \sum_{k=1}^{\infty} N^{k-1}\alpha_0(\Lambda_k)\cdot\alpha_0(2\pi i\cdot e^k) \\
&= 2\pi i + \sum_{k=1}^{\infty} N^{k-1} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!}\Lambda_{k-j}\cdot 2\pi i\cdot \sum_{i=0}^{\infty} \frac{N^i}{i!} e^{k+1} \\
&= 2\pi i + \sum_{l=1}^{\infty}\sum_{k=1}^{l} N^{k-1}\frac{(-1)^j}{j!}\Lambda_{k-j}\cdot\frac{N^{l-k}}{(l-k)!}\cdot 2\pi i\cdot e^l \\
&= 2\pi i + \sum_{l=1}^{\infty} N^{l-1}\sum_{k=1}^{l}\frac{1}{(l-k)!}\sum_{i=1}^{k}\frac{(-1)^{k-i}}{(k-i)!}\Lambda_i\cdot 2\pi i\cdot e^l \\
&= 2\pi i + \sum_{l=1}^{\infty} N^{l-1}\sum_{i=1}^{l}\Lambda_i \underbrace{\sum_{j=0}^{l-i}\frac{(-1)^{l-j-i}}{(l-j-i)!j!}}_{\neq 0 \text{ only for } i=l} 2\pi i\cdot e^l \\
&= 2\pi i + \sum_{l=1}^{\infty} N^{l-1}\Lambda_l\cdot 2\pi i\cdot e^l = f\,. \\
\alpha_1(f) &= \alpha_1(2\pi i) + \sum_{k=1}^{\infty} N^{k-1}\alpha_1(\Lambda_k)\cdot\alpha_1(2\pi i\cdot e^k) \\
&= 2\pi i + 2\pi i\cdot e + \sum_{k=1}^{\infty}\Lambda_k\cdot 2\pi i\cdot e^k - 2\pi i\cdot e = f\,.
\end{aligned}$$

q.e.d.

So $\mathrm{For}_{\mathbb{Q}}(pol(0,i,K_N))$ is the pro–local system over $\mathbb{Q}$ sitting inside the holomorphic pro–bundle $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}} \times \prod_{k\in\mathbb{N}_0}\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}$ given by the inverse P_N of the pro–matrix valued function

$$P_N^{-1} := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdot s \\
0 & 1 & 0 & 0 & 0 & \cdot s \\
\Lambda_1 & \frac{N}{2\pi i}\log & 1 & 0 & 0 & \cdot s \\
N\Lambda_2 & \frac{1}{2!}\left(\frac{N}{2\pi i}\log\right)^2 & \frac{N}{2\pi i}\log & 1 & 0 & \cdot s \\
N^2\Lambda_3 & \frac{1}{3!}\left(\frac{N}{2\pi i}\log\right)^3 & \frac{1}{2!}\left(\frac{N}{2\pi i}\log\right)^2 & \frac{N}{2\pi i}\log & 1 & \cdot s \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}.$$

Hence P_N is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot s \\ 0 & 1 & 0 & 0 & 0 & \cdot s \\ \frac{1}{2\pi i}\mathrm{Li}_1 & -\frac{N}{2\pi i}\log & 1 & 0 & 0 & \cdot s \\ -\frac{N}{(2\pi i)^2}\mathrm{Li}_2 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log\right)^2 & -\frac{N}{2\pi i}\log & 1 & 0 & \cdot s \\ \frac{N^2}{(2\pi i)^3}\mathrm{Li}_3 & \frac{1}{3!}\left(-\frac{N}{2\pi i}\log\right)^3 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log\right)^2 & -\frac{N}{2\pi i}\log & 1 & \cdot s \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

Again we note what this matrix notation means:
if the canonical basis of $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}} \times \prod_{k\in\mathbb{N}_0} \mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}$ is $(f, 2\pi i\cdot e_k \,|\, k \in \mathbb{N}_0)$, then $P_N(f)$ and all $P_N(2\pi i\cdot e_k)$ are multivalued sections of the pro–local system $\mathrm{For}_{\mathbb{Q}}(pol)$.
We note explicitly the formula for $P_N(f)$:

$$P_N(f) = f + \sum_{k=1}^{\infty}(-N)^{k-1}\frac{1}{(2\pi i)^k}\mathrm{Li}_k\cdot 2\pi i\cdot e_k \,.$$

We want to extend the mixed structure $\mathcal{L}og(i, K_N)(1)$ described in 3.2 to the whole of $\mathrm{For}_{\mathbb{Q}}(pol)$.
By 3.2, for $p \leq -2$, the vector bundle $\mathcal{F}^p/\mathcal{F}^{p+1}(pol)$ is of rank one and generated by the image of $2\pi i\cdot e_{-p-1}$.
$\mathcal{F}^{-1}(pol)$ must be of rank two, and

$$2\pi i\cdot e_0 \in \Gamma(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}\,,\ \mathcal{F}^{-1}(pol))\,.$$

Theorem 3.5: If we let f be a section of $\mathcal{F}^{-1}$, then these data define an admissible pro–variation of Hodge structure on $\mathbb{P}^1_{\mathbb{C}}\setminus\{0,1,\infty\}$.
It coincides with $pol(0, i, K_N)$.
Proof: We have defined the underlying local system, the Hodge filtration by holomorphic sub–vector bundles and the weight filtration by rational sub–local systems. The quotient by W_{-3} is easily seen to be $\mathbb{Q}(1)\oplus\mathbb{Q}(1)$. Since the weight and Hodge filtrations induce variations of Hodge structure on W_{-3} and on the quotient by W_{-3}, [GS], Observation 1.16 tells us that we have indeed defined a graded–polarizable variation of Hodge structure as soon as we have checked Griffiths–transversality.
To achieve this, we must show that

$$\nabla(f) \in \mathcal{F}^{-2} \otimes_{\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}} \Omega^1_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}\,.$$

In terms of the matrix P_N, the connection ∇ is given by

$$P_N \circ d(P_N^{-1})\,.$$

Its first column is $P_N(d\Lambda)$, where Λ is the vector

$$\begin{pmatrix} 1 \\ 0 \\ \Lambda_1 \\ N\Lambda_2 \\ N^2\Lambda_3 \\ \vdots \end{pmatrix}.$$

Letting $\mathrm{Li}_0(t) := \dfrac{t}{1-t}$, we have

$$\begin{aligned} d\Lambda_k(t) &= \frac{1}{(-2\pi i)^k}\left(\sum_{i=1}^{k-1}\left(-\frac{(-\log)^{k-i-1}(t)}{(k-i-1)!\,t}\mathrm{Li}_i(t)\right.\right. \\ &\qquad \left.\left.+\frac{(-\log)^{k-i}(t)}{(k-i)!\,t}\mathrm{Li}_{i-1}(t)\right) + \frac{1}{t}\mathrm{Li}_{k-1}(t)\right) dt \\ &= \frac{1}{(-2\pi i)^k}\cdot\frac{1}{(k-1)!}\cdot(-\log(t))^{k-1}\frac{1}{1-t}dt \\ &= -\frac{1}{(k-1)!}\cdot\frac{1}{2\pi i}\cdot\left(\frac{1}{2\pi i}\log(t)\right)^{k-1}\frac{1}{1-t}dt\,. \end{aligned}$$

Hence

$$d\Lambda(t) = -\frac{1}{2\pi i}\cdot\frac{1}{1-t}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{N}{2\pi i}\log(t) \\ \frac{1}{2!}\left(\frac{N}{2\pi i}\log(t)\right)^2 \\ \frac{1}{3!}\left(\frac{N}{2\pi i}\log(t)\right)^3 \\ \vdots \end{pmatrix} dt\,,$$

and

$$\nabla(f)(t) = P_N(d\Lambda)(t) = -\frac{1}{2\pi i}\cdot\frac{1}{1-t}\cdot 2\pi i\cdot e_1(t)\,dt$$

so $\nabla(f)$ is in fact an element of $\mathcal{F}^{-2}\otimes_{\mathcal{O}}\Omega^1$.
So we have a graded–polarizable variation of Hodge structure.

A calculation similar to the above yields

$$\nabla(2\pi i\cdot e_k)(t) = \frac{N}{2\pi i}\cdot\frac{1}{t}\cdot 2\pi i\cdot e_{k+1}(t)\,dt \quad \text{for all } k\in\mathbb{N}_0\,.$$

It follows that our choice of the definition of $\mathcal{F}^{-1}$ is in any case limited: since $\mathcal{F}^{-1}$ maps isomorphically to $(\mathbb{Q}(1) \otimes \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}$, it must be generated by two global sections, namely $2\pi i{\cdot}e_0$ and a section $\widetilde{f}$ congruent to f modulo $\langle 2\pi i{\cdot}e_k \,|\, k \in \mathbb{N}\rangle_{\mathcal{O}_{\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}}}$.

The above formula implies that if we want Griffiths–transversality to be satisfied, then $\widetilde{f}$ must necessarily be of the form

$$\widetilde{f} = f + \sum_{k=1}^{\infty} g_k{\cdot}2\pi i{\cdot}e_k\,,$$

where the g_k are single valued holomorphic functions on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ satisfying

$$\frac{d}{dt}g_{k+1}(t) = -\frac{N}{2\pi i}{\cdot}\frac{g_k(t)}{t} \quad \text{for all } k \in \mathbb{N}\,.$$

In particular, if g_1 is constant, then all g_k are zero since there is no single valued function on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$, whose derivative is $t \mapsto \dfrac{1}{t}$.

To prove admissibility, we claim that is suffices to show that the quotient by W_{-5} of the variation defined by $\widetilde{f}$ is admissible if and only if $g_1 = 0$. Namely, our choice of $\mathcal{F}^{-1}$ would then be the only one leading to a variation of Hodge structure, which is admissible modulo W_{-5}. But we *know* that *pol* exits and defines an admissible extension of our data.

The quotient by W_{-5} is described by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2\pi i}\mathrm{Li}_1 + g_1 & -\frac{N}{2\pi i}\log & 1 \end{pmatrix}$$

and defines an extension of $\mathbb{Q}(1) \oplus \mathbb{Q}(1)$ by $\mathbb{Q}(2)$, which is the push-out via the summation morphism of two extensions of $\mathbb{Q}(1)$ by $\mathbb{Q}(2)$, the more interesting one being

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2\pi i}\mathrm{Li}_1 + g_1 & 1 \end{pmatrix}.$$

By Theorem 3.7 below, this matrix defines an admissible variation if and only if the non–vanishing function

$$\exp\left(2\pi i{\cdot}\left(\frac{1}{2\pi i}\mathrm{Li}_1 + g_1\right)\right)$$

is meromorphic on the whole of $\mathbb{P}^1(\mathbb{C})$.

This is the case if and only if $\exp(2\pi i g_1)$ is meromorphic, which implies that g_1 is constant. q.e.d.

In order to conclude the proof of 3.5, we need the following two results. Denote by $MH_{\mathbb{Q}}$ the category of graded–polarizable mixed $\mathbb{Q}$–Hodge structures.

Theorem 3.6: For any $k \in \mathbb{N}$, there is a canonical isomorphism

$$\mathbb{C}/(2\pi i)^k\mathbb{Q} \xrightarrow{\sim} \mathrm{Ext}^1_{MH_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(k)) \, .$$

It sends the class of $z \in \mathbb{C}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^k} \cdot z & 1 \end{pmatrix} .$$

As before, this means that if e_0 and e_k are the usual base vectors of $\mathbb{Q} \subset \mathbb{C}$ and $(2\pi i)^k\mathbb{Q} \subset \mathbb{C}$, then the Hodge structure corresponding to z is specified by

$$F^0 := \langle e_0 \rangle_{\mathbb{C}} \ , \quad W_{-2k} \otimes_{\mathbb{Q}} \mathbb{C} := \langle e_k \rangle_{\mathbb{C}} \, ,$$

and the rational structure is given by

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^k} \cdot z & 1 \end{pmatrix} e_0 = e_0 - \frac{1}{(2\pi i)^k} \cdot z e_k \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^k} \cdot z & 1 \end{pmatrix} e_k = e_k \, .$$

Proof: [J2], Lemma 9.2 and Remark 9.3.a), or [B3], § 1. Our normalization of the isomorphism coincides with that of Jannsen. It differs from Beilinson's by multiplication by -1. q.e.d.

Theorem 3.7: Let $X/\mathbb{C}$ be a smooth proper variety, $U \subset X$ Zariski–open and dense.

a)
$$g \longmapsto \begin{pmatrix} 1 & 0 \\ -\frac{1}{2\pi i} \cdot \log g & 1 \end{pmatrix}$$

defines an isomorphism

$$\Gamma(U(\mathbb{C}), \mathcal{O}^*_{U(\mathbb{C})}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \, ,$$

the Ext–group being the one in the category of graded–polarizable variations of $\mathbb{Q}$–Hodge structure over $U(\mathbb{C})$.

b) Under the isomorphism in a), the extension defines an admissible variation if and only if g is algebraic, i.e., meromorphic on the whole of $X(\mathbb{C})$.

Proof: a) is left to the reader.
b) We check the conditions of [Ka], (1.8).

Let g be a non–vanishing holomorphic function on the punctured disc $\triangle^*$ and consider the variation given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2\pi i}\cdot\log g & 1 \end{pmatrix} .$$

We have the universal covering map

$$\mathfrak{g} : \mathfrak{h} \longrightarrow \triangle^*, z \longmapsto \exp(2\pi i z)$$

from the upper half plane.

$$G : \mathfrak{h} \longrightarrow \mathbb{C}$$

is chosen such that $\exp(G) = g \circ q$.

Hence the logarithm N of the monodromy transformation is

$$N := \begin{pmatrix} 0 & 0 \\ -n_g & 0 \end{pmatrix} ,$$

where $G(z+1) - G(z) = 2\pi i n_g$ for all $z \in \mathfrak{h}$.

Observe that the monodromy is unipotent, hence the condition [Ka], (1.8.2) of quasi–unipotency is fulfilled.

By [SZ], Proposition 2.14, the weight filtration of N relative to W exists, hence [Ka], (1.8.3) also holds unconditionally.

It remains to check [Ka], (1.8.4), i.e., the extendability of the Hodge filtration to the canonical extension.

By definition ([D], proof of Proposition 5.2.b)), if e_1 and e_2 denote the canonical base vectors of $\mathcal{O}^2_{\triangle^*}$, so that

$$e_1 - \frac{\log g}{2\pi i} e_2 \quad \text{and} \quad e_2$$

constitute a basis of multivalued sections of the local system underlying our variation, then a basis of sections of the canonical extension is given by

$$\left(z \longmapsto e_1 - \frac{G(z)}{2\pi i} + n_g z \, , \; e_2 \right) .$$

Hence [Ka], (1.8.4) is equivalent to the existence of the limit

$$\lim_{z \to i\infty} \left(G(z) - 2\pi i n_g z \right) .$$

Writing down the Laurent series of g around zero, it is easy to see that this limit exists if and only if g is meromorphic at zero.

It remains to show that meromorphicity on $X(\mathbb{C})$ of a holomorphic function g on $U(\mathbb{C})$ can be checked via the curve test. By [D], II, Propositions 2.24 and 2.19, we may assume that $X \setminus U$ is a divisor with normal crossings. Our claim then follows from [D], II, Lemme 4.1.1. q.e.d.

Theorem 3.5 tells us that $pol(0, i, K_N)$ is fully described by

$$P_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot s \\ 0 & 1 & 0 & 0 & 0 & \cdot s \\ \frac{1}{2\pi i}\mathrm{Li}_1 & -\frac{N}{2\pi i}\log & 1 & 0 & 0 & \cdot s \\ -\frac{N}{(2\pi i)^2}\mathrm{Li}_2 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log\right)^2 & -\frac{N}{2\pi i}\log & 1 & 0 & \cdot s \\ \frac{N^2}{(2\pi i)^3}\mathrm{Li}_3 & \frac{1}{3!}\left(-\frac{N}{2\pi i}\log\right)^3 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log\right)^2 & -\frac{N}{2\pi i}\log & 1 & \cdot s \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

meaning that the underlying holomorphic vector bundle is trivial, that its weight and Hodge filtrations are given by

$$\begin{aligned} \mathcal{H}^{-1,-1}(pol) &= \langle f, 2\pi i \cdot e_0 \rangle_{\mathcal{O}^{\text{diff.}}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}} , \\ \mathcal{H}^{p,p}(pol) &= \langle 2\pi i \cdot e_{-p-1} \rangle_{\mathcal{O}^{\text{diff.}}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}} \end{aligned}$$

and the formulae

$$\begin{aligned} W_n \hat{\otimes}_{\mathbb{Q}} \mathcal{O}^{\text{diff.}}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}} &= \prod_{p' \le \frac{n}{2}} \mathcal{H}^{p',p'} , \\ \mathcal{F}^p \otimes_{\mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}} \mathcal{O}^{\text{diff.}}_{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}} &= \bigoplus_{p' \ge p} \mathcal{H}^{p',p'} , \end{aligned}$$

and that $P_N(f)$ and all $P_N(2\pi i \cdot e_k)$ are multivalued flat rational sections.
In particular, P_N^{-1} is the period matrix of pol.

Observe that in the category of variations of Tate–Hodge structure, it is always possible to describe one–extensions by a matrix of multivalued holomorphic functions. Namely, by an argument similar to the one having occurred in the proof of 3.5, the underlying vector bundle together with the Hodge filtration and the induced weight filtration can always be trivialized as a bifiltered holomorphic bundle. So we only need to see how the rational structure differs from the one of the trivial extension.
The general situation differs sharply from the one considered here, as we shall see in part III.

We compare our definition of the classical polylogarithm, for $N = 1$, with the ones already existent:
it coincides with the extension $\mathcal{P}$ defined in [BLp], 2.5 and discussed in [BLp], 3.1. In the notation used there,
$B = \operatorname{Spec}(\mathbb{C})$, $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$ and $X = \mathbb{P}^1_{\mathbb{C}} \backslash \{0, \infty\}$.
Because of weight reasons, we have a surjection

$$\operatorname{Ext}^1_{\operatorname{Sh}(\mathbb{P}^1_{\mathbb{C}} \backslash \{0,1,\infty\})}(\mathbb{Q}(1), W_{-4}(j^* \mathcal{L}og(1))) \longrightarrow\!\!\!\!\rightarrow \operatorname{Ext}^1_{\operatorname{Sh}(\mathbb{P}^1_{\mathbb{C}} \backslash \{0,1,\infty\})}(\mathbb{Q}(1), j^* \mathcal{L}og(1))\,,$$

one of the pre–images of *pol* being given by the matrix P'_N obtained by removing the second row and column of P_N.
This extension, twisted by (-1), coincides with the one of [BLpp], § 4. It differs from the ones considered in [BD1], 3.3 and [BD2] by the factor -1. Also, in these two sources, the normalization of the base of global sections of the vector bundle underlying $\mathcal{L}og(i, K_N)$ differs from ours: there, the base $((-1)^{k+1} e_k \mid k \in \mathbb{N}_0)$ is used.

As indicated at the end of § 1, it is an easy matter to write down isomorphisms between polylogarithms of different levels, i.e., for different $N \in \mathbb{N}$. The reason why we chose to use different notations is that we wanted to keep the parametrization given by the canonical construction. It turns out to be convenient when we spell out what norm compatibility ([W4], Theorem 5.2) means in our situation:
let $N, M \in \mathbb{N}$. The morphism

$$[\cdot 1]_{K_{NM}, K_N} : M^{K_{NM}}(P_0, \mathfrak{X}_0) \longrightarrow M^{K_N}(P_0, \mathfrak{X}_0)$$

is given by

$$t \longmapsto t^M : \mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}}\,.$$

We have the extension

$$pol(0, i, K_{NM})\Big|_{\mathbb{G}_{m,\mathbb{C}} \backslash \mu_{M,\mathbb{C}}} \in \operatorname{Ext}^1_{\operatorname{Sh}(\mathbb{G}_{m,\mathbb{C}} \backslash \mu_{M,\mathbb{C}})}(\mathbb{Q}(1), \mathcal{L}og(i, K_{NM})(1)\Big|_{\mathbb{G}_{m,\mathbb{C}} \backslash \mu_{M,\mathbb{C}}})\,.$$

Proposition 3.8: Under the norm map N_{K_{NM}, K_N} (see [W4], Theorem 5.2), this extension is mapped to the one described by the matrix–valued function which sends t to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot s \\ 0 & 1 & 0 & 0 & 0 & \cdot s \\ \frac{1}{2\pi i}\sum\limits_{s^M=t}\mathrm{Li}_1(s) & -\frac{N}{2\pi i}\log(t) & 1 & 0 & 0 & \cdot s \\ -\frac{NM}{(2\pi i)^2}\sum\limits_{s^M=t}\mathrm{Li}_2(s) & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log(t)\right)^2 & -\frac{N}{2\pi i}\log(t) & 1 & 0 & \cdot s \\ \frac{(NM)^2}{(2\pi i)^3}\sum\limits_{s^M=t}\mathrm{Li}_3(s) & \frac{1}{3!}\left(-\frac{N}{2\pi i}\log(t)\right)^3 & \frac{1}{2!}\left(-\frac{N}{2\pi i}\log(t)\right)^2 & -\frac{N}{2\pi i}\log(t) & 1 & \cdot s \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

For $k \in \mathbb{N}$, the multivalued function

$$t \longmapsto \sum_{s^M=t} \mathrm{Li}_k(s)$$

is the branch

$$\widetilde{\mathfrak{X}}^0 \longrightarrow \mathbb{C}$$

taking the value

$$\sum_{s^M=p(\widetilde{y})} \mathrm{Li}_k(s)$$

at $\widetilde{y}$ near $\widetilde{x}$, where as usual we take the power series expression for Li_k in $|s| < 1$.

Proof: left to the reader. q.e.d.

Corollary 3.9: (Distribution property.)

For k and $M \in \mathbb{N}$, we have the following equality of multivalued functions on $\mathbb{P}^1(\mathbb{C})\setminus\{0, 1, \infty\}$:

$$M^{k-1} \sum_{s^M=t} \mathrm{Li}_k(s) = \mathrm{Li}_k(t) .$$

Proof: Of course, this identity is well known and easy to prove by looking at the power series expression of Li_k.

However, this is not the proof we have in mind. By [W4], Theorem 5.2, the matrix appearing in 3.8 and P_N describe the same one–extension of variations of Hodge structure. In order to see that this implies that the two matrices are actually *equal*, we argue as follows: denote by $(f, 2\pi i\cdot e_k \mid k \in \mathbb{N}_0)$ the basis of the bifiltered holomorphic vector bundle underlying $pol(0, i, K_N)$. Similarly, let $(\widetilde{f}, 2\pi i\cdot e_k \mid k \in \mathbb{N}_0)$ be the basis of the vector bundle belonging to the variation described in 3.8.

The isomorphism between the two objects sends $2\pi i\cdot e_k$ to $2\pi i\cdot e_k$. It maps f to

a section, which on the one hand is equal to $\widetilde{f}$ modulo W_{-3} and which on the other hand lands in $\mathcal{F}^{-1}$.
So this section must be $\widetilde{f}$ itself. It follows that the section, whose coordinates with respect to $(f, 2\pi i\cdot e_k \mid k \in \mathbb{N}_0)$ are given by the first column of the matrix in 3.8 is rational. So there are numbers $q_n \in \mathbb{Q}$ such that

$$\frac{(-NM)^{k-1}}{(2\pi i)^k} \sum_{s^M=t} \mathrm{Li}_k(s) = \frac{(-N)^{k-1}}{(2\pi i)^k} \mathrm{Li}_k(s) + \sum_{n=1}^{k} q_n \frac{1}{(k-n)!} \left(\frac{-N}{2\pi i} \log(t) \right)^{k-n}$$

for all k. By using indiction on k one sees, forming the limit $t \to 0$, that all q_n are zero. q.e.d.

It remains to study values at Levi sections, i.e., spectra of cyclotomic fields.
Let $u = \dfrac{b}{f} \in U_0(\mathbb{Q}) = \mathbb{Q}$ with coprime integers b, f,

$$d := \left| \frac{f\cdot N}{gcd(b,N)} \right| .$$

Recall the embedding

$$i_u : (G_0, \mathcal{H}_0) \hookrightarrow (P_0, \mathfrak{X}_0)\,,$$

which on group level is given by

$$i_u : G_0 \longrightarrow P_0, a \longmapsto \begin{pmatrix} 1 & 0 \\ u(1-a) & a \end{pmatrix}\,,$$

$L_{u,N} = i_u^{-1}(K_N) = \ker(\hat{\mathbb{Z}}^* \longrightarrow (\mathbb{Z}/d\mathbb{Z})^*)$
and $[i_u] : M^{L_{u,N}}(G_0, \mathcal{H}_0) = \mathrm{Spec}(\mathbb{Q}(\mu_d)) \longrightarrow M^{K_N}(P_0, \mathfrak{X}_0) = \mathbb{G}_{m,\mathbb{Q}}$.

We formulate the splitting principle ([W4], Proposition 6.1):

Lemma 3.10: $i_u^*\hat{\mathfrak{U}}(\mathrm{Lie}\, U_0)$ splits canonically into a direct product

$$i_u^*\hat{\mathfrak{U}}(\mathrm{Lie}\, U_0) = \prod_{k\in\mathbb{N}_0} \mathbb{Q}(k)\,,$$

the base vector of $\mathbb{Q}(k)$ mapping to e^k under the natural projection

$$i_u^* W_{-2k}(\hat{\mathfrak{U}}(\mathrm{Lie}\, U_0)) \longrightarrow \mathbb{Q}(k)$$

being given by

$$e^k \exp(ue)\,.$$

Proof: This is a direct calculation using 2.1. q.e.d.

As in [W4], § 6, we assume that $u \notin K_N \cap U_0(\mathbb{A}_f)$, i.e., that N does not divide b or that f is not ± 1. So $d > 1$, and $[i_u]$ factors through $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$.

$[i_u]^* pol$ is an element of

$$\prod_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mu_d^{\text{prim.}}} \operatorname{Ext}^1_{MH_{\mathbb{Q}}}(\mathbb{Q}(1), \mathbb{Q}(k+1)) = \prod_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mu_d^{\text{prim.}}} \operatorname{Ext}^1_{MH_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(k)) \, .$$

Here, we used the isomorphism

$$\mu_{d,\mathbb{C}}^{\text{prim.}} \xrightarrow{\sim} M^{L_{u,N}}(\mathbb{C})$$

occurring in the definition of the canonical model.
Also, we allowed ourselves to forget the component "$k = 0$" as there are no nontrivial extensions in $MH_{\mathbb{Q}}$ of $\mathbb{Q}(0)$ by itself.

Theorem 3.11: Under the isomorphism of 3.6,

$$[i_u]^* pol(-1) = \left(-(-N)^{k-1} \operatorname{Li}_k \left(\zeta^{\frac{b}{gcd(b,N)}} \right) \right)_{\zeta, k} \in \prod_{k \in \mathbb{N}} \bigoplus_{\zeta \in \mu_{d,\mathbb{C}}^{\text{prim.}}} \mathbb{C}/(2\pi i)^k \mathbb{Q} \, .$$

Proof: The extension *pol* takes the "value" $P_N(\zeta)$ over a root of unity $\zeta \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.
By 3.10, $2\pi i \cdot e_k(\zeta)$ is the base vector of $\mathbb{Q}(k+1)$, with respect to which we calculate the extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(k+1)$. Hence this extension is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -(-N)^{k-1} \frac{1}{(2\pi i)^k} \operatorname{Li}_k(\zeta) & 1 \end{pmatrix} .$$

Now apply 1.5.c). q.e.d.

Remarks: a) Observe that $[i_u]^* pol(-1)$ lands in

$$\prod_{k \in \mathbb{N}} \left(\bigoplus_{\zeta \in \mu_{d,\mathbb{C}}^{\text{prim.}}} \mathbb{C}/(2\pi i)^k \mathbb{Q} \right)^+ ,$$

the superscript $^+$ denoting the fixed part of the involution c given by complex conjugation on $\mathbb{C}$ as well as on $\mu_{d,\mathbb{C}}^{\text{prim.}}$. But this is what one should expect:
if $X/\mathbb{R}$ is a smooth variety, then the category of graded–polarizable variations of $\mathbb{Q}$–MHS on $X(\mathbb{C})$ carries a natural involution c. It is defined by associating to $\mathbf{V}$ the variation $c(\mathbf{V})$ given by first pulling back $\mathbf{V}$ via complex conjugation on $X(\mathbb{C})$ and then exchanging the Hodge filtration and its conjugate.

It can be proven that c respects the subcategory of admissible variations. By [W3], § 6, $\mathcal{L}og(i, K_N)$ is the Hodge component of a mixed system of smooth sheaves. In particular, there is an isomorphism $c(\mathcal{L}og(i, K_N)(1)) \xrightarrow{\sim} \mathcal{L}og(i, K_N)(1)$. So c acts on $\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\})}(\mathbb{Q}(1), j^*\mathcal{L}og(i, K_N)(1))$.
By [W4], Corollary 2.2, the class of *pol* is fixed by c. But then so are its pull–backs to roots of unity.

b) If we fix $d > 1$ and $k \geq 2$ and let N and $u = \dfrac{b}{f}$ run through all possible combinations satisfying

$$d = \Big| \frac{f \cdot N}{gcd(b, N)} \Big| ,$$

then the k–th components of the $[i_u]^* pol(-1)$ generate a rational sub–vector space of

$$\left(\bigoplus_{\zeta \in \mu^{\mathrm{prim.}}_{d,\mathbb{C}}} \mathbb{C}/(2\pi i)^k \mathbb{Q} \right)^+$$

which induces a $\mathbb{Q}$–structure of

$$\left(\bigoplus_{\zeta \in \mu^{\mathrm{prim.}}_{d,\mathbb{C}}} \mathbb{C}/(2\pi i)^k \mathbb{R} \right)^+ .$$

As explained in [B1], § 7 and [N], Part II, this is exactly the $\mathbb{Q}$–structure given by the regulator map

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\mu_d)), \mathbb{Q}(k)) \to H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}(\mu_d))_{\mathbb{R}}, \mathbb{R}(k)) = \left(\bigoplus_{\zeta \in \mu^{\mathrm{prim.}}_{d,\mathbb{C}}} \mathbb{C}/(2\pi i)^k \mathbb{R} \right)^+$$

occurring in Beilinson's conjectures.

At least philosophically, this is again what is to be expected: the right hand side of 3.11 is an image of an element of motivic cohomology in Deligne cohomology of $\mathrm{Spec}(\mathbb{Q}(\mu_d))$. The latter can be interpreted as an Ext^1 in the category of "variations" of $\mathbb{R}$–Hodge structure over $\mathrm{Spec}(\mathbb{Q}(\mu_d))(\mathbb{C})$ fixed under c, while one would like to think of motivic cohomology as an Ext^1 in the not yet existent category of mixed motives. Since *pol* should be expected to come, via the Hodge realization functor, from a one–extension in the category of mixed motivic sheaves on $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, its restrictions to spectra of cyclotomic fields should be motivic as well.

§ 4 The l–adic version of pol

As in § 3, we give a description of the logarithmic pro–sheaf first:
fix $N \in \mathbb{N}$ and recall the isomorphism

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow M^{K_N}(P_0, \mathfrak{X}_0)$$

of 1.4.
Let $l \in \mathbb{N}$ be a prime. We describe the Kummer torsor C_l on $\mathbb{G}_{m,\mathbb{Q}}$:
for $n \in \mathbb{N}$, let $C_{l,n}$ be the sheaf of finite sets on $\mathbb{G}_{m,\mathbb{Q}}$

$$C_{l,n} := \mathrm{Mor}_{[l^n]}(\_ , \mathbb{G}_{m,\mathbb{Q}}) \, .$$

Here, the subscript $[l^n]$ indicates that we consider $\mathbb{G}_{m,\mathbb{Q}}$ as a scheme over $\mathbb{G}_{m,\mathbb{Q}}$ via the morphism

$$[l^n] : t \longmapsto t^{l^n} \, .$$

$C_{l,n}$ is a torsor under $\mathbb{Z}/l^n\mathbb{Z}(1)$.
So $C_l := \varprojlim_n C_{l,n}$ is a $\mathbb{Z}_l(1)$–torsor.
Define the pro–sheaf

$$R_l := \mathbb{Q}_l[[C_l]] := \varprojlim_n \mathbb{Q}_l[C_{l,n}] \, .$$

It has a natural structure of $\mathbb{Q}_l[[\mathbb{Z}_l(1)]]$–torsor.
Furthermore, observe that there is a global section

$$1 \in \Gamma(\mathrm{Spec}(\mathbb{Q}), [i]^* R_l)$$

given by the projective system $(\ldots, 1, 1, 1)$ of roots of 1.
By [W2], Theorem 3.5.i), there is a unique morphism

$$\varphi_l : \mathcal{L}og(i, K_N) = \mu_{K_N,l}(\hat{\mathfrak{U}}(\mathrm{Lie}\, U_0)) \longrightarrow R_l$$

sending 1 to 1. Note that by the remark following [W2], Theorem 3.5, we may apply the result without having checked mixedness for R_l.
Let r be the following element of the stalk of C_l at 1:

$$r := \left(\exp\left(-\frac{2\pi i}{l^n} \right) \right)_{n \in \mathbb{N}} \, .$$

Recall that the embedding

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

belonging to the definition of the canonical model allows us to define elements of $\mathbb{G}_m(\overline{\mathbb{Q}})$ by their image in $\mathbb{G}_m(\mathbb{C})$.

Theorem 4.1: φ_l is an isomorphism. For $k \in \mathbb{N}_0$, it sends e^k to

$$\left(\frac{1}{N}\log(r)\right)^k = \left(-\frac{1}{N}\sum_{m=1}^{\infty}\frac{(1-r)^m}{m}\right)^k .$$

Note that $(C_l)_{\overline{1}} = \varprojlim_n \mathbb{Q}_l[\mu_{l^n}]$ carries the structure of a topological ring. By a theorem of Serre ([Wa], Theorem 7.1), it is topologically isomorphic to the ring of power series in $(r-1)$. In particular, the series $\log(r)$ converges.

Proof: We may check the assertion after applying base change to $\mathrm{Spec}(\mathbb{C})$. Then the algebraic fundamental group is the profinite completion of the topological one. The generator γ of $\pi_1(M^{K_N}(\mathbb{C}), 1)$ acts by mapping 1 to $\exp(Ne)$. The corresponding topological generator of $\pi_1(M^{K_N}_{\mathbb{C}}, 1)$ acts by mapping 1 to r.
Hence the above formula. Together with [Wa], Theorem 7.1, it shows that φ_l is an isomorphism. q.e.d.

So $pol(0, i, K_N)(1)$ is a one–extension of $\mathbb{Q}(0)$ by $R_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$. As before, weight reasons imply that it is the push–out via the monomorphism of $R_l(1)$ into R_l given by

$$s \longmapsto \left(\frac{1}{N}\log(r)\right)\cdot s$$

of an extension of $\mathbb{Q}(0)$ by $R_l(1)\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$, which we describe now:
for $m, n \in \mathbb{N}$, consider the étale covering maps

$$[l^n] : \mathbb{P}^1_{\mathbb{Q}}\setminus\{0, \infty, \zeta \,|\, \zeta^{l^n} = 1\} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}\setminus\{0, 1, \infty\}$$

and

$$\begin{aligned} \psi^m_n : \mathbb{P}^1_{\mathbb{Q}}\setminus\{0, \infty, \eta \,|\, (1-\eta^{l^m})^{l^n} = 1\} &\longrightarrow \mathbb{P}^1_{\mathbb{Q}}\setminus\{0, \infty, \zeta \,|\, \zeta^{l^n} = 1\}, \\ t &\longmapsto 1 - t^{l^m} . \end{aligned}$$

Let $\widetilde{C}^m_{l,n}$ be the image in the category of étale presheaves of sets of

$$(\psi^m_n)_* : \mathrm{Hom}_{[l^n]\circ\psi^m_n}(\_, \mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}) \longrightarrow C_{l,n\,|\,\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}} .$$

Its sheafification is $C_{l,n\,|\,\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$ itself.
Let $\widetilde{T}^m_{l,n}$ be the presheaf

$$U \longmapsto \{(f_g)_{g\in\widetilde{C}^m_{l,n}(U)} \,|\, f_g \in \mathrm{Hom}_{[l^n]\circ\psi^m_n}(U, \mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}), \psi^m_n \circ f_g = g\} .$$

It is easily seen to be a torsor under $\mathbb{Z}/l^m\mathbb{Z}(1)[\widetilde{C}^m_{l,n}]$.
Hence its sheafification $T^m_{l,n}$ is a torsor under $\mathbb{Z}/l^m\mathbb{Z}(1)[[C_{l,n}]]\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$.

There are transition morphisms

$$\begin{aligned} &T^{m+1}_{l,n} \longrightarrow T^m_{l,n}\ ,\quad (f_g)_g \longmapsto ([l]\circ f_g)_g \quad \text{and} \\ &T^m_{l,n+1} \longrightarrow T^m_{l,n}\ ,\quad (f_h)_{h\in\widetilde{C}^m_{l,n+1}} \longmapsto \left(\prod_{[l]\circ h=g} f_h\right)_{g\in\widetilde{C}^m_{l,n}} \end{aligned}$$

compatible with the respective torsor structures giving rise to an étale sheaf $T_l := \varprojlim_{n,m} T^m_{l,n}$, which is a torsor under $\mathbb{Z}_l(1)[[C_l]]\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$.
After tensoring with $\mathbb{Q}_l$, we get an $R_l(1)\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$–torsor Q_l.

Theorem 4.2: *pol*$(0,i,K_N)(1)$ is the push–out of the extension corresponding to Q_l under the usual bijection

$$\begin{aligned} \left\{ \begin{array}{c} \text{extension classes} \\ 0\to\mathbf{V}\to\mathbf{E}\xrightarrow{\pi}\mathbb{Q}_l(0)\to 1 \end{array} \right\} &\xrightarrow{\sim} \{\mathbf{V}\text{–torsors}\} \\ [\mathbf{E}] &\longmapsto \pi^{-1}(1) \end{aligned}$$

via

$$R_l(1)\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}} \hookrightarrow R_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}},\ s\mapsto \left(\frac{1}{N}\log(r)\right)s\,.$$

Proof: Under the projection

$$R_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}} \longrightarrow R_l/W_{-3}(R_l)\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$$

the topological version of *pol*(1) is pushed out to the extension corresponding to the $\mathbb{Z}_l(1)$–torsor

$$(t\mapsto 1-t)^*(C_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}})$$

pushed out via

$$\mathbb{Z}_l(1)\hookrightarrow\mathbb{Q}_l(1)\hookrightarrow R_l/W_{-3}(R_l)\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}\,,\ 1\mapsto\frac{1}{N}\log(r)\,.$$

This follows from 2.2 and comparison.
So if we show that the push–out of T_l via the augmentation

$$\varepsilon(1):\mathbb{Z}_l(1)[[C_l]]\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}} \longrightarrow \mathbb{Z}_l(1)$$

is isomorphic to $(t\mapsto 1-t)^*(C_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}})$, then our claim follows from the rigidity principle ([W4], Theorem 2.3.a)).

Recall that the push–out of T_l is defined by

$$\varepsilon(1)_*(T_l) := (\mathbb{Z}_l(1) \times T_l)/\sim,$$

where $\sim$ denotes the equivalence relation

$$(z, f) \sim (z - \varepsilon(1)(\lambda), \lambda f)$$

for a section λ of $\mathbb{Z}_l(1)[[C_l]]\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}$. $\mathbb{Z}_l(1)$ acts on the first component.
Now note that there are morphisms

$$\begin{aligned} \widetilde{T}^m_{l,n} &\longrightarrow (t \mapsto 1-t)^*(C_{l,m}\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}}), \\ (f_g)_g &\longmapsto \prod_g f_g \end{aligned}$$

inducing

$$T^m_{l,n} \longrightarrow (t \mapsto 1-t)^*(C_{l,m}\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}})\,.$$

It can be checked stalkwise that the induced morphism

$$\varepsilon(1)_*(T_l) \longrightarrow (t \mapsto 1-t)^*(C_l\,|_{\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}})$$

is an isomorphism. q.e.d.

We proceed to determine the values at Levi sections.

Fix $q \in \mathbb{Q}$, and let $\zeta := \exp(2\pi i q) \in \mathbb{G}_m(\overline{\mathbb{Q}})$.
The stalk $(C_l)_{\overline{\zeta}}$ of C_l at ζ is the Galois–module and $\mathbb{Z}_l(1)$–torsor

$$\{(x_n)_{n\in\mathbb{N}} \,|\, x^l_{n+1} = x_n, x^l_1 = \zeta\}$$

of projective systems of l^n–th roots of ζ. So $(R_l)_{\overline{\zeta}} = \mathbb{Q}_l[[(C_l)_{\overline{\zeta}}]]$.
From 4.1 and the splitting principle 3.10, we know that as a Galois module, $(R_l)_{\overline{\zeta}}$ is the direct product of the $\mathbb{Q}_l(k), k \in \mathbb{N}_0$.

Lemma 4.3: Let $r = \left(\exp\left(-\frac{2\pi i}{l^n}\right)\right)_{n\in\mathbb{N}}$ as before,

$$r_q := \left(\exp\left(\frac{2\pi i}{l^n} q\right)\right)_{n\in\mathbb{N}} \in (C_l)_{\overline{\zeta}}\,.$$

Then $\mathbb{Q}_l(k) \subset (R_l)_{\overline{\zeta}}$ is generated by

$$(\log(r))^k \exp(q\log(r))\cdot r_q\,.$$

Again, $(\log(r))^k \exp(q\log(r))$ is a well–defined element of

$$(C_l)_{\overline{1}} = \mathbb{Q}_l[[r-1]] = \mathbb{Q}_l[[\log(r)]]\,.$$

Proof: It suffices to prove the claim for $k = 0$.
If $p = 1 + dn$, where d is a denominator of $q, n \in \mathbb{Z}$, then

$$r_q^p = r^{-bn} \cdot r_q \, ,$$

where $q = \dfrac{b}{d}$, and

$$\exp(q \log(r^p)) = \exp(q(p-1)\log(r)) \cdot \exp(q \log(r)) \, .$$

But $q(p-1) = bn$ is an integer, so

$$\exp(q(p-1)\log r) = r^{bn} \, .$$

q.e.d.

We want to compare the projection

$$pr_k : (R_l)_{\overline{\zeta}} \longrightarrow\!\!\!\!\rightarrow \mathbb{Q}_l(k)$$

of Lemma 4.3 to the one given by sending

$$\underline{x} \in (C_l)_{\overline{\zeta}} \quad \text{to} \quad \frac{1}{d^k} \cdot (\underline{x}^d)^{\otimes k} \in \mathbb{Q}_l(k) \, .$$

Here, as in the proof of 4.3, $d \in \mathbb{Z} \backslash \{0\}$ is such that $\zeta^d = 1$, and the factor $\dfrac{1}{d^k}$ is introduced to make the projection independent of d.
Call the latter projection pr'_k.
More conceptually, one may interpret $(R_l)_{\overline{\zeta}}$ as the set of $\mathbb{Q}_l$–valued measures on $(C_l)_{\overline{\zeta}}$. Then pr'_k has an expression as a certain integral. Our definition is $(k+1)!$ times the one appearing in [BD1], 3.6.3.ii).

Lemma 4.4: Under pr'_k, the element $(\log(r))^n \exp(q \log(r)) \cdot r_q$ is mapped to

$$\delta_{k,n} \cdot (-1)^k k! \cdot r_1^{\otimes k} \, .$$

Proof: Under pr_k, $(\log(r))^n \exp(q \log(r)) \cdot r_q$ is mapped to

$$\delta_{k,n} \cdot (\log(r))^k \exp(q \log(r)) \cdot r_q \, .$$

Hence $(\log(r))^n \cdot r_q$ is mapped to 0 if $n > k$, and to

$$\frac{(-q)^{k-n}}{(k-n)!} \cdot (\log(r))^k \exp(q \log(r)) \cdot r_q$$

if $n \leq k$.

We conclude that $r^n \cdot r_q = \sum_{m=0}^{\infty} \frac{n^m}{m!} (\log(r))^m \cdot r_q$ is mapped to

$$\sum_{m=0}^{k} \frac{n^m (-q)^{k-m}}{m!(k-m)!} \cdot (\log(r))^k \exp(q \log(r)) \cdot r_q = \frac{(n-q)^k}{k!} \cdot (\log(r))^k \exp(q \log(r)) \cdot r_q$$

under pr_k.

On the other hand,

$$\begin{aligned} pr'_k(r^n \cdot r_q) &= \frac{1}{d^k} \cdot (r^{nd} \cdot r_q^d)^{\otimes k} \\ &= \frac{1}{d^k} \cdot (r_1^{-nd} \cdot r_1^b)^{\otimes k} \\ &= \frac{(b-nd)^k}{d^k} \cdot r_1^{\otimes k} = (q-n)^k \cdot r_1^{\otimes k} \,. \end{aligned}$$

q.e.d.

The choice of a geometric point of $\mathrm{Spec}(\mathbb{Q}(\mu_d))$ gives an isomorphism between $\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(\mathrm{Spec}(\mathbb{Q}(\mu_d)))$ and the category of continuous $\mathbb{Q}_l$–modules under the Galois group of $\mathbb{Q}(\mu_d)$, that are mixed and weight graded–polarizable.

So we think of $\mathrm{Ext}^1_{\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(\mathrm{Spec}(\mathbb{Q}(\mu_d)))}(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$ as sitting inside $H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k))$. We aim at a description of the latter group:

$$\begin{aligned} & H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k)) \\ = & \left(\varinjlim_{m \in \mathbb{N}} H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Q}_l(k)) \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty d})/\mathbb{Q}(\mu_d))} \\ = & \left(\varinjlim_{m \in \mathbb{N}} H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Z}_l(k)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty d})/\mathbb{Q}(\mu_d))} \\ = & \left(\varinjlim_{m \in \mathbb{N}} \left(\varprojlim_{n \in \mathbb{N}} H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Z}/l^n\mathbb{Z}(k)) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty d})/\mathbb{Q}(\mu_d))} . \end{aligned}$$

In the last equality, we have used [J1], (3.1).

Observe that for $m \geq n$, we have

$$\begin{aligned} & H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Z}/l^n\mathbb{Z}(k)) \\ = & \ H^1_{\mathrm{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Z}/l^n\mathbb{Z}(1)) \otimes_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}(k-1) \\ = & \ \mathbb{Q}(\mu_{l^m d})^* / (\mathbb{Q}(\mu_{l^m d})^*)^{l^n} \otimes_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}(k-1) \,. \end{aligned}$$

Also, since the transition maps for the direct limit are all injective,

$H^1_{\text{cont.}}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k))$ is contained in

$$\left(\left(\varprojlim_{n \in \mathbb{N}} \varinjlim_{m \in \mathbb{N}} H^1_{\text{cont.}}(\mathbb{Q}(\mu_{l^m d}), \mathbb{Z}/l^n\mathbb{Z}(k)) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty d})/\mathbb{Q}(\mu_d))} ,$$

which equals

$$\left(\left(\varprojlim_{n \in \mathbb{N}} \mathbb{Q}(\mu_{l^\infty d})^* / (\mathbb{Q}(\mu_{l^\infty d})^*)^{l^n} \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l(k-1) \right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty d})/\mathbb{Q}(\mu_d))} .$$

Now let the notations be as in 3.11.

Theorem 4.5: Let $\zeta \in \mathbb{G}_m(\mathbb{C})$ be a geometric point of $\operatorname{Spec}(\mathbb{Q}(\mu_d))$.
Then under the above injection,

$$[i_u]^* pol(0, i, K_N)(-1) \in \prod_{k \in \mathbb{N}} \operatorname{Ext}^1_{\operatorname{Et}^{l,m}_{\mathbb{Q}_l}(\operatorname{Spec}(\mathbb{Q}(\mu_d)))}(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$$

is mapped to

$$((-N)^{k-1} \cdot \frac{1}{d^{k-1}} \cdot \frac{1}{(k-1)!} \cdot \sum_{\epsilon^{l^n} = \zeta^{\overline{\frac{b}{gcd(b,N)}}}} ([1-\epsilon] \otimes (\epsilon^d)^{\otimes(k-1)}))_{n \in \mathbb{N}, k \in \mathbb{N}} \cdot$$

Proof: $\qquad d^{k-1} \cdot pr'_{k-1}(1) : (R_l(1))_{\overline{\zeta}} \longrightarrow\!\!\!\!\rightarrow \mathbb{Q}_l(k)$
maps $z \cdot \underline{x}$ to $z \cdot (\underline{x}^d)^{\otimes(k-1)}$,
hence respects the integral structures

$$\mathbb{Z}_l(1)[[C_{l,\overline{\zeta}}]] \quad \text{and} \quad \mathbb{Z}_l(k) \, .$$

Assume we managed to show:
$(*)$ the push–out of the $\mathbb{Z}/l^n\mathbb{Z}(1)[[(C_{l,n})_{\overline{\zeta}}]]$–torsor $(T^n_{l,n})_{\overline{\zeta}}$ with respect to the map $d^{k-1} \cdot pr'_{k-1}(1)$ is the $\mathbb{Z}/l^n\mathbb{Z}(k)$–torsor

$$\sum_{\epsilon^{l^n} = \zeta} (\epsilon^d)^{\otimes(k-1)} (C_{l,n})_{\overline{1-\epsilon}} \, .$$

Then the claim would follow from 3.10, 4.1–4.4 and the fact that the torsor $(C_{l,n})_{\overline{1-\epsilon}} \in H^1_{\text{cont.}}(\mathbb{Q}(\mu_{l^n d}), \mathbb{Z}/l^n\mathbb{Z}(1))$ is mapped to the class of $1-\epsilon$ under

$$H^1_{\text{cont.}}(\mathbb{Q}(\mu_{l^n d}), \mathbb{Z}/l^n\mathbb{Z}(1)) \xrightarrow{\sim} \mathbb{Q}(\mu_{l^n d})^* / (\mathbb{Q}(\mu_{l^n d})^*)^{l^n} \, .$$

In order to prove $(*)$, recall that we defined the push–out of torsors in the proof of 4.2. So $(d^{k-1} \cdot pr'_{k-1}(1))_*(T^n_{l,n})_{\overline{\zeta}}$ is defined, as are the $(\epsilon^d)^{\otimes(k-1)}(C_{l,n})_{\overline{1-\epsilon}}$:

$$(\epsilon^d)^{\otimes(k-1)} : \mathbb{Z}/l^n\mathbb{Z}(1) \longrightarrow \mathbb{Z}/l^n\mathbb{Z}(k)$$

is Galois–equivariant since we only consider the action of the Galois group of $\mathbb{Q}(\mu_{l^n d})$.
If $C_1, \ldots, C_r$ are torsors under R, then the sum is defined as the push–out via

$$\sum : R^r \longrightarrow R$$

of the R^r–torsor $C_1 \times \ldots \times C_r$.
Explicitly, it is $(C_1 \times \ldots \times C_r)/\sim$, the equivalence relation being induced by the action of $\ker(\sum)$. R then acts on any of the factors.
This being said, we leave it to the reader to check that the morphism

$$\begin{aligned} (\widetilde{T}^n_{l,n})_{\overline{\zeta}} &\longrightarrow \sum_{\epsilon^{l^n}=\zeta} (\epsilon^d)^{\otimes(k-1)}(C_{l,n})_{\overline{1-\epsilon}}, \\ (\gamma_\epsilon)_\epsilon &\longmapsto [((\epsilon^d)^{\otimes(k-1)} \cdot \gamma_\epsilon)_\epsilon] \end{aligned}$$

induces the desired isomorphism. q.e.d.

Remarks: a) Because of [W4], Theorem 6.2, the elements in 4.5 for different levels N are norm compatible. Again, this can be seen by a direct calculation.
b) Observe that $[i_u]^* pol(-1)_k \in H^1_{\text{cont.}}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k))$ lies in the subgroup called $H^1_g(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k))$ in [BK], § 5. It is the group of those classes of cocycles mapping to $H^1_f(\mathbb{Q}(\mu_d)_\nu, \mathbb{Q}_l(k))$ for almost all finite places ν. This group in turn is defined in [BK], § 3. If $k \geq 2$ and $\nu \nmid l$, then it is rather easy to see that $H^1_{\text{cont.}}(\mathbb{Q}(\mu_d)_\nu, \mathbb{Q}_l(k))$ is zero and hence, that the condition is empty. Likewise, for $k \geq 2$ and $\nu \mid l$, the condition is empty because of the table in [BK], Example 3.9. So for $k \geq 2$, we have $H^1_{\text{cont.}} = H^1_g = H^1_{f,\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)})}$ in the notation of [BK], § 5. For $k \geq 1$, a straightforward computation for $\nu \nmid l$ and [BK], Example 3.9 for $\nu \mid l$ show that $H^1_f(\mathbb{Q}(\mu_d)_\nu, \mathbb{Q}_l(1))$ is the subgroup of

$$H^1_{\text{cont.}}(\mathbb{Q}(\mu_d)_\nu, \mathbb{Q}_l(1)) = \left(\varprojlim_n \mathbb{Q}(\mu_d)^*_\nu / (\mathbb{Q}(\mu_d)^*_\nu)^{l^n} \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

of elements coming from $(o_{\mathbb{Q}(\mu_d)_\nu})^* \otimes_{\mathbb{Z}} \mathbb{Q}$.

By Theorem 4.5, we have

$$[i_u]^* pol(-1)_1 = \left(\prod_{\epsilon^{l^n} = \zeta^{\frac{b}{gcd(b,N)}}} (1-\epsilon) \right)_{n \in \mathbb{N}} = \left(1 - \zeta^{\frac{b}{gcd(b,N)}} \right)_{n \in \mathbb{N}}.$$

So if d has more than one prime factor, $[i_u]^* pol(-1)_1$ lies in $H^1_{f,\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)})} \subset H^1_g$.

If $d = p^n$, then there is exactly one place ν_p dividing p. Hence $[i_u]^* pol(-1)_1$ is an element of $H^1_{f,\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)})\setminus\{\nu_p\}} \subset H^1_g$.

Now let $k \geq 2$. We claim that the $[i_u]^* pol(-1)_k$, for all N and $u = \dfrac{b}{f}$ such that $d = \Big|\dfrac{f \cdot N}{gcd(b,N)}\Big|$, define a $\mathbb{Q}$–structure of $H^1_g(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k))$.

By [J3], Lemma 4 and [M], II, Proposition 2.9, we have

$$H^1_g(\mathbb{Q}(\mu_d), \mathbb{Q}_l(k)) = H^1_{\text{ét.}}\Big(\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)}) \left[\frac{1}{l}\right], \mathbb{Q}_l(k)\Big) .$$

For $l \neq 2$, our claim follows from [So1], Theorem 1 and the theory of cyclotomic elements in K–theory ([So2], Théorème 3). For $l = 2$, we proceed as follows:
i) Show that, as for $l \neq 2$, the space H^1_g is of dimension

$$d(k) := \begin{cases} 0 & , \quad d = 2, k \text{ even} \\ 1 & , \quad d = 2, k \text{ odd} \\ \frac{1}{2}\varphi(d) & , \quad d > 2 \end{cases} ,$$

where we let $\varphi(d) := \sharp(\mathbb{Z}/d\mathbb{Z})$.
ii) Show that $d(k)$ of the $[i_u]^* pol(-1)_k$ suffice to generate the $\mathbb{Q}$–vector space generated by all $[i_u]^* pol(-1)_k$.
iii) Show that the $[i_u]^* pol(-1)_k$ generate the $\mathbb{Q}_2$–vector space H^1_g.

For i), observe that by [J3], proof of Lemma 1 and [M], II, Proposition 2.9, the dimension of $H^1_g(\mathbb{Q}(\mu_d), \mathbb{Q}_2(k))$ equals the corank of the étale cohomology group $H^1_{\text{ét.}}\Big(\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)}) \left[\frac{1}{2}\right], \mathbb{Q}_2/\mathbb{Z}_2(k)\Big)$. By [So3], 1.2 and [So3], Proposition 2, this corank is greater or equal to $d(k)$. Furthermore, equality holds if $H^2_{\text{ét.}}\Big(\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)}) \left[\frac{1}{2}\right], \mathbb{Q}_2/\mathbb{Z}_2(k)\Big)$ is torsion. This in turn follows from [So4], Theorem 2, whose proof can be modified to give an analogous statement for $l = 2$, with $k!$ possibly replaced by $2^m k!$.
From the formula in 4.5, it is easy to conclude that we need only consider those pairs (N, u) with $N = 1$ and $f = d$, the b forming a set of representatives of $(\mathbb{Z}/d\mathbb{Z})^*$. Furthermore, we have

$$\left[i_{\frac{b}{d}}\right]^* pol(-1)_k = (-1)^{k-1} \left[i_{-\frac{b}{d}}\right]^* pol(-1)_k .$$

This shows ii). Finally, observe that since the part of [So1] from page 384 onwards works for arbitrary l once we know that $H^2_{\text{ét.}}\Big(\mathrm{Spec}(o_{\mathbb{Q}(\mu_d)}) \left[\frac{1}{l}\right], \mathbb{Q}_l/\mathbb{Z}_l(k)\Big)$

is torsion, all we have to show is the validity of [So2], Théorème 3 for arbitrary primes. The proof of this result can actually be simplified: [G], Theorem 3.1 and the remark following it, together with elementary class field theory show that the characteristic ideals of the Iwasawa modules $\overline{E}/\overline{C}$ and $\mathrm{Gal}(L_\infty/F_\infty)^+$ (in Soulé's notation) coincide up to a power of l. This reduces us to showing [So2], Théorème 2 for arbitrary primes. This in turn follows from [S], § 5, Corollary 4 and § 6, Lemma 1, which are also valid for $l = 2$ and totally imaginary number fields.[†] For $F = \mathbb{Q}(\mu_d) = \mathbb{Q}$, the group $\mathrm{Gal}(L_\infty/F_\infty)^\wedge(i)^G$ (in the notation of [So2], page 247) is finite since already the invariants of $\mathrm{Gal}(L_\infty/F_\infty)^\wedge(i)$ under a subgroup of G of index 2 are finite.

This shows iii), and proves our claim.

§ 5 Remarks on the Tamagawa number conjecture for Tate motives

Assume we are prepared to accept the existence of a motivic formalism, i.e., a theory of mixed motivic sheaves admitting the usual six functors on the level of derived categories, Hodge– and l–adic realization functors into the categories of algebraic mixed Hodge modules and l–adic mixed perverse sheaves (see [W2], § 4) compatible with the six functors, an isomorphism between motivic cohomology and Ext–groups of mixed motivic sheaves and the compatibility of the realization functors with the regulators.

Assume also that a decent motivic analogue of the canonical construction ([W3]) is available.

Then the same proofs as in [W4], § 1 yield a motivic version of the polylogarithm, and because of the motivic rigidity principle, its realizations must be the Hodge and l–adic versions of *pol* described in §§ 3 and 4.

Because of the motivic splitting principle, the elements in 3.11 and 4.5 must be the respective regulators of the *same* elements in motivic cohomology. Observe that this is precisely what is needed to complete the proof of [BK], Theorem 6.1, i.e., the Tamagawa number conjecture modulo powers of 2 for Tate motives $\mathbb{Q}(k)$ with $k \geq 3$ odd.

Thanks to Kato's work in [BK], § 6, it can be shown that the conjecture holds if we replace the rational structure Φ of [BK], (5.11) by the rational structure

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[†] The assumption on the number fields is needed in order to have cohomological 2–dimension of the absolute Galois group equal to two.

$\Phi_{pol,k}$ given as follows: by [W4], Corollary 2.2, there is a mixed system of smooth sheaves pol, whose Hodge– and l–adic components are those of §§ 3 and 4. Fix $d > 1$ and $k \geq 2$, and let N and $u = \frac{b}{f}$ run through all possible combinations satisfying $d = \left|\frac{f \cdot N}{gcd(b,N)}\right|$. Let $[i_u]_{L_{u,N},K_N} : \mathrm{Spec}(\mathbb{Q}(\mu_d)) \hookrightarrow \mathbb{P}^1_{\mathbb{Q}} \backslash \{0,1,\infty\}$ be as before. Define $\Phi_{pol,k} \subset \mathrm{Ext}^1_{MS^s_{\mathbb{Q}}(\mathrm{Spec}(\mathbb{Q}(\mu_d)))}(\mathbb{Q}(0), \mathbb{Q}(k))$ to be the $\mathbb{Q}$–vector space of one–extensions in the category of mixed systems of smooth sheaves ([W3], § 6) on $\mathrm{Spec}(\mathbb{Q}(\mu_d))$ generated by the k–th components of the classes of all $[i_u]^*_{L_{u,N},K_N} pol(-1)$.

We define $d(k)$ as before, i.e.,

$$d(k) = \begin{cases} 0 & , \ d = 2, k \text{ even} \\ 1 & , \ d = 2, k \text{ odd} \\ \frac{1}{2}\varphi(d) & , \ d > 2 \end{cases} .$$

Theorem 5.1: $\Phi_{pol,k}$ has dimension $d(k)$.

Proof: Since the image of the Hodge component defines a $\mathbb{Q}$–structure of $\left(\bigoplus_{\zeta \in \mu_d^{\mathrm{prim.}}} \mathbb{C}/(2\pi i)^k \mathbb{R}\right)^+$ (see the remark at the end of § 3), which is of dimension $d(k)$ over $\mathbb{R}$, the dimension of $\Phi_{pol,k}$ is at least $d(k)$. Alternatively, we can use the remark at the end of § 4. In order to see that the dimension is at most $d(k)$, we again have to see that the pairs (N, u) with $N = 1$ and $f = d$, the b forming a set of representatives of $(\mathbb{Z}/d\mathbb{Z})^*$, generate $\Phi_{pol,k}$, and that

$$\left[i_{\frac{b}{d}}\right]^* pol(-1)_k = (-1)^{k-1} \left[i_{-\frac{b}{d}}\right]^* pol(-1)_k .$$

The second claim follows from the fact that pol is mapped to $-pol$ under the map $t \mapsto t^{-1}$: see the remark at the end of § 1.

The proof of the first claim runs along similar lines: for N and $u = \frac{b}{f}$ such that $d = \left|\frac{f \cdot N}{gcd(b,N)}\right|$, set $N' := 1$, $f' := d$, $b' := \frac{b}{gcd(b,N)}$ and $u' := \frac{b'}{f'} = \frac{u}{N}$. So $L_{u',N'} = i_{u'}^{-1}(K_1) = i_u^{-1}(K_N) = L_{u,N}$, and we have a commutative diagram

$$\begin{tikzcd}
M^{L_{u,N}} \arrow[r, hook, "{[i_u]}"] \arrow[dr, hook, "{[i_{u'}]}"'] & M^{K_N} \arrow[d, "{[\varphi_N]}", "\wr"'] \\
& M^{K_1}
\end{tikzcd}$$

Here, φ_N is the automorphism of $(P_0, \mathfrak{X}_0)$ defined at the end of § 1. So from the identification of $[\varphi_N]^* pol(0, i, K_1)$ and $N \cdot pol(0, i, K_N)$ we conclude that $[i_u]^* pol(-1)_k = N^{k-1} \cdot [i_{u'}]^* pol(-1)_k$. q.e.d.

As shown by Kato, Theorems 3.11 and 4.5 then imply

Theorem 5.2: If $k \geq 3$ is odd, then the Tamagawa number conjecture is true modulo a power of 2 for the motivic pair

$$\Big(V = H^0((\mathrm{Spec}\mathbb{Q})(\mathbb{C}), \mathbb{Q}(k)) \ , \ D = H^0_{DR}(\mathrm{Spec}\mathbb{Q})\Big) \ ,$$

equipped with the $\mathbb{Q}$–structure $\Phi_{pol,k}$ for $d = 2$.

Proof: [BK], § 6. Observe the relation $c_k(1) = \dfrac{2^{k-1}}{1 - 2^{k-1}} c_k(-1)$. q.e.d.

Index of Notations

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