THE HODGE FILTRATION AND CYCLIC HOMOLOGY

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Abstract. We relate the "Hodge filtration" of the cohomology of a complex algebraic variety X to the "Hodge decomposition" of its cyclic homology. If X is smooth and projective, $HC_n^{(i)}(X)$ is the quotient of the Betti cohomology $H^{2i-n}(X(\mathbb{C});\mathbb{C})$ by the $(i+1)^{st}$ piece of the Hodge filtration.

Introduction

The cyclic homology $HC_*(A)$ of an algebra A has been intensely developed in the last decade. Recently this theory has been extended to schemes (in [WG] and [W]), where something new happens: The cyclic homology of a smooth complex projective variety X is equivalent to the Hodge filtration of its classical cohomology.

This result was announced in [W, 9.8.19], and forms the core (§4) of the present paper. Here is a more precise statement. If X is a smooth projective variety then the quotient $H^m/F^{i+1}H^m$ of $H^m_{an}(X,\mathbb{C})$ by the Hodge layer $F^{i+1}H^m_{an}(X,\mathbb{C})$ is naturally isomorphic to the component $HC^{(i)}_{2i-m}(X)$ of $HC_{2i-m}(X)$; this isomorphism is compatible with the S maps in cyclic homology, and $H^m_{an}(X,\mathbb{C}) \cong HC^{(i)}_{2i-m}(X)$ for $i \geq m$. For this reason, I call the underlying decomposition $HC_n(X) = \prod HC_n^{(i)}(X)$ the *Hodge decomposition* of cyclic homology; it is also called the *λ-decomposition* [FT][L].

The Hodge filtration of $H^m_{an}(X,\mathbb{C})$ is described by the rows of the "Hodge diamond" associated to a smooth projective variety X. The columns of the Hodge diamond have a similar interpretation; they describe the "Hodge-type" decomposition $HH_n(X) = \prod HH_n^{(i)}(X)$ of Hochschild homology [GS].

We also give some localization sequences for cyclic homology. As an application, consider the coordinate ring A of a smooth affine curve over k. If the curve has genus g and n rational points at infinity, we show that $HC_1(A) = k^{2g+n-1}$.

Here is an outline of this paper. The Hochschild result is proven in §1 and revisited in §4. The cyclic homology of mixed complexes of sheaves is developed in §2. An important example is the mixed complex $(\Omega^*_X, 0, d)$; its periodic cyclic homology is a product of copies of the classical *de Rham cohomology* $H^*_{dR}(X/k)$ of X over k (see [GdR]). This will be our segue into our main result, since when X is smooth the cyclic homology of $(\Omega^*_X, 0, d)$ will agree with the cyclic homology of

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X, and it is well-known [GdR] that $H^*_{dR}(X/\mathbb{C}) \cong H^*_{an}(X,\mathbb{C})$; Of course, if X is singular then the calculations of [BH] show that $H^*_{dR}(X/\mathbb{C})$ does not always equal $H^*_{an}(X,\mathbb{C})$.

The decomposition of cyclic homology is developed in §3. The most important result (3.4) is an extension of a theorem of Feigin and Tsygan [FT]: for every quasi-projective variety X over $\mathbb{C}$ the analytic cohomology $H^m_{an}(X,\mathbb{C})$ is isomorphic to the component $HP^{(i)}_{2i-n}(X)$ of periodic cyclic homology. We also give some localization sequences (3.7) which allow us to better describe the cyclic homology of affine curves (see 3.6 and 5.5).

The last section (§5) is devoted to an analysis of the cyclic homology of singular varieties. We show that cyclic homology defines a natural filtration G^iH^m on $H^m(X,\mathbb{C})$, which is contained in—but in general smaller than—the Hodge filtration of Deligne. It suggests that the cyclic homology of a smooth hypercover of X, which is directly related to Deligne's Hodge filtration, is just as useful as the cyclic homology of X. Unfortunately there seems no way to translate this idea into the universe of noncommutative algebras.

Notation. We will fix a commutative ring k containing $\mathbb{Q}$, and consider quasi-compact quasi-separated schemes X over k; the reader may restrict to noetherian schemes over a field of characteristic zero without much loss of generality. (All noetherian schemes are quasi-compact and separated by [EGA, $0_{IV}(1.2.8)$].)

We will write Ω_X for the sheaf $\Omega_{X/k}$ of relative Kähler differentials on X. 'Tot' means the product total complex. We will freely identify chain complexes C_* with cochain complexes C^* via $C_n = C^{-n}$. The translation $C[p]$ of C is indexed as in [W, 1.2.8]: $C[p]_n = C_{n+p}$, $C[p]^n = C^{n-p}$.

We will freely refer to the "Cartan-Eilenberg" hypercohomology $\mathbb{H}^*(X,\mathcal{C}^*)$ of a cochain complex $\mathcal{C}^*$ of sheaves (say of k-modules) which is not bounded below. This is the hypercohomology used in [T], [WG] and [W1], and it differs slightly from the derived category hypercohomology of [Sp]. (See the appendix to [W1] for a discussion.) To define it, choose an injective Cartan-Eilenberg resolution $\mathcal{C}^* \to I^{**}$; its global sections ΓI^{**} form a double complex of injective k-modules. The *hypercohomology complex* $\mathbb{H}(X,\mathcal{C}^*)$ is just the total complex $\operatorname{Tot}\Gamma I^{**}$, and the hypercohomology of $\mathcal{C}^*$ is simply its cohomology: $\mathbb{H}^n(X,\mathcal{C}) = H^n\mathbb{H}(X,\mathcal{C})$.

§1. **Hochschild homology**

We begin with a discussion of the Hodge decomposition for Hochschild homology, where things are less intricate.

Let X be a quasi-compact quasi-separated scheme over a fixed commutative ring k. To define the Hochschild homology of X, we may sheafify the usual Hochschild complex $C_*(A) = A^{\otimes *+1}$ in the Zariski topology to obtain a chain complex $\mathcal{C}_*$ of sheaves.

$$\mathcal{C}_*: \qquad \cdots \to \mathcal{O}_X^{\otimes *+1} \to \cdots \to \mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{O}_X \to 0$$

As in [WG] [W1], the Hochschild homology of X over k is defined as the hypercohomology of this complex, reindexed as a cochain complex by $\mathcal{C}^n = \mathcal{C}_{-n}$:

$$HH_n(X) = \mathbb{H}^{-n}(X,\mathcal{C}_*) = \mathbb{H}^{-n}(X,\mathcal{C}^{-*}). \tag{1.0}$$

For an affine scheme $X = \operatorname{Spec} A$, this $HH_n(X)$ agrees with the classical group $HH_n(A)$ by [WG, 4.1]. Although X can have negative Hochschild homology—see (1.1) below—we will see in 1.2 that it is bounded below. For example, when X is noetherian over k then $HH_n(X) = 0$ for $n < -\dim(X)$ [WG, 4.3].

We now assume that k contains $\mathbb{Q}$. In this case, for each commutative algebra A there is a natural decomposition of $C_*(A)$ as a product (or direct sum) of chain subcomplexes $\mathcal{C}_*^{(i)}(A)$, $i \geq 0$.

$$C_*(A) = A \times C_*^{(1)}(A) \times C_*^{(2)} \times \cdots \times C_*^{(i)}(A) \times \cdots \qquad \text{with } C_n^{(i)}(A) = 0 \text{ if } i > n.$$

This gives $HH_n(A) = \prod_{i=0}^n HH_n^{(i)}(A)$, where $HH_n^{(i)}(A) = H_n C_*^{(i)}(A)$. Following [GS], we shall call this the *Hodge decomposition*; it is called the *λ–decomposition* in [L].

For schemes we set $HH_n^{(i)}(X) = \mathbb{H}^{-n}(X, \mathcal{C}_*^{(i)})$; we will see in 1.3 below that for any affine scheme $X = \operatorname{Spec}(A)$ we have $HH_n^{(i)}(X) = HH_n^{(i)}(A)$. When $i = 0$, the hypercohomology of $\mathcal{C}_*^{(0)} = \mathcal{O}_X$ is just cohomology:

$$HH_n^{(0)}(X) = \begin{cases} H^0(X, \mathcal{O}_X) & \text{if } n = 0, \\ H^{-n}(X, \mathcal{O}_X) & \text{if } n \text{ is negative,} \\ 0 & \text{if } n \text{ is positive.} \end{cases} \tag{1.1}$$

We will prove in 1.3 that $HH_n(X)$ is the product of the $HH_n^{(i)}(X)$. In order to do so, we introduce the sheafification $\mathcal{HH}_n^{(i)}$ of the presheaf $HH_n^{(i)}$. These sheaves are summands of $\mathcal{HH}_n$, so they are quasi-coherent by [WG, 0.4]. Moreover $\mathcal{HH}_n^{(i)} = 0$ for $n < i$, and $\mathcal{HH}_n^{(n)} \cong \Omega_X^n$.

Proposition 1.2. *There is a fourth quadrant spectral sequence for each i, which is bounded and converges if X is quasi-compact and quasi-separated:*

$$E_2^{pq} = H^p(X, \mathcal{HH}_{-q}^{(i)}) \Rightarrow HH_{-p-q}^{(i)}(X).$$

Moreover, if X is noetherian then $HH_n^{(i)}(X) = 0$ if $i > n + \dim(X)$.

Proof. (Cf. [WG, 4.2].) This is just the hypercohomology spectral sequence [EGA, 0_{III}(12.4)][T, 1.46] associated to the complex $\mathcal{C}_*^{(i)}$. When X is quasi-compact and quasi-separated, there is a uniform bound d on the cohomology of quasi-coherent sheaves. (See [EGA, III(1.4.12) and IV(1.7.21)].) Since $E_2^{pq} = 0$ if $p > d$ or $q > -i$, we have $HH_n^{(i)}(X) = 0$ if $n > d - i$.

Proposition 1.3. *When X is quasi-compact and quasi-separated, there is a finite decomposition for each n:*

$$HH_n(X) = HH_n^{(0)}(X) \times HH_n^{(1)}(X) \times \cdots \times HH_n^{(i)}(X) \times \cdots$$

We call this the Hodge decomposition of $HH_n(X)$. When $X = \operatorname{Spec}(A)$ we have $HH_n^{(i)}(X) = HH_n^{(i)}(A)$, and the Hodge decompositions of $HH_n(X)$ and $HH_n(A)$ agree.

Proof. We have to show that the natural map $p_X : HH_n(X) \to \prod HH_n^{(i)}(X)$ is an isomorphism, since only finitely many of the $HH_n^{(i)}(X)$ in the product are nonzero for each n by 1.2. For an affine scheme $X = \operatorname{Spec} A$, we argue as in [WG, 4.1]; the spectral sequence collapses along the line $p = 0$ to yield $HH_n^{(i)}(X) = HH_n^{(i)}(A)$, and we recover the usual Hodge decomposition of $HH_n(A)$. We now proceed as in [W1, 0.5]. If X is separated, the result follows from induction on the size of an affine cover, the Mayer-Vietoris sequence and the 5-lemma. Another induction, on the size of a quasi-affine cover, proves the result for all quasi-separated X.

The following two results follow from the hypercohomology spectral sequence.

Corollary 1.4. *(announced in [W, 9.8.19]) If X is smooth over k, then for all i and n*

$$HH_n^{(i)}(X) = H^{i-n}(X, \Omega_X^i).$$

Proof. If X is smooth over k then $\mathcal{HH}_n^{(i)} = 0$ unless $n = i$. Therefore the spectral sequence 1.2 degenerates along the row $q = -i$.

Corollary 1.5. *(Cf. [V, 4.2]) If X is locally a complete intersection, then*

$$HH_n^{(i)}(X) = 0 \qquad \text{whenever} \quad i < n/2.$$

Proof. If A is a locally complete intersection over k, the cotangent complex $\mathbb{L}_{A/k}$ has projective dimension ≤ 1 by [Q, 5.4]. As in [Q], this means that the André-Quillen homology $D_n(A/k)$ vanishes for $n > 1$, and the variant groups $D_n^{(i)}(A/k) = H_n(\Lambda^i \mathbb{L}_{A/k})$ vanish whenever $n > i$. Using the dictionary (see [Ro]) that $HH_n^{(i)}(A)$ equals $D_{n-i}^{(i)}(A/k)$, this means that $\mathcal{HH}_n^{(i)} = 0$ whenever $n > 2i$. Hence the spectral sequence 1.2 is zero below the row $q = -2i$, and we may read the result off from this.

Remark. It is suggestive to write $D_n(X/k)$ for $HH_{n+1}^{(1)}(X)$, since this is a natural extension of André-Quillen homology to schemes over k.

Example 1.6. Let X be a reduced (singular) curve over a perfect field k. Then $\mathcal{HH}_0^{(0)} = \mathcal{O}_X$ and $\mathcal{HH}_1^{(1)} = \Omega_X$; all the other sheaves $\mathcal{HH}_n^{(i)}$ are skyscraper sheaves supported on the (finite) singular locus. So the spectral sequence 1.2 degenerates to yield $HH_{-1}(X) = HH_{-1}^{(0)}(X) = H^1(X, \mathcal{O}_X)$, $HH_0(X) = HH_0^{(0)}(X) \oplus HH_0^{(1)}(X)$,

$$HH_0^{(0)}(X) = H^0(X, \mathcal{O}_X) \quad \text{and} \quad HH_0^{(1)}(X) = H^1(X, \Omega_X),$$

$HH_n^{(i)}(X) = H^0(X, \mathcal{HH}_n^{(i)})$ for $n \geq 1$. In particular, $HH_n^{(i)}(X) = 0$ if $i > n > 0$, and

$$HH_1(X) = HH_1^{(1)}(X) = H^0(X, \Omega_X).$$

§2. Mixed complexes of sheaves

For purposes of studying the Hodge decomposition of cyclic homology, it is useful to introduce the notion of cyclic hyperhomology for mixed complexes $(\mathcal{M}, b, B)$ of sheaves on a site X. By a *mixed complex* of sheaves we mean a nonnegative chain complex $\mathcal{M}_*$ of sheaves with differential b, together with a map $B : \mathcal{M}_n \to \mathcal{M}_{n+1}$ such that $B^2 = Bb + bB = 0$.

Definition 2.0. Let $(\mathcal{M}, b, B)$ be a mixed complex of sheaves on X. The *Hochschild hyperhomology* of $\mathcal{M}$ is $HH_n(\mathcal{M}) = \mathbb{H}^{-n}(X, (\mathcal{M}, b))$. The *cyclic hyperhomology* of $\mathcal{M}$ is defined to be $HC_n(\mathcal{M}) = \mathbb{H}^{-n}(X, \operatorname{Tot}\mathcal{B}_{**}(\mathcal{M}))$, where $\mathcal{B}_{**}(\mathcal{M})$ is Connes' (b, B) double complex (of sheaves) for $\mathcal{M}$, as in [L, 2.5.13] or [W, 9.8.2].

The usual homological yoga yields SBI sequences (2.1.3) connecting $HH_*(\mathcal{M})$ and $HC_*(\mathcal{M})$, because hypercohomology is a cohomological functor. (See the appendix to [W1], [T, 1.35] or [WG, A.4].) In addition, the column filtration on $\mathcal{B}_{**}$ gives rise to a generalization of Connes' spectral sequence: $E^1_{pq} = HH_{q-p}(\mathcal{M}) \Rightarrow HC_{p+q}(\mathcal{M})$, which converges if X has finite cohomological dimension by [D, 1.4].

2.1. Because inverse limits of sheaves are poorly behaved, we need care in defining periodic and negative cyclic hyperhomology. Write $\mathbb{H}$ for the hypercohomology complex $\mathbb{H}(X, \operatorname{Tot}\mathcal{B}_{**}(\mathcal{M}))$, so that by definition $HC_n(\mathcal{M})$ is $H^{-n}(\mathbb{H})$. We may arrange that the map $\mathbb{H}[2] \to \mathbb{H}$ induced by the usual map $S : \mathcal{B}_{**}[2] \to \mathcal{B}_{**}$ is onto. This allows us to form both a surjective tower of complexes of sheaves $\{\operatorname{Tot}\mathcal{B}_{**}[2p]\}$ and a surjective tower of complexes of injective k-modules

$$\cdots \xrightarrow{S} \mathbb{H}\,[2p] \xrightarrow{S} \mathbb{H}\,[2p-2] \xrightarrow{S} \cdots \xrightarrow{S} \mathbb{H} \tag{2.1.1}$$

Write $\operatorname{holim}_p \mathbb{H}\,[2p]$ for the inverse limit of the tower (2.1.1); this chain complex is quasi-isomorphic to the usual homotopy limit by [BK, XI.4.1], [T, 5.6] or [BN, 2.3.1]. We define the *periodic cyclic hyperhomology* of $\mathcal{M}$ to be $HP_n(\mathcal{M}) = H^{-n}\operatorname{holim}_p\mathbb{H}\,[2p]$. It is related to cyclic homology by the usual Milnor sequence for the tower (2.1.1); see [W, 3.5.8], [BK, IX.3.1] or [T, 5.41].

Milnor sequence 2.1.2. *For any X and $\mathcal{M}$ we have an exact sequence*

$$0 \to \varprojlim{}^1 HC_{n+2i+1}(\mathcal{M}) \to HP_n(\mathcal{M}) \to \varprojlim HC_{n+2i}(\mathcal{M}) \to 0.$$

The *negative cyclic hyperhomology* $HN_*(\mathcal{M})$ of $\mathcal{M}$ is defined as the (co)homology of the kernel of the natural surjection $\operatorname{holim}_p\mathbb{H}\,[2p] \to \mathbb{H}\,[-2]$; The resulting long exact hypercohomology sequence is compatible with the usual SBI sequence:

$$\begin{array}{ccccccc} \cdots HP_{n+1}(\mathcal{M}) & \xrightarrow{S} & HC_{n-1}(\mathcal{M}) & \xrightarrow{B} & HN_n(\mathcal{M}) & \xrightarrow{I} & HP_n(\mathcal{M}) \cdots \\ \downarrow & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\ \cdots HC_{n+1}(\mathcal{M}) & \xrightarrow{S} & HC_{n-1}(\mathcal{M}) & \xrightarrow{B} & HH_n(\mathcal{M}) & \xrightarrow{I} & HC_n(\mathcal{M}) \cdots . \end{array} \tag{2.1.3}$$

Main Example 2.2 (HC of a scheme). Let X be a scheme over k. In §1 we introduced the chain complex $\mathcal{C}_* = \mathcal{O}_X^{\otimes *+1}$ and wrote $HH_n(X)$ for $HH_n(\mathcal{C}_*)$. In fact, $\mathcal{C}_*$ forms a mixed complex; the map B is the sheafification of the usual operator B in cyclic homology. We shall write $HC_*(X)$ for $HC_n(\mathcal{C}_*)$, as was done in [WG] and [W1]. We shall also write $HP_*(X)$ for $HP_n(\mathcal{C}_*)$ and $HN_*(X)$ for $HN_n(\mathcal{C}_*)$.

If $X = \operatorname{Spec}(A)$ is affine then $HC_*(X) = HC_*(A)$ by [W1]. By (2.1.2) we also have $HP_*(X) = HP_*(A)$ and therefore also $HN(X) = HN_*(A)$.

Example 2.3 (Sheaves). At the other extreme, we may regard any sheaf $\mathcal{M}$ as a mixed complex, concentrated in degree 0. Clearly $HH_n(\mathcal{M}) = H^{-n}(X,\mathcal{M})$. In this case $\operatorname{Tot}\mathcal{B}_{**} = \prod_{i\geq 0}\mathcal{M}[-2i]$, so $HC_n(\mathcal{M}) = \prod_{i\geq 0} H^{2i-n}(X,\mathcal{M})$. (See [W1, A.4].) Also for all $n \geq 0$ the Milnor sequence 2.1.2 yields $HP_n(\mathcal{M}) = HC_n(\mathcal{M})$.

Another consequence of our homological yoga is this. A short exact sequence of mixed complexes gives rise to short exact sequences of double complexes $\mathcal{B}_{**}$, and hence to long exact sequences for HH_*, HC_*, HP_* and HN_*. We also have the following general principle.

Lemma 2.5. *1) If $(\mathcal{C}, b, B)$ is a mixed complex whose underlying chain complex of sheaves $(\mathcal{C}, b)$ is acyclic, then*

$$HH_n(\mathcal{C}) = HC_n(\mathcal{C}) = HP_n(\mathcal{C}) = HN_n(\mathcal{C}) = 0 \quad \textit{for all } n.$$

2) Let $f : \mathcal{L} \to \mathcal{M}$ be a morphism of mixed complexes of sheaves on X. If f is a quasi-isomorphism on the underlying chain complexes of sheaves, then f induces an isomorphism on HH_*, HC_*, HP_* and HN_*.

Proof. 1) The double complex $\mathcal{B}_{**}$ associated to the mixed complex $\mathcal{C}$ has acyclic columns, and this implies (see [W, 2.7.3]) that the total complexes of sheaves is acyclic. By the hypercohomology spectral sequence, the hypercohomology of these acylic complexes vanishes. This gives vanishing of HH_* and HC_*; the vanishing of HP_* and HN_* is a formal consequence. Part 2) follows, since the mapping cone $\mathcal{C}$ of f is a mixed complex, acyclic as a cochain complex.

Example 2.6. (Normalized mixed complex) Let $\bar{\mathcal{C}}_*$ be the normalized chain complex $\mathcal{O}_X \otimes \bar{\mathcal{O}}_X^{\otimes n}$ associated to the complex $\mathcal{C}_*$ of 2.2; it too is a mixed complex and is quasi-isomorphic to $\mathcal{C}_*$ [L, 1.1.14] [W, 9.8.4]. Lemma 2.5 assures us that we could have defined cyclic homology using $\bar{\mathcal{C}}_*$ instead of $\mathcal{C}_*$.

Example 2.7. (de Rham theory) Setting $b = 0$, we may regard the de Rham cochain complex (Ω_X^*, d) as a trivial mixed complex $(\Omega_X^*, 0, d)$ [W, 9.8.8]. Since $(\Omega_X^*, 0)$ is the product of the sheaves $\Omega_X^p[-2p]$, its hypercohomology is the product of their cohomologies ([W1, A.4]). This yields the following product decomposition

$$HH_n(\Omega_X^*, 0, d) = H^0(X, \Omega_X^n) \times H^1(X, \Omega_X^{n+1}) \times \cdots \times H^p(\Omega_X^{n+p}) \times \cdots$$

As X is quasi-compact and quasi-separated (e.g. noetherian), this is a finite decomposition [EGA, III(1.4.12)].

There is a similar decomposition for the cyclic homology of $(\Omega_X^*, 0, d)$ and its variants. As a cochain complex, $\operatorname{Tot}\mathcal{B}_{**}(\Omega_X^*, 0, d)$ is the product of the translations $\Omega_X^{\leq i}[-2i]$ of its brutal truncations

$$\Omega_X^{\leq i}: \quad 0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^i \to 0.$$

As each term of $\operatorname{Tot}(\mathcal{B}_{**})$ only involves finitely many of these truncated complexes, $HC_n(\Omega_X^*, 0, d)$ is $\prod_{i=0}^{\infty} \mathbb{H}^{2i-n}(X, \Omega_X^{\leq i})$.

Recall from [Dix, 6.2] that the *de Rham cohomology* of X over k is defined to be $H^*_{dR}(X/k) = \mathbb{H}^*(X, \Omega^*_X)$. As $\mathbb{H}^n(X, \Omega^*_X) = \mathbb{H}^n(X, \Omega^{\le i}_X)$ for $i > n$, it follows from 2.1.2 that

$$HP_n(\Omega^*_X, 0, d) = \prod_{i=-\infty}^{\infty} H^{2i-n}_{dR}(X/k) \tag{2.7.1}$$

while $HN_n(\Omega^*_X, 0, d)$ is the product of the $\mathbb{H}^{2i-n}(X, \Omega^{\ge i}_X)$.

Hodge-to-de Rham spectral sequence 2.8. The filtration of the cochain complex Ω^*_X by the subcomplexes $\Omega^{\ge i}_X$ gives rise to a spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \Rightarrow H^*_{dR}(X/k)$$

called the *Hodge-to-de Rham* spectral sequence in [Dix, 6.4]. Since this is the same filtration as the column filtration on $\mathcal{B}_{**}(\Omega^*_X, 0, d)$, it follows that Connes' spectral sequence is the product of copies of the Hodge-to-de Rham spectral sequence.

When X is smooth and projective, the i^{th} piece of the associated *Hodge filtration* on $H^*_{dR}(X/k)$ is the image of the map $\mathbb{H}^*(X, \Omega^{\ge i}_X) \to \mathbb{H}^*(X, \Omega^*_X)$. The Hodge filtration on the cohomology of X is more complicated in the general case; see [D].

If X is smooth and projective over a field k of characteristic zero, the spectral sequence degenerates at E_1 (by Hodge theory on the Kähler manifold $X(\mathbb{C})$) to yield the usual "Hodge decomposition":

$$H^n_{dR}(X/k) = \prod_{p+q=n} H^q(X, \Omega^p_X).$$

§3. **The Hodge decomposition of cyclic homology**

We now apply the theory of mixed complexes to study the cyclic homology of X. For simplicity, we write $\mathcal{B}_{**}$ for Connes' (b, B) double complex $\mathcal{B}_{**}(\bar{\mathcal{C}}_*)$ (2.6); $HC_*(X)$ is the hypercohomology of $\mathcal{B}_{**}$. Recall that the B operator of Connes takes the reduced complex $\bar{\mathcal{C}}^{(i)}_n$ to $\bar{\mathcal{C}}^{(i+1)}_{n+1}$ [L, 4.6.7] [W, 9.8.15]. Therefore $\mathcal{B}_{**}$ breaks up as a product (and direct sum) of first quadrant double complexes $\mathcal{B}^{(i)}_{**}$ $(i \ge 0)$ with

$$\mathcal{B}^{(i)}_{pq} = \bar{\mathcal{C}}^{(i-p)}_{q-p}.$$

We set $HC^{(i)}_n(X) = \mathbb{H}^{-n}(X, \operatorname{Tot} \mathcal{B}^{(i)}_{**})$.

Lemma/Definition 3.0. There is a finite product decomposition for each n:

$$HC_n(X) = HC^{(0)}_n(X) \times HC^{(1)}_n(X) \times \cdots \times HC^{(i)}_n(X) \times \cdots$$

which we shall call the *Hodge decomposition* of $HC_n(X)$.

For an affine scheme $X = \operatorname{Spec} A$ we have $HC^{(i)}_n(X) = HC^{(i)}_n(A)$, and this is the usual Hodge decomposition of $HC_n(X) = HC_n(A)$. Of course, since $\mathcal{B}^{(0)}_{**} = \mathcal{O}_X$, we have $HC^{(0)}_*(X) = HH^{(0)}_*(X) = H^{-*}(X, \mathcal{O}_X)$, as in (1.1).

Proof. Given Proposition 1.3, the proof of [W1, 2.5] applies to show that for an affine scheme we have $HC_n^{(i)}(X) = HC_n^{(i)}(A)$. The result now follows from [W1, 0.5], as in the proof of 1.3.

The S map sends $\mathcal{B}_{**}^{(i)}$ into a translation of $\mathcal{B}_{**}^{(i-1)}$ up and to the right, which we write as $\mathcal{B}_{**}^{(i-1)}[-2]$. As in 2.1, we form the tower of hypercohomology complexes and form the inverse (= homotopy) limit $L_* = \mathrm{holim}_p \mathbb{H}(X, \mathrm{Tot}\, \mathcal{B}_{**}^{(i+p)})[2p]$. Then we define $HP_n^{(i)}(X)$ to be the (co)homology of L_*. We also define $HN_n^{(i)}(X)$ to be the (co)homology of the kernel in order to have SBI sequences like (2.1.3).

As with cyclic homology, we have decompositions $HP_n(X) = \prod HP_n^{(i)}(X)$ and $HN_n(X) = \prod HN_n^{(i)}(X)$, but this time the products are over all integers i. Of course, when $X = \mathrm{Spec}\,(A)$ is affine we see from 3.0 and (2.1.2) that $HP_n^{(i)}(X) = HP_n^{(i)}(A)$ and $HN_n^{(i)}(X) = HN_n^{(i)}(A)$.

The affine version of the following result is due to Feigin and Tsygan [FT, 6.5].

Proposition 3.1. *If X is locally a complete intersection, then*

$$HC_n^{(i)}(X) = 0 \qquad \textit{whenever} \quad i < n/2.$$

Proof. We proceed by induction on i, having just checked the case $i = 0$. The SBI sequence decomposes to yield the exact sequence

$$HH_n^{(i)}(X) \xrightarrow{I} HC_n^{(i)}(X) \xrightarrow{S} HC_{n-2}^{(i-1)}(X).$$

If $n > 2i$ then the outside terms vanish, by induction and by 1.5. Done.

We now turn to the relation with de Rham cohomology. As in the affine case, there is a morphism of mixed complexes of sheaves $e : \mathcal{C}_* \to (\Omega_X^*, 0, d)$ defined by $e(a_0 \otimes ...) = a_0\, da_1 \wedge \cdots \wedge da_n/n!$. This map sends $\mathcal{C}_*^{(i)}$ to Ω_X^i [W, 9.8.12,15]. By 2.6, it sends $\mathcal{B}_{**}^{(i)}$ to $\Omega_X^{\leq i}[-2i]$. Therefore e induces maps

$$\text{(3.2)} \qquad HH_n^{(i)}(X) \to H^{i-n}(X, \Omega_X^i) \quad ... \quad \text{and} \quad HP_n^{(i)}(X) \to H_{dR}^{2i-n}(X/k).$$

Theorem 3.3. *If X is smooth over k (and $\mathbb{Q} \subseteq k$) then $HH_n^{(i)}(X) \cong H^{i-n}(X, \Omega_X^i)$,*

$$HP_n^{(i)}(X) \cong H_{dR}^{2i-n}(X/k),$$

$$HC_n^{(i)}(X) \cong \mathbb{H}^{2i-n}(X, \Omega_X^{\leq i}) \quad \textit{and} \quad HN_n^{(i)}(X) \cong \mathbb{H}^{2i-n}(X, \Omega_X^{\geq i}).$$

Proof. Combine 1.4 and (3.2) with 2.5(2), 2.7 and 3.0.

If X is smooth over $k = \mathbb{C}$, Grothendieck's Comparison Theorem [GdR] gives an isomorphism between $HP_n^{(i)}(X)$ and the Betti cohomology $H_{top}^{2i-n}(X(\mathbb{C}); \mathbb{C})$ of the underlying complex manifold $X(\mathbb{C})$.

Example 3.3.1 (Smooth projective curves). If X is a smooth curve then $HC_n^{(0)}(X) = H^{-n}(X, \mathcal{O}_X)$ and $HC_n^{(1)}(X) \cong HP_n^{(1)}(X) \cong H_{dR}^{2-n}(X/k)$. In particular, if X is projective with genus g then $HC_1(X) = HC_1^{(1)}(X) \cong k^{2g}$ and $HC_{-1}(X) = HC_{-1}^{(0)}(X) \cong k^g$. We will return to this calculation in Examples 3.6 and 4.4.1.

When X is a singular variety over a field of characteristic 0, the de Rham cohomology $H_{dR}^*(X/k)$ is no longer the relevant theory. The useful theory is Hartshorne's "algebraic de Rham cohomology" [H], which we write as $H_{cris}^*(X/k)$ because (as observed in [Gcr]) it agrees with the cristalline cohomology of the structure sheaf in the crystalline topos for X. When the field is $\mathbb{C}$, Hartshorne proved in [H, IV.1.1] that $H_{cris}^*(X/k)$ is isomorphic to the Betti cohomology $H_{top}^*(X(\mathbb{C}); \mathbb{C})$ of the underlying topological space $X(\mathbb{C})$. If $X = \mathrm{Spec}\,(A)$, Feigin and Tsygan proved that $HP_n^{(i)}(A) \cong H_{cris}^{2i-n}(A/k)$ (see [FT, 6.1][E][KW]). The following theorem promotes their result to schemes.

Comparison Theorem 3.4. *When X is a quasi-projective scheme over a field k of characteristic zero, the periodic cyclic homology of X is naturally isomorphic to the crystalline cohomology of X:*

$$HP_n^{(i)}(X) \cong H_{cris}^{2i-n}(X/k).$$

In particular, if $k = \mathbb{C}$ it is isomorphic to the Betti cohomology of $X(\mathbb{C})$:

$$HP_n^{(i)}(X) \cong H_{top}^{2i-n}(X(\mathbb{C}); \mathbb{C}).$$

Proof. Using the Comparison Theorem for the Čech cohomology spectral sequences (and [H, II.7.3]), the global version follows from the affine version of Feigin and Tsygan, once we construct a natural map from $HP_n^{(i)}(X)$ to $H_{cris}^{2i-n}(X/k)$. This construction will be modelled on the construction of [FT, 6.1.1(1)]. The reader may wish to compare our construction with the analysis in [Em][KW].

Embed X in a smooth variety S. Each affine open $\mathrm{Spec}\,(R)$ of S contains an affine open $\mathrm{Spec}\,(A)$ of X. By standard techniques we can construct a quasi-isomorphism $\mathcal{P} \to \mathcal{O}_X$, where $\mathcal{P} = Sym_S(\oplus V_i)$ is a free differential graded-commutative $\mathcal{O}_S$-algebra on a sequence $V_1, V_2, \ldots$ of vector bundles on S.

The algebra $\Omega_{\mathcal{P}}$ of differential forms may be thought of as a second quadrant homology double complex containing Ω_S on the x-axis; Feigin and Tsygan observed in [FT, 6.3.2] that, after index shifting, its brutal truncations form models for the cyclic homology of each affine open subset of X. In [FT, 6.3.3] they extend the filtration F^p on Ω_S^* to a filtration J^p such that the maps $\Omega_S^*/F^p \to \Omega_{\mathcal{P}}/J^p$ are all quasi-isomorphisms. This yields a natural chain of morphisms between towers of complexes whose p^{th} term is

$$\mathrm{Tot}\,\mathcal{B}_{**}^{(p)}(\mathcal{O}_X)[2p] \xleftarrow{\sim} \mathrm{Tot}\,\sigma_p\Omega_{\mathcal{B}} \to \mathrm{Tot}\,\sigma_p\Omega_{\mathcal{B}}/J^p \xleftarrow{\sim} \Omega_S^*/F^p\Omega_S^*.$$

Now take hypercohomology to get a chain of morphisms between towers of k-module complexes. The homotopy limit of the first tower represents the component

$HP^{(0)}_*(X)$ of the periodic cyclic homology of X. The homotopy limit of the last tower is just the hypercohomology of the inverse limit $\hat{\Omega}^*_S$, which by [H, II.1] represents the algebraic de Rham cohomology $H^*_{cris}(X/k) = H^*(\hat{X}, \hat{\Omega}^*_S)$. This yields the desired map from $HP^{(i)}_n(X) \cong HP^{(0)}_{n-2i}(X)$ to $H^{2i-n}_{cris}(X/k)$.

Corollary 3.4.1. *(Emmanouil [Em]) If X is quasi-projective over a field of characteristic zero, then every tower* $\cdots \xrightarrow{S} HC_{n+2}(X) \xrightarrow{S} HC_n(X)$ *satisfies the Mittag-Leffler condition and hence* $HP_n(X) = \varprojlim_p HC_{n+2p}(X)$.

Proof. By [H, II.6.1] the $H^{2i-n}(X/k) = HP^{(i)}_n(X)$ are finite-dimensional vector spaces; by [H, II.7.2.1] they are nonzero for only finitely many i. Hence each $HP_n(X)$ is finite-dimensional. The vector spaces $\varprojlim{}^1 HC_{n+2p+1}(X)$ are also finite by Lemma 2.1.2. The result now follows from a theorem of B. Gray (see [Em]).

Localization of smooth schemes 3.5. Let X be a smooth scheme of dimension d over k. Hartshorne [H, II.3] defines the *de Rham homology* $H^{dR}_q(Z)$ of a closed subscheme Z to be hypercohomology with supports $\mathbf{H}^{2d-q}_Z(X, \Omega^*_X)$. Thus it is the relative term in the long exact sequence

$$\cdots \to H^{dR}_q(Z) \to H^{2d-q}_{dR}(X) \to H^{2d-q}_{dR}(X-Z) \to H^{dR}_{q-1}(Z) \to H^{2d-q+1}_{dR}(X) \to \cdots .$$

Using 3.3, it is also the relative term in the periodic cyclic homology sequence:

$$(3.5.1) \quad HP^{(i)}_{n+1}(X{-}Z) \to H^{dR}_{\substack{2d-2i\\+n}}(Z) \to HP^{(i)}_n(X) \to HP^{(i)}_n(X{-}Z) \to H^{dR}_{\substack{2d-2i\\+n-1}}(Z)$$

If in addition Z is also smooth, of codimension c (and hence dimension $d-c$), then the above sequence simplifies into the more attractive sequence:

$$(3.5.2) \quad HP^{(i)}_{n+1}(X{-}Z) \to HP^{(i-c)}_n(Z) \to HP^{(i)}_n(X) \to HP^{(i)}_n(X{-}Z) \to HP^{(i-c)}_{n-1}(Z)$$

Example 3.6 (Smooth affine curves). Let A be the coordinate ring of a smooth affine curve X obtained from a smooth projective curve $\bar{X}$ by removing $n \geq 1$ points, whose corresponding residue fields are $k_1, \dots, k_n$. Suppose for simplicity that $\bar{X}$ is connected with genus g, so that $HC_1(\bar{X}) = HP_1(\bar{X}) = k^{2g}$ by 3.3.1. Since $HC_1(A) = HP_1(A)$ and the maps $H^{dR}_0(k_i) \to H^{dR}_0(\bar{X})$ are the trace maps $k_i \to k$ by [H, II.3.1], the Localization Sequence (3.5.2) (or [H, II.3.3]) translates into the exact sequence

$$(3.6.1) \qquad 0 \to k^{2g} \to HC_1(A) \to \coprod_{i=1}^n k_i \xrightarrow{\text{trace}} k \to 0.$$

The image of the first map contains the g global holomorphic differentials on $\bar{X}$, considered as elements of $HC_1(A) = \Omega_A/dA$. In particular, if $k = \mathbb{C}$ (or if $k_i = k$ for all i) then the dimension of $HC_1(A)$ is $2g + n - 1$.

As an exercise, the reader might want to check this for the hyperelliptic plane curve $A = \mathbb{C}[x, y]/(y^2 = f(x))$, f a degree d polynomial with distinct roots. In this case the genus is $[\frac{d-1}{2}]$, and there are either 1 or 2 points on $\bar{X} - X$. Thus $\dim HC_1(A) = d - 1$.

To see the effect of the trace, consider the coordinate ring of the circle $A = \mathbb{R}[x, y]/(x^2 + y^2 = 1)$. Since there is 1 point at infinity with residue field $\mathbb{C}$, and $g = 0$, this yields $HC_1(A) = \mathbb{R}(1)$, the kernel of the trace $\mathbb{C} \to \mathbb{R}$.

Localization Theorem 3.7. *Let X be a d-dimensional quasi-projective scheme X, and suppose that Z is a closed subscheme disjoint from the singular locus of X. Then the relative term in the localization sequence for periodic cyclic homology is Hartshorne's algebraic de Rham homology, with a shift of $s = 2d - 2i$. That is, the sequence (3.5.1) is exact:*

$$\cdots HP^{(i)}_{n+1}(X{-}Z) \to H^{dR}_{s+n}(Z) \to HP^{(i)}_n(X) \to HP^{(i)}_n(X{-}Z) \to H^{dR}_{s+n-1}(Z) \cdots .$$

If in addition Z is smooth, the relative term is also the periodic cyclic homology of Z, with the Hodge index shifted by the codimension c of Z in X. That is, the sequence (3.5.2) is exact:

$$\cdots HP^{(i)}_{n+1}(X{-}Z) \to HP^{(i-c)}_n(Z) \to HP^{(i)}_n(X) \to HP^{(i)}_n(X{-}Z) \to HP^{(i-c)}_{n-1}(Z) \cdots .$$

Proof. Set $Y = \mathrm{Sing}(X)$. It suffices by 3.5 to show that the relative terms agree for the maps $H^*_{cris}(X/k) \to H^*_{cris}(X - Z/k)$ and $H^*_{cris}(X - Y/k) \to H^*_{cris}(X - Y - Z/k)$. This follows from the Mayer-Vietoris sequence of [H, II.7.3]:

$$\cdots H^*_{cris}(X) \to H^*_{cris}(X-Y) \oplus H^*_{cris}(X-Z) \to H^*_{cris}(X-Y-Z) \to H^*_{cris}(X) \cdots .$$

More directly, one could embed X in a smooth variety S and set $U = S - Y$. The two relative terms in question are the hypercohomology groups with supports: $\mathbb{H}^*_Z(\hat{S}, \hat{\Omega}^*_S)$ and $\mathbb{H}^*_Z(\hat{U}, \hat{\Omega}^*_U)$. By excision, these are isomorphic, so we are done.

Remark. If Z is projective but not smooth, then by [H, II.5.1] we know that $H^{dR}_q(Z)$ and $H^q_{cris}(Z/k)$ are dual vector spaces. We leave the transformation of the localization sequences of 3.7 in the general case to the reader.

If Z is not even projective, then by [H, IV.1.2] the de Rham homology of Z over $\mathbb{C}$ is the same as the Borel-Moore homology of the locally compact topological space $Z(\mathbb{C})$; this is the same as homology with locally compact supports because the coefficients are taken in $\mathbb{C}$.

Chern characters 3.8 We conclude this section with a discussion of Chern characters. There is a Chern character $ch : K_0(X) \to HN_0(X)$ for schemes, generalizing the usual affine Chern character $ch : K_0(A) \to HN_0(A)$ [L, 8.3.8]. It is described in [WG, 4.4], and is obtained by sheafifying the higher Chern character [L, 11.4.2] and taking hypercohomology as in [T].

The composition of ch with $HN_0(X) \to HH_0(X)$ yields a natural generalization to schemes of the classical Hattori-Stallings trace map $K_0(A) \to HH_0(A)$. We shall call it the *Dennis trace map*, since it is obtained by sheafifying the Dennis trace map (for higher K-theory) and taking hypercohomology as in [T]. We will see below (in 4.5) that for X smooth and projective the Dennis trace map completely determines the classical Chern classes of X.

The composition of ch with $HN_0(X) \to HP_0(X)$ is a generalization to schemes of the Connes-Karoubi Chern character. Note that $HP_0(X) = \prod H^{2i}_{cris}(X/k)$ by the Comparison Theorem 3.4.

Proposition 3.8.1. *For quasiprojective schemes, the Connes-Karoubi Chern character $K_0(X) \xrightarrow{ch} HP_0(X) = \prod H^{2i}_{cris}(X/k)$ agrees with the Chern character associated to the Chern classes constructed by Hartshorne in [H, II.7.9].*

Proof. If X is smooth, this is [W2, Thm.1]. The general case follows from Jouanalou's device and [W2, Lemme 6].

§4. Hodge filtration for Smooth Projective Varieties

If X is a projective variety over a field k of characteristic 0, there is a connection between the filtration on $H^m(X/k)$ defined by the system of maps

$$H^m(X/k) \to \cdots \to HC^{(i)}_{2i-m}(X) \to HC^{(i-1)}_{2i-m-2}(X) \to \cdots$$

defined by the comparison theorems in §3, and the *Hodge filtration* $H^m = F^0 \supseteq F^1 \supseteq \cdots$ on $H^m_{top}(X(\mathbb{C}), \mathbb{C})$ defined by Deligne in [D]. This connection is clearest for smooth X, where $H^m(X/k)$ is de Rham cohomology, and is given by the following result.

Proposition 4.1 (Smooth projective varieties). *If X is a smooth projective variety over a field k (and char$(k) = 0$) then:*

a) *The i^{th} part $F^iH^m(X/k)$ of the Hodge filtration on $H^m_{dR}(X/k)$ is isomorphic to $HN^{(i)}_{2i-m}(X)$, via the map $I : HN^{(i)}_{2i-m}(X) \to HP^{(i)}_{2i-m}(X) \cong H^m_{dR}(X/k)$.*

b) *The SBI sequences break up into short exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & HN_n(X) & \xrightarrow{I} & HP_n(X) & \xrightarrow{S} & HC_{n-2}(X) & \longrightarrow & 0 \\ & & \text{onto}\downarrow & & \downarrow\text{onto} & & \downarrow = & & \\ 0 & \longrightarrow & HH_n(X) & \xrightarrow{I} & HC_n(X) & \xrightarrow{S} & HC_{n-2}(X) & \longrightarrow & 0 \end{array}$$

c) *$HP^{(i)}_n(X)$ is the product of the $H^q(X, \Omega^p_X)$, $p + q = 2i - n$, and*

$$HN^{(i)}_n(X) = \prod_{\substack{p+q=2i-n \\ p\geq i}} H^q(X, \Omega^p_X), \qquad HC^{(i)}_n(X) = \prod_{\substack{p+q=2i-n \\ p\leq i}} H^q(X, \Omega^p_X).$$

Proof. In this case the Hodge-to-de Rham spectral sequence 2.8 degenerates. Thus $HP^{(i)}_n(X)$, $HC^{(i)}_n(X)$ and $HN^{(i)}_n(X)$ are direct products, as described in c). Part b) follows by taking the product over the Hodge components. Assertion a) about the i^{th} part of the Hodge filtration of $H^m_{dR}(X/k)$ follows from this and 2.8.

Remark 4.2. When X is a smooth affine variety over $\mathbb{C}$ none of 4.1 holds, even though we still have $H^*_{dR}(X/\mathbb{C}) \cong H^*_{top}(X(\mathbb{C}); \mathbb{C})$ by [GdR]. It is well-known that the SBI sequence breaks up differently, with $HC^{(n)}_n(X) \cong \Omega^n_A/d\Omega^{n-1}_A$. Moreover the Hodge filtration (see [D, 3.2.2]) is more complicated and has less of a relation to cyclic homology. Consider for example a smooth affine curve $X = \operatorname{Spec} A$ of genus $g \neq 0$ over $\mathbb{C}$. The Hodge filtration on $H^1_{dR}(X/\mathbb{C})$ is nontrivial by [D, 8.2.5] yet, as we see from Example 3.6, the map from $HN^{(1)}_1(X) = \Omega_A$ to $HP^{(1)}_1(X) = H^1_{dR}(X) \cong \Omega_A/dA$ is onto with kernel $A/\mathbb{C}$. In this case, the Hodge-to-de Rham spectral sequence 2.8 degenerates only at E_2, unlike the spectral sequence for the Hodge filtration, which degenerates at E_1 by [D, 3.2.13].

The Hodge Diamond 4.3. Let X be a smooth projective variety. It is traditional to display the Hodge numbers $h^{pq} = \dim H^q(X, \Omega^p_X)$ of X in the shape of a diamond, so that $n = p - q$ is constant in each column and $m = p + q$ is constant in each row, with the $p = (m+n)/2$ increasing from left to right as well as down to up. (See examples 4.4 below.) In this format, the m^{th} row displays the classical Hodge decomposition of $H^m(X(\mathbb{C}); \mathbb{C})$, as well as the Hodge filtration; $\dim F^i H^m$ is the sum of the terms in the m^{th} row which lie in columns $n \geq (2i - m)$.

The Hodge decomposition of $HH_n(X)$ is given by the n^{th} column of the Hodge Diamond. Indeed, the entry in row $m = 2i - n$ of the n^{th} column is $h^{i-n,n}$, which by 1.4 is the dimension of $HH_n^{(i)}(X)$. Serre Duality implies that $HH_{+n}(X)$ and $HH_{-n}(X)$ are dual vector spaces.

The rows of the Hodge diamond describe the Hodge decomposition of $HC_n(X)$ as follows. Fix n and i, and set $m = 2i - n$. By Proposition 4.1, the sum of the terms in the m^{th} row to the left and including the n^{th} column gives the dimension of $HC_n^{(i)}(X)$. Thus the collection of all terms in columns $\leq n$ gives $HC_n(X)$, and the Hodge decomposition of $HC_n(X)$ is given by the rows of this truncated diamond. Of course, the S maps correspond to truncation, and the sum of *all* the terms in the m^{th} row is the dimension of $HP_n^{(i)}(X) \cong H^m(X(\mathbb{C}); \mathbb{C})$.

Example 4.4.1 (Smooth projective curves). Let X be a smooth projective curve of genus g over a field k of characteristic 0. From the Hodge diamond

	1		$H^2(X;\mathbb{C})$
g		g	$H^1(X;\mathbb{C})$
	1		$H^0(X;\mathbb{C})$
HH_{-1}	HH_0	HH_1	

and 4.1 we see that $HH_{+1}(X) \cong HH_{-1}(X) \cong k^g$, $HP_1(X) = HC_1(X) = k^{2g}$ but $HC_{-1}(X) = k^g$, and that

$$HP_0(X) = HC_0(X) \cong HH_0(X) \cong k \oplus k, \quad \text{but} \quad HC_{-2}(X) \cong k.$$

Example 4.4.2 (Smooth projective surfaces). Consider a smooth projective surface X of genus g and irregularity q. The Hodge diamond looks like this:

		h^{22}			$H^4(X;\mathbb{C})$
	h^{12}		h^{21}		$H^3(X;\mathbb{C})$
h^{02}		h^{11}		h^{20}	$H^2(X;\mathbb{C})$
	h^{01}		h^{10}		$H^1(X;\mathbb{C})$
		h^{00}			$H^0(X;\mathbb{C})$
HH_{-2}	HH_{-1}	HH_0	HH_1	HH_2	

Since $g = h^{02} = h^{20}$ and $q = h^{01} = h^{12} = h^{10} = h^{21}$ we have

$$HH_2(X) \cong HH_{-2}(X) \cong k^g, \qquad HH_1(X) \cong HH_{-1}(X) \cong k^{2q},$$

and $HH_0(X) = k \oplus HH_0^{(1)}(X) \oplus k$ with $HH_0^{(1)}(X) = H^1(X, \Omega_X) = k^{h^{11}}$. The classical group of divisors modulo numerical equivalence is free abelian of rank h^{11} and injects into $HH_0^{(1)}(X)$ under the Chern class c_1. The sequence of S-maps

$$H^2(X;\mathbb{C}) = HC_2^{(2)}(X) \twoheadrightarrow HC_0^{(1)}(X) \twoheadrightarrow HC_{-2}^{(0)}(X) = k^g$$

give the Hodge filtration of $H^2(X;\mathbb{C})$.

Proposition 4.5. *When X is a smooth projective variety, the classical Chern character $K_0(X) \to \prod H_{dR}^{2i}(X)$ factors through the Dennis trace map*

$$K_0(X) \to HH_0(X) = \prod_{i=0}^{\dim X} H^i(X, \Omega_X^i).$$

Proof. Since the classical Chern classes

$$c_i : K_0(X) \to H_{dR}^{2i}(X)$$

land in the submodules $HH_0^{(i)}(X) = H^i(X, \Omega_X^i)$, it follows from 3.8.1 that the image of *ch* lies in the submodule $HH_0(X) = \prod H^i(X, \Omega_X^i)$ of $HN_0(X)$. Since the composition of *ch* with $HN_0(X) \to HH_0(X)$ is the Dennis trace map, we are done.

§5 Hodge Structure for Singular Varieties

Let X be a singular projective variety over $\mathbb{C}$. The *Hodge filtration* F^iH^m on $H^m(X, \mathbb{C}) = H_{top}^m(X(\mathbb{C}), \mathbb{C})$ was defined by Deligne in [D, 8.2.2] as a part of the mixed Hodge structure on $H^m(X;\mathbb{Z})$. To define it, we choose a smooth proper hypercovering $\pi : X_\bullet \to X$, as in [D, 6.2.8], and observe that $H^m(X, \mathbb{C}) = \mathbb{H}^m(X_\bullet; \Omega_{X_\bullet}^*)$ by [D, 5.3.5(V)]. The Hodge filtration is given by the hypercohomology of the brutal truncations $\Omega_{X_\bullet}^{\geq i}$ of the simplicial cochain complex $\Omega_{X_\bullet}^*$ [D, 8.1.8]. We remark that the mixed Hodge structure on $H^m(X, \mathbb{Z})$ is defined in [D, 8.1.12, 8.1.19 and 8.2.2], using the cohomological mixed Hodge structure on the constant sheaf $\mathbb{Z}$.

Theorem 5.1. *Let X be a projective variety over $\mathbb{C}$, and fix m, n with $m+n = 2i$.*
a) The image of $HN_n^{(i)}(X) \to HP_n^{(i)}(X) \cong H^m(X, \mathbb{C})$ is contained in $F^iH^m(X, \mathbb{C})$.
b) $HC_n^{(i)}(X)$ surjects onto $H^m(X, \mathbb{C})/F^{i+1}H^m$.
c) If $i \geq m$ then $H^m(X, \mathbb{C}) = HP_n^{(i)}(X)$ injects into $HC_n^{(i)}(X)$.

Proof. The mixed complex $\bar{\mathcal{C}}_*^{(i)}(X_\bullet)$ is isomorphic to the de Rham mixed complex $\Omega_{X_\bullet}^*$ of Example 2.7, by the maps of (3.2). Hence $\operatorname{Tot} \mathcal{B}_{**}^{(i)}(X_\bullet)$ is quasi-isomorphic to the truncated de Rham complex $\Omega_{X_\bullet}^{\leq i}[-2i]$ on $X_\bullet$. Hence there are natural maps

$$HN_n^{(i)}(X) \to \mathbb{H}^m(X, \Omega_X^{\geq i}) \to \mathbb{H}(X_\bullet, \Omega_{X_\bullet}^{\geq i}) = F^iH^m(X, \mathbb{C}),$$

$$HC_n^{(i)}(X) = \mathbb{H}^{-n}(X, \operatorname{Tot} \mathcal{B}_{**}^{(i)} X) \to \mathbb{H}^{-n}(X_\bullet, \operatorname{Tot} \mathcal{B}_{**}^{(i)}(X_\bullet))$$

$$= \mathbb{H}^{2i-n}(X_\bullet, \Omega_{X_\bullet}^{\leq i}) = H^m(X, \mathbb{C})/F^{i+1}H^m.$$

Part (c) follows from (b) and the vanishing of $F^{m+1}H^m(X, \mathbb{C})$ [D, 8.2.4(iii)].

Corollary 5.2. *Let X be a connected singular projective variety with c irreducible components. If $\dim X = d$ then $HP_0^{(d)}(X) = \mathbb{C}^c$ injects into $HC_0^{(d)}(X)$.*

Proof. The Hodge filtration on $H^{2d}(X,\mathbb{C})$ has $F^{d+1}H^{2d} = 0$, $F^dH^{2d} = H^{2d} = \mathbb{C}^c$.

Definition 5.3. The *SBI filtration* $H^m = G^0 \supseteq G^1 \supseteq \cdots$ on $H^m = H^m(X,\mathbb{C})$ is defined by: G^iH^m is the image of $HN_n^{(i)}(X) \to HP_n^{(i)}(X) \cong H^m(X,\mathbb{C})$, $m+n = 2i$. Thus $H^m/G^{i+1}H^m$ is the image of H^m in $HC_n^{(i)}(X)$.

By 5.1, the SBI filtration is always contained in the Hodge filtration of a projective variety, and $G^{m+1}H^m = 0$. By 4.2, the SBI filtration coincides with the Hodge filtration for smooth projective varieties. The SBI and Hodge filtrations agree on H^0, where they are trivial. The SBI filtration also coincides with the Hodge filtration on $H^{2\dim X}$ by 5.2. This raises the following question.

Question 5.3.1. When does the SBI filtration coincide with the Hodge filtration on a projective variety X? Equivalently, when is the image of $H^m(X,\mathbb{C})$ in $HC_n^{(i)}(X)$ exactly $H^m/F^{i+1}H^m$? The next example shows that the two filtrations are not always equal.

Example 5.4. (H^1 of Projective Curves) Let X be a connected, reduced singular projective curve. Then H^1 injects into $HC_1^{(1)}(X)$ by 5.1(c), so $G^2H^1 = F^2H^1 = 0$. We claim that the inclusion $G^1H^1 \subseteq F^1H^1$ is not always equality, i.e., that the image of F^1H^1 in $H^1/G^1H^1 \subseteq HC_{-1}^{(0)}(X) = H^1(X,\mathcal{O}_X)$ can be nontrivial.

For this, we need to analyze the mixed Hodge structure on $H^1(X,\mathbb{C})$. If X has c irreducible components then so does the normalization $\tilde{X}$ of X; let $g_1,\dots,g_c$ be the genera of these components. The singular locus Y of X is finite, and so is its inverse image $\tilde{Y} = Y \times_X \tilde{X}$ in $\tilde{X}$. By [H, II.4.4] and 4.4.1, $HP_1(X) = H^1(X,\mathbb{C})$ fits into an extension

$$0 \to W_0 \to HP_1(X) \to \oplus\mathbb{C}^{2g_i} \to 0, \qquad \dim W_0 = 1 + |\tilde{Y}| - |Y| - c.$$

By [D, 8.2.5], W_0 is the weight 0 subspace for the mixed Hodge structure of $H^1(X,\mathbb{C})$. In fact, W_0 is the subspace H^{00} and its dimension is the Hodge number h^{00} [D, 8.2.4]. In this case the Hodge filtration has $F^1H^1 = \oplus\mathbb{C}^{g_i}$ by [D, 8.2.5].

Since X is reduced and proper, $HC_0^{(0)}(X) = H^0(X,\mathcal{O}_X) = \mathbb{C}$. The SBI sequence

$$HC_0^{(0)}(X) \xrightarrow{B} HH_1^{(1)}(X) \to HC_1^{(1)}(X) \xrightarrow{S} HC_{-1}^{(0)}(X) \xrightarrow{B} HH_0^{(1)}(X)$$

translates via Example 1.6 into the exact sequence

$$\mathbb{C} \xrightarrow{0} H^0(X,\Omega_X) \to HC_1^{(1)}(X) \xrightarrow{S} H^1(X,\mathcal{O}_X) \xrightarrow{d} H^1(X,\Omega_X).$$

Comparing with 4.4.1, we see that the image of F^1H^1 in $H^1/G^1 \subseteq H^1(X,\mathcal{O}_X)$ is isomorphic to the cokernel of $H^0(X,\Omega_X) \to H^0(\tilde{X},\Omega_{\tilde{X}})$.

Let X be the projective cusp obtained from the elliptic curve $y^2z = x^3 - xz^2$ by glueing the origin $(0:0:1)$ to itself. Then $Y = \operatorname{Spec}\mathbb{C}$ and $\tilde{Y} = \operatorname{Spec}\mathbb{C}[\varepsilon]$, and the canonical 1-form $\omega = dy$ generating $H^0(\tilde{X},\Omega_{\tilde{X}}) = \mathbb{C}$ cannot come from $H^0(X,\Omega_X)$ because it maps to $d\varepsilon \neq 0$ in $\Omega_{\tilde{Y}/Y}$. For this curve X we therefore have $G^1H^1 = 0$ but $F^1H^1 \cong \mathbb{C}$.

Example 5.5. Now let $U = \mathrm{Spec}\,(A)$ be a singular affine curve, obtained from a (singular) projective curve X by removing n smooth points. The cyclic homology of X is given by the previous example. Suppose for simplicity that U is connected and has c irreducible components. Since $H^m(U) = 0$ for $m \geq 2$, the localization sequence of 3.7 and 3.4 yields $HP_0(A) = HP_0^{(0)}(A) = k$, and fits $HP_1(A) = HP_1^{(1)}(A)$ into an exact sequence

$$0 \to HP_1(X) \to HP_1(A) \to k^n \xrightarrow{\delta} k^c \to 0,$$

where δ is the incidence matrix (of points and irreducible components). Since δ is onto, $HP_1(A) \cong HP_1(X) \oplus k^{n-c}$. Thus $HP_*(A)$ completely determines $HP_*(X)$.

Every vector space $HC_n(A)$ is finite-dimensional (when A is a curve). This is a consequence of Example 3.6, the analytic isomorphisms sequence of [GRW, A.1], and the calculations of [GRW]. The method of [GRW, §6] allows us to compute the cyclic homology of all seminormal affine curves, and in principle of all curves.

A lower bound for $HC_{2i-1}(A)$ is given by $HP_1(A) = HP_1^{(1)}(A)$, because the map $HP_1(A) \to HC_i^{(2i-1)}(A) \to HC_1(A)$ is an injection. (This theorem is proven in [Em, Remark 4] [KW, 5.4]; it follows from the main theorem of [BH] and the fact that $HC_1(A)$ contains $H^1_{dR}(A/\mathbb{C})$.) And the calculations of [GRW] show that $HC_1(A)$ is bigger than $HP_1^{(1)}(A)$ in general.

Remark 5.6. For curves, the subgroup $H^1(X/k) = HP_1(X)$ is the obstruction to naïve étale descent for periodic cyclic homology in the sense of [WG]. This is illustrated by the coordinate ring A of a line and a parabola in the plane. In this case $n = c = 2$ and we have $HP_1(A) = k$, with $HC_{2p+1}(A) = HP_1(A) \oplus k^2$ for all $p \geq 0$. (This example was discussed in [WG, 3.1].)

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