The canonical construction of mixed sheaves on mixed Shimura varieties

Jörg Wildeshaus
Mathematisches Institut
der Universität Münster
Einsteinstr. 62
D–48149 Münster*

---

*current address: Dept. of Math. Sciences, South Road, GB–Durham DH1 3LE
e–mail: wildesh@math.uni-muenster.de

Introduction

Given a pure Shimura variety $M^L(G,\mathcal{H})$, it is rather well known how to construct functors associating to a representation $\mathbf{V}$ of G an l–adic sheaf $\mu_{L,l}(\mathbf{V})$ on $M^L(G,\mathcal{H})$ and a variation of Hodge structure $\mu_{L,\infty}(\mathbf{V})$ on $M^L(G,\mathcal{H})(\mathbb{C})$. Milne's results on canonical models of standard principal bundles ([M], III, §§ 4, 5) allow one to show that the vector bundle underlying $\mu_{L,\infty}(\mathbf{V})$ has a model over the reflex field $E(G,\mathcal{H})$, and that the flat connection and the weight and Hodge filtrations descend to this model, giving rise to a bifiltered flat vector bundle $\mu_{L,DR}(\mathbf{V})$ on $M^L(G,\mathcal{H})$.

In this article, we study the analogous functors in the context of mixed Shimura varieties as defined in [P1]. Let W denote the unipotent radical of the underlying group P. The universal envelope of Lie$W$, completed with respect to the augmentation ideal, is denoted by $\hat{\mathfrak{U}}(\mathrm{Lie}W)$. Since $\mathrm{Rep}_{\mathbb{Q}}(P)$ is generated by $\mathrm{Rep}_{\mathbb{Q}}(P/W)$ and the finite–dimensional subquotients of $\hat{\mathfrak{U}}(\mathrm{Lie}W)$, it appears natural to consider the values of the "canonical construction" functors on $\hat{\mathfrak{U}}(\mathrm{Lie}W)$. For example, admissibility ([Ka]) of the variations of Hodge structure coming from representations of P/W follows automatically from Schmid's Nilpotent Orbit Theorem ([Sch], Theorem 4.9), since all these variations are merely direct sums of their weight–graded parts. By contrast, $\mu_{K,\infty}(\hat{\mathfrak{U}}(\mathrm{Lie}W))$ is as mixed as one can get by applying the canonical construction. The proof of admissibility of this pro–variation is one of the main results of this work.

The central observation, that will simplify our task, is that $\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}W))$ coincides with the generic pro–sheaf of [W2], § 3 for the relative situation given by the projection $[\pi] : M^K(P,\mathfrak{X}) \longrightarrow M^{\pi(K)}(P/W,\mathcal{H})$ to the underlying pure Shimura variety. It therefore has a lot of desirable properties, which will enable us to show that the canonical construction is just as well behaved as in the pure case. We decided to rename $\mu_{K,-}(\hat{\mathfrak{U}}(\mathrm{Lie}W))$ and call it the *logarithmic pro–sheaf.* The motivation for reserving this name of the generic pro–sheaf for the context of Shimura varieties is the following: the simplest non–trivial case of a mixed Shimura variety is given by the trivial torus $\mathbb{G}_{m,\mathbb{Q}}$ over the "pure Shimura variety" Spec($\mathbb{Q}$). The entries of the period matrix of the logarithmic pro–variation of Hodge structure are essentially powers of the multivalued function $\frac{1}{2\pi i}\cdot log$ on $\mathbb{G}_m(\mathbb{C})$. Since its values at roots of unity are rational numbers, the fibres of $\mu_{K,\infty}(\hat{\mathfrak{U}}(\mathrm{Lie}W))$ at such roots of unity are canonically equal to the direct product of their weight–graded objects. Now $\mathbb{G}_{m,\mathrm{tors}}$ is precisely the union of the *pure* sub–Shimura varieties of $\mathbb{G}_m$, and the above "splitting principle" of the logarithmic pro–sheaf over this union is in fact prototypical for all mixed Shimura varieties. In this sense, we like to think of the splitting principle as being a generalization of the fact that

$$log(\mathbb{G}_m(\mathbb{C})_{\mathrm{tors}}) \subset 2\pi i \cdot \mathbb{Q} \, .$$

On the other hand, we don't expect the generic pro–sheaf for arbitrary morphisms to split over a Zariski–dense subset unless the fibres of the morphism in question are of a specific shape.

§ 1 starts with a collection of results of [P1], which we hope is self–contained enough to provide non–experts with an idea of the basic concepts underlying the theory of mixed Shimura varieties.

We then recall the Hodge version of the canonical construction. While the definition is rather straightforward, by far the best part of § 2 is taken up by the proof of the fact mentioned earlier, that $\mu_{K,\infty}(\hat{\mathfrak{U}}(\mathrm{Lie}W))$ is the generic sheaf for $[\pi]$ (Theorem 2.1). The proof of admissibility of all variations $\mu_{K,\infty}(\mathbf{V})$ is then a rather formal matter (Theorem 2.2).

In § 3, we define the de Rham version of μ_K. Theorem 2.1 allows us to use the results of [W2], § 3 and hence reduce ourselves to the pure case, which is covered by [M], III, §§ 4, 5.

§ 4 treats the λ–adic component of μ_K. Again, the definition of the λ–adic sheaves poses no problem. We state a conjecture analogous to [P2], Conjecture 5.4.1, which amounts to saying that the sheaves $\mu_{K,\lambda}(\mathbf{V})$ are mixed in the sense of [D2], VI (Conjecture 4.2). By proving the λ–adic version of Theorem 2.1 (Theorem 4.4), we are able to show that 4.2 holds for the mixed Shimura variety if and only if it holds for the underlying pure Shimura variety (Theorem 4.6).

For a functor with values in mixed systems of smooth sheaves ([W2], § 2), we need to define an admissible variation of Hodge structure not just for the canonical embedding σ_0 of $E(P, \mathfrak{X})$ into $\mathbb{C}$, but for any such embedding. This forces us to generalize Milne's and Shih's results on conjugates of pure Shimura varieties ([M], II, §§ 4, 5, 7) to the mixed case (§ 5). For the sake of completeness, we also include a description of complex conjugation on the $\mathbb{C}$–valued points of a

Shimura variety, whose reflex field is real (Lemma 5.11, Corollary 5.12).
In § 6, we associate to a representation $\mathbf{V}$ of P its conjugates ${}^{\tau,x}\mathbf{V}$, which are representations of the groups ${}^{\tau,x}P$ obtained by the process of twisting defined in § 5. We show that if 4.2 holds then the canonical constructions of all the ${}^{\tau,x}\mathbf{V}$ fit together to define a mixed system of smooth sheaves $\mu_{K,MS}(\mathbf{V})$ on $M^K(P,\mathfrak{X})$.
In §§ 2–4, we included results on the compatibility of $\mu_{K,-}$ and $\mu_{\pi(K),-}$ with higher direct images of $[\pi]_*$ and group cohomology of W (Theorems 2.3, 3.5 and 4.7).

This article is a revised and extended version of § 5 of my doctoral thesis ([W1]).
I would like to thank C. Deninger for his generosity and constant support, and T. Scholl for valuable comments. I am obliged to the organizers of the Oberwolfach Arbeitstagung on Shimura varieties in Spring 1992. What I learned while preparing myself for that conference stimulated the study of the canonical construction for mixed Shimura varieties.
Finally, I am most grateful to Mrs. G. Weckermann for skillfully TEXing my manuscript.

§ 1 Mixed Shimura data and mixed Shimura varieties

We recall the definition and basic properties of mixed Shimura varieties. Our exposition follows [P1].

Let $P/\mathbb{Q}$ be a connected algebraic group,
$W := R_u(P)$ its unipotent radical,
$G := P/W$, $\pi : P \twoheadrightarrow G$,
$U \leq W$ a normal subgroup of P,
$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ the Deligne torus, $w : \mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbb{S}$ the weight,
$\mathfrak{X}$ a homogeneous space under $P(\mathbb{R}) \cdot U(\mathbb{C})$,
$h : \mathfrak{X} \longrightarrow \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ a $P(\mathbb{R}) \cdot U(\mathbb{C})$–equivariant map with finite fibres.
Write h_x for $h(x)$.
Let $V := W/U$, $\pi_m : P \twoheadrightarrow P/U$.

Definition: ([P1], Definition 2.1.)
$(P, \mathfrak{X})$ is called *mixed Shimura data* if the following holds for some (hence all) $x \in \mathfrak{X}$:

i) $\pi_m \circ h_x : \mathbb{S}_{\mathbb{C}} \longrightarrow (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

ii) $\pi \circ h_x \circ w : \mathbb{G}_{m,\mathbb{R}} \longrightarrow G_{\mathbb{R}}$ is a cocharacter of the center $Z(G)_{\mathbb{R}}$ of $G_{\mathbb{R}}$.

iii) $\mathrm{Ad}_P \circ h_x$ induces on $\mathrm{Lie}\, P$ a mixed graded–polarizable $\mathbb{Q}$–Hodge structure ($\mathbb{Q}$–MHS) of type

$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}\,.$$

iv) the weight filtration on $\mathrm{Lie}\, P$ is given by

$$W_n(\mathrm{Lie}\, P) = \begin{cases} 0 & , n \leq -3 \\ \mathrm{Lie}\, U & , n = -2 \\ \mathrm{Lie}\, W & , n = -1 \\ \mathrm{Lie}\, P & , n \geq 0 \end{cases}.$$

v) int $(\pi(h_x(\sqrt{-1})))$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.

vi) $G^{\mathrm{ad}}_{\mathbb{R}}$ has no nontrivial factors of compact type, that are defined over $\mathbb{Q}$.

vii) $Z(G)$ acts on U and on V through a torus, that is an almost direct product of a $\mathbb{Q}$–split torus with a torus of compact type defined over $\mathbb{Q}$.

Because of weight reasons, the algebraic group V is abelian, and U is contained in $Z(W)$.
If $W = 1$ then $(P, \mathfrak{X})$ is called pure.

Actually, in order to be able to define the canonical construction, we shall restrict ourselves to those mixed Shimura data satisfying

vii)' $Z(G)^0$ is an almost direct product of a $\mathbb{Q}$–split torus with a torus of compact type defined over $\mathbb{Q}$.

This condition implies that any real cocharacter of $Z(G)$ is defined over $\mathbb{Q}$.

Again, because of weight reasons, $\pi : P \longrightarrow G$ is injective on $Z(P)$, so $Z(P)^0$ is a torus of the same type. As explained in [P1], § 1, these axioms imply

Theorem 1.1: Let $F \subset \mathbb{R}$ be a field.

a) There is a canonical $P(\mathbb{R}) \cdot U(\mathbb{C})$–invariant complex structure on $\mathfrak{X}$.

b) There is a tensor functor

$$\mathrm{Rep}_F(P) \longrightarrow \left\{ \begin{array}{l} \text{graded–polarizable variations} \\ \text{of } F\text{–Hodge structure on } \mathfrak{X} \end{array} \right\} .$$

c) For every irreducible $\mathbf{V} \in \mathrm{Rep}_F(P)$, which is pure of some weight n, there is a representation of P on $F(-n) := (2\pi\sqrt{-1})^{-n} F \subset \mathbb{C}$ and a P–equivariant bilinear form

$$\Psi : \mathbf{V} \times \mathbf{V} \longrightarrow F(-n)$$

such that for all $x \in \mathfrak{X}$ either Ψ or $-\Psi$ is a polarization of the corresponding MHS on $\mathbf{V}$. Here, $\mathbf{V}$ is called pure of weight n if for some (hence all) $x \in \mathfrak{X}$,

$$h_x \circ w : \mathbb{G}_{m,\mathbb{C}} \longrightarrow P_{\mathbb{C}}$$

acts on $\mathbf{V}_{\mathbb{C}}$ by

$$z \longmapsto (\text{multiplication by } z^{-n}) .$$

Proof: [P1], 1.18. q.e.d.

The functor

$$\mathrm{Rep}_F(P) \longrightarrow \left\{ \begin{matrix} \text{graded–polarizable variations} \\ \text{of } F\text{–Hodge structure on } \mathfrak{X} \end{matrix} \right\}$$

is as natural as it could be:
$x \in \mathfrak{X}$ gives a map $h_x : \mathbb{S}_{\mathbb{C}} \longrightarrow P_{\mathbb{C}}$, i.e., a $\mathbb{Z}^2$–grading on $\mathbf{V}_{\mathbb{C}}$ for any representation $\mathbf{V}$ of P. By definition, the underlying local system is constant, and $x \longmapsto h_x$ defines the weight and Hodge filtrations. More precisely, for $x \in \mathfrak{X}$, we have

$$W_{n,x}(\mathbf{V})_{\mathbb{C}} = \bigoplus_{p+q\leq n} H_x^{p,q}(\mathbf{V}),\ F_x^p(\mathbf{V}_{\mathbb{C}}) = \bigoplus_{p'\geq p} H_x^{p',q}(\mathbf{V}),$$

where $H_x^{p,q}(\mathbf{V})$ is the eigenspace of the cocharacter $(z_1, z_2) \longmapsto {z_1}^{-p}{z_2}^{-q}$ of $\mathbb{S}_{\mathbb{C}}$ under the action of $\mathbb{S}_{\mathbb{C}}$ on $\mathbf{V}_{\mathbb{C}}$ given by h_x.

The complex structure on $\mathfrak{X}$ is unique with respect to the requirement that the Hodge filtration of any $\mathbf{V} \in \mathrm{Rep}_F(P)$ vary holomorphically ([P1], Proposition 1.7.a)). Griffiths transversality is a direct translation of axiom iii), and graded–polarizability follows from 1.1.c).

Whenever we have a normal subgroup $P_0 \leq P$, we can define quotient mixed Shimura data $(P, \mathfrak{X})/P_0$, whose underlying algebraic group is P/P_0 and which have a universal property ([P1], Proposition 2.9).
In particular, we write $(G, \mathcal{H}) := (P, \mathfrak{X})/W$.

As in the classical case, one defines mixed Shimura varieties, or rather, their topological spaces of $\mathbb{C}$–valued points, as follows:
let $\mathbb{A}_f$ denote the ring of finite adeles over $\mathbb{Q}$, and let $K \leq P(\mathbb{A}_f)$ be open and compact. Set

$$M^K(\mathbb{C}) := M^K(P, \mathfrak{X})(\mathbb{C}) := P(\mathbb{Q})\backslash(\mathfrak{X} \times (P(\mathbb{A}_f)/K))$$

where $P(\mathbb{Q})$ acts on both factors from the left.
We have

Proposition 1.2:

a) $M^K(\mathbb{C}) = \bigcup_{i=1}^n \Gamma(p_{f,i})\backslash(\mathfrak{X}_i^0 \times p_{f,i}K/K) = \bigcup_{i=1}^n \Gamma(p_{f,i})\backslash\mathfrak{X}_i^0$,
where the union is finite and disjoint, $\mathfrak{X}_i^0$ denotes a connected component of $\mathfrak{X}$,
$p_{f,i} \in P(\mathbb{A}_f)$, and $\Gamma(p_{f,i}) := \mathrm{Stab}_{P(\mathbb{Q})}(\mathfrak{X}_i^0) \cap p_{f,i} \cdot K \cdot p_{f,i}^{-1} \leq P(\mathbb{Q})$ is an arithmetic subgroup.

b) For any i, the group $\Gamma(p_{f,i})$ acts properly discontinuously on $\mathfrak{X}_i^0$. $M^K(\mathbb{C})$ is a normal complex space, whose singularities are at most quotient singularities by finite groups.

c) If K is neat, then for any i, the group $\Gamma(p_{f,i})$ acts freely on $\mathfrak{X}_i^0$, so $M^K(\mathbb{C})$ is a complex manifold.[†]

Proof: [P1], 3.2 and Proposition 3.3 including its proof. There, it is shown that b) and c) are true modulo $\Gamma(p_{f,i}) \cap Z(P)(\mathbb{Q})$. But $Z(P)$ injects into $Z(G)$ since $\operatorname{Lie} W$ is of weight ≤ -1, and by vii)', any arithmetic subgroup of $Z(G)(\mathbb{Q})$ is finite.

Note that by convention [P1], 0.4, the usage of the term "properly discontinuous" in [P1] differs from the usual one. So we cannot quote [P1], Proposition 3.3 directly. q.e.d.

Remark: We note that in order to get the conclusion of 1.2.c) for a fixed K, we need only assume that any subgroup of $P(\mathbb{Q})$ of the shape

$$\operatorname{Stab}_{P(\mathbb{Q})}(\mathfrak{X}^0) \cap p_f \cdot K \cdot p_f^{-1}$$

is neat.

The conclusions of this article, in particular 1.3, 1.4, 4.1 and the calculation of the Galois group preceding 4.1 remain valid under this weaker assumption as the proofs of the relevant results of [P1] (Lemma 3.11, Corollary 3.12.a)) and [P2] (Proposition 3.3.3) run through without any problems.

A morphism $\varphi : (P_1, \mathfrak{X}_1) \longrightarrow (P_2, \mathfrak{X}_2)$ of mixed Shimura data consists of a morphism $\varphi : P_1 \longrightarrow P_2$ and a $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$–equivariant map $\psi : \mathfrak{X}_1 \longrightarrow \mathfrak{X}_2$ such that

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r, "\psi"] \arrow[d, "h_1"'] & \mathfrak{X}_2 \arrow[d, "h_2"] \\ \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}) \arrow[r, "\varphi_*"] & \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, P_{2,\mathbb{C}}) \end{tikzcd}$$

commutes.

If ψ is injective and φ is a closed immersion, then the morphism is called an embedding.

[†] For the definition of neatness, see [P1], 0.6 or [P2], 3.2.

If $K_i \leq P_j(\mathbb{A}_f)$, $i = 1, 2$, satisfy $\varphi(K_1) \leq K_2$, then there is a canonical map

$$[\varphi](\mathbb{C}) = [\varphi]_{K_1,K_2}(\mathbb{C}) : M^{K_1}(P_1, \mathfrak{X}_1)(\mathbb{C}) \longrightarrow M^{K_2}(P_2, \mathfrak{X}_2)(\mathbb{C}) \,,$$

which is holomorphic ([P1], 3.4.b)).

Similarly, if $p_f \in P(\mathbb{A}_f)$ and $K' \leq p_f \cdot K \cdot p_f^{-1}$, we have

$$[\cdot p_f](\mathbb{C}) = [\cdot p_f]_{K',K}(\mathbb{C}) : M^{K'}(P, \mathfrak{X})(\mathbb{C}) \longrightarrow M^K(P, \mathfrak{X})(\mathbb{C}),$$

which is holomorphic, finite and surjective ([P1], 3.4.a)).

We now turn to two of the main results of [P1]:

By [P1], Corollary 8.14 and § 9, every $M^K(P, \mathfrak{X})(\mathbb{C})$ is the set of $\mathbb{C}$–valued points of a quasi–projective variety $M^K(P, \mathfrak{X})_{\mathbb{C}}$ over $\mathbb{C}$.
By [P1], Theorem 11.18, each $M^K(P, \mathfrak{X})_{\mathbb{C}}$ admits a *canonical model* $M^K(P, \mathfrak{X})$, which is a normal quasi–projective variety over a number field $E(P, \mathfrak{X})$, the so–called *reflex field* of $(P, \mathfrak{X})$, which is given together with fixed embeddings $\overline{\sigma_0} : \overline{E(P, \mathfrak{X})} \hookrightarrow \mathbb{C}$ and $\sigma_0 := \overline{\sigma_0} \mid_{E(P,\mathfrak{X})}$.
(For a definition of both the reflex field and the canonical model, see [P1], Definitions 11.1 and 11.5.)

By [P1], Definition 11.5.a) and Proposition 11.10, all the above holomorphic maps $[\varphi](\mathbb{C})$ and $[\cdot p_f](\mathbb{C})$ come from algebraic morphisms $[\varphi]_{\mathbb{C}}$ and $[\cdot p_f]_{\mathbb{C}}$, that descend to the reflex field of the source. These morphisms will be denoted by the symbols $[\varphi]$ and $[\cdot p_f]$ respectively. If K is neat, then $[\cdot p_f]$ is étale.

By [P1], Corollary 3.12.a), 3.14, 3.22 and Corollary 3.12.b), up to an error obtained by dividing out the action of a finite group, we may think of

$$[\pi] : M^K(P, \mathfrak{X}) \longrightarrow M^{\pi(K)}(G, \mathcal{H})$$

as a torus–torsor over an abelian scheme over $M^{\pi(K)}(G, \mathcal{H})$.
The abelian scheme is of relative dimension $\frac{1}{2} \dim V$, the torus–torsor is of relative dimension $\dim U$ over the abelian scheme.

We need to be more precise since we want to show that, possibly up to the geometrical connectedness of $M^K(P, \mathfrak{X})$, we are in the situation studied in [W2], § 3.

Fix once and for all a Levi section $i : G \longrightarrow P$ of π.
It is not difficult to see that $\mathcal{H} = (W(\mathbb{R}) \cdot U(\mathbb{C}))\backslash\mathfrak{X}$: see the remark following [P1], Proposition 2.9. Or look at the proof of [P1], Proposition 2.9 and use [P1], Lemma 1.17 and Corollary 2.12, together with the connectedness of the topological group $W(\mathbb{R}) \cdot U(\mathbb{C})$.
Next, if $x \in \mathfrak{X}$, then $i \circ \pi \circ h_x$ and h_x are conjugate under $W(\mathbb{R}) \cdot U(\mathbb{C})$: namely, the map

$$\begin{aligned} W(\mathbb{R}) \cdot U(\mathbb{C}) &\longrightarrow \left\{ \begin{matrix} \text{Levi decompositions of } P_{\mathbb{C}}, \text{ that,} \\ \text{modulo } U_{\mathbb{C}}, \text{ are defined over } \mathbb{R} \end{matrix} \right\}, \\ p &\longmapsto p \cdot \mathrm{Cent}_{P_{\mathbb{C}}}(h_x \circ w) \cdot p^{-1} \end{aligned}$$

is a bijection; see the proof of [P1], Proposition 1.16.b). Since $\mathrm{Lie}(W)$ has negative weights, the group $\mathrm{Cent}_{P_{\mathbb{C}}}(i \circ \pi \circ h_x \circ w)$ defines such a decomposition, hence

$$\mathrm{Cent}_{P_{\mathbb{C}}}(i \circ \pi \circ h_x \circ w) = p \cdot \mathrm{Cent}_{P_{\mathbb{C}}}(h_x \circ w) \cdot p^{-1} = \mathrm{Cent}_{P_{\mathbb{C}}}(\mathrm{int}(p) \circ h_x \circ w)$$

for some $p \in W(\mathbb{R}) \cdot U(\mathbb{C})$. But $i \circ \pi \circ h_x$ and $\mathrm{int}(p) \circ h_x$ both lift $\pi \circ h_x$. Since they land in the same Levi subgroup, they are equal.

This, together with [P1], Corollary 2.12 and the connectedness of $W(\mathbb{R}) \cdot U(\mathbb{C})$ shows that $i : G \longrightarrow P$ can be extended to an embedding

$$i : (G, \mathcal{H}) \longrightarrow (P, \mathfrak{X})$$

which is uniquely determined by the following properties:

a) $\pi \circ i = \mathrm{id}_{(G,\mathcal{H})}$.

b) for any $x \in \mathfrak{X}$, the element $i \circ \pi(x)$ lies in the same connected component as x, i.e., $i \circ \pi(x)$ and x are conjugate under $W(\mathbb{R}) \cdot U(\mathbb{C})$.[†]

Remark: Note that the existence of such a splitting i on the level of Shimura data shows that the following holds for any $\mathbf{V} \in \mathrm{Rep}_F(P)$ and $x \in \mathfrak{X}$:
the decomposition

$$W_{n,x}(\mathbf{V})_{\mathbb{C}} = \bigoplus_{p+q \leq n} H_x^{p,q}(\mathbf{V}), \; F_x^p(\mathbf{V}_{\mathbb{C}}) = \bigoplus_{p' \geq p} H_x^{p',q}(\mathbf{V})$$

[†] Note that by [P1], Proposition 2.17.b), there is exactly one morphism i with property a). That it has property b) can also be seen from the proof there.

corresponding to the action of $\mathbb{S}_{\mathbb{C}}$ on $\mathbf{V}_{\mathbb{C}}$ given by h_x is the unique decomposition satisfying

$$\overline{H^{q,p}} \equiv H^{p,q} \bmod \bigoplus_{\substack{p'<p\\ q'<q}} H^{p',q'} .$$

Namely, if $x \in i(\mathcal{H})$, then by axiom i), h_x is defined over $\mathbb{R}$.
Therefore, we even have the equality

$$\overline{H_x^{q,p}(\mathbf{V})} = H_x^{p,q}(\mathbf{V}) .$$

It clearly also holds for any $P(\mathbb{R})$–translate of x. So let $u \in U(\mathbb{C})$. Because of the Hodge–type of $\operatorname{Lie} U$, we have

$$H^{p,q}_{u(x)}(\mathbf{V}) \equiv H_x^{p,q}(\mathbf{V}) \bmod \bigoplus_{\substack{p'<q\\ q'<p}} H_x^{p',q'}(\mathbf{V}) .$$

By induction on the smallest weight of $\mathbf{V}$, one shows finally

$$\bigoplus_{\substack{p'<p\\ q'<q}} \overline{H_x^{q',p'}(\mathbf{V})} = \bigoplus_{\substack{p'<p\\ q'<q}} H_x^{p',q'}(\mathbf{V}) .$$

The uniqueness of a decomposition as above is proven in [CKS], Theorem 2.13.

We use i to write $P = W \rtimes G$.
Fix a neat open compact subgroup L of $G(\mathbb{A}_f)$ and an open compact subgroup K^W of $W(\mathbb{A}_f)$, which is stable under conjugation by $i(L)$.
Set $K := K^W \rtimes L$.
We want to study

$$[\pi] = [\pi]_{K,L} : M^K(P, \mathfrak{X}) \longrightarrow M^L(G, \mathcal{H}) .$$

Writing

$$\begin{aligned} (P_a, \mathfrak{X}_a) &:= (P, \mathfrak{X})/U , \\ \pi_m \ &: \ (P, \mathfrak{X}) \longrightarrow (P_a, \mathfrak{X}_a) , \\ \pi_a \ &: \ (P_a, \mathfrak{X}_a) \longrightarrow (G, \mathcal{H}) \end{aligned}$$

such that $\pi = \pi_a \circ \pi_m$, we have

$$M^K(P, \mathfrak{X}) \xrightarrow{[\pi_m]} M^{\pi_m(K)}(P_a, \mathfrak{X}_a) \xrightarrow{[\pi_a]} M^L(G, \mathcal{H}) .$$

By [P1], Corollary 3.12.a), 3.14 and the algebraicity of the group structure, $[\pi_a]$ is an abelian scheme, the zero section being given by

$$[\pi_m] \circ [i] : M^L(G, \mathcal{H}) \longrightarrow M^{\pi_m(K)}(P_a, \mathfrak{X}_a) .$$

Assume temporarily that $(P,\mathfrak{X})$ is irreducible, i.e. ([P1], 2.13), that there does not exist a proper normal subgroup of P defined over $\mathbb{Q}$, through which h_x factors for some (hence any) $x \in \mathfrak{X}$.
By [P1], Proposition 2.14, P acts on $\operatorname{Lie} U$ through a scalar character $P \longrightarrow \mathbb{G}_m$. This implies that any subgroup of U is normal in P. Choose a basis $(u_1,\ldots,u_l)$ of $U = \operatorname{Lie} U$. View $U_j := \langle u_j \rangle_{\mathbb{Q}}$ as a quotient of U and consider the projections

$$\pi'_{m,j,U} : U \longrightarrow\!\!\!\!\rightarrow U_j \, .$$

Write

$$(P_j,\mathfrak{X}_j) := (P,\mathfrak{X})/\ker(\pi'_{m,j,U}) \, , \; \pi'_{m,j} : (P,\mathfrak{X}) \longrightarrow (P_j,\mathfrak{X}_j) \, .$$

As explained above, the mixed Shimura data $(P_j,\mathfrak{X}_j)$ admit morphisms

$$\pi''_{m,j} : (P_j,\mathfrak{X}_j) \longrightarrow (P_a,\mathfrak{X}_a)$$

such that $\pi''_{m,j} \circ \pi'_{m,j} = \pi_m$ for all j.

It is clear that P is the fibre product of the P_j over P_a. As before, we have $\mathfrak{X}_j = (\ker(\pi'_{m,j}))(\mathbb{C}) \backslash \mathfrak{X}$ and $\mathfrak{X}_a = U(\mathbb{C}) \backslash \mathfrak{X}$. Since the action of $U(\mathbb{C})$ on $\mathfrak{X}$ is faithful, $\mathfrak{X}$ is the fibre product of the $\mathfrak{X}_j$ over $\mathfrak{X}_a$.

It follows that $(P,\mathfrak{X})$ is the fibre product of the $(P_j,\mathfrak{X}_j)$ over $(P_a,\mathfrak{X}_a)$ in the category of mixed Shimura data (compare [P1], 2.20).

Now if $K^U := K^W \cap U(\mathbb{A}_f) = K \cap U(\mathbb{A}_f)$ happens to be of the shape

$$K^{U_1} \times \ldots \times K^{U_l} \, ,$$

where $K^{U_j} := K \cap U_j(\mathbb{A}_f)$, then K is the fibre product of the $\pi'_{m,j}(K)$ over $\pi_m(K)$.
We claim that for given K, it is always possible to choose the basis $(u_1,\ldots,u_l)$ in such a way that $K^U = K^{U_1} \times \ldots \times K^{U_l}$: use the fact that

$$\Lambda \longmapsto \Lambda \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p$$

gives a bijective correspondence between the set of $\mathbb{Z}$–lattices of rank l in $U(\mathbb{Q})$ and the set of open compact subgroups of $U(\mathbb{A}_f)$. The claim follows from the basic theory of principal divisors for free modules over $\mathbb{Z}$ of finite rank.

So we assume that our basis has this property.
By [P1], Lemma 3.11 together with the remark preceding its proof, $M^K(P,\mathfrak{X})$ is the fibre product of the $M^{\pi'_{m,j}(K)}(P_j,\mathfrak{X}_j)$ over $M^{\pi_m(K)}(P_a,\mathfrak{X}_a)$.

More precisely, we get a morphism of $M^K(P,\mathfrak{X})$ to the fibre product, which by [P1], Lemma 3.11 is an isomorphism after $\otimes_{E(P,\mathfrak{X})}\mathbb{C}$, hence is an isomorphism altogether.

Now let $j \in \{1,\ldots,l\}$. We want to study

$$[\pi''_{m,j}] : M^{\pi'_{m,j}(K)}(P_j,\mathfrak{X}_j) \longrightarrow M^{\pi_m(K)}(P_a,\mathfrak{X}_a)\,.$$

By [P1], Corollary 3.12.b) (with $\varphi := \pi''_{m,j}$ and $(P_*,\mathfrak{X}_*) := (\mathbb{G}_m, h(\mathcal{H}_0))$ as in [P1], 2.8, using the map $(P_j,\mathfrak{X}_j) \longrightarrow (P_*,\mathfrak{X}_*)$ given by [P1], Proposition 2.14), $[\pi''_{m,j}]$ is an algebraic $\mathbb{G}_m$–torsor: the holomorphic maps occurring in the proof of [P1], Corollary 3.12.b) are all algebraic, and the unipotent extension

$$(P',\mathfrak{X}') \longrightarrow (P_*,\mathfrak{X}_*)$$

occurring in the claim of [P1], Corollary 3.12.b) is precisely $(P_0, h(\mathfrak{X}_0))$ as in [P1], 2.24. On the level of Shimura varieties, this arrow turns into the morphism of schemes over $\mathbb{Q}$

$$\mathbb{G}_{m,\mathbb{Q}} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(E_L) \longrightarrow \mathrm{Spec}(E_L)$$

for a number field E_L. This claim is proven in [P1], Proposition 11.15: the automorphism of order 2 of $(P_0,\mathfrak{X}_0)$ already comes from $(\mathbb{G}_m,\mathcal{H}_0)$, hence only affects the field $M^{K^*}(\mathbb{G}_m,\mathcal{H}_0)$ in [P1], Proposition 11.14.

So each $[\pi''_{m,j}]$ is the total space of a line bundle over $M^{\pi_m(K)}(P_a,\mathfrak{X}_a)$ with the zero section removed.

Theorem 1.3: Let $(P,\mathfrak{X})$ be mixed Shimura data, $i : (G,\mathcal{H}) \hookrightarrow (P,\mathfrak{X})$ as above, $L \leq G(\mathbb{A}_f)$ neat, open and compact, $K^W \leq W(\mathbb{A}_f)$ open, compact and stable under conjugation by $i(L)$, $K := K^W \rtimes L$.
Then $[\pi] : M^K(P,\mathfrak{X}) \longrightarrow M^L(G,\mathcal{H})$ factors into

$$M^K(P,\mathfrak{X}) \xrightarrow{[\pi_m]} M^{\pi_m(K)}(P_a,\mathfrak{X}_a) \xrightarrow{[\pi_a]} M^L(G,\mathcal{H})\,,$$

where $[\pi_a]$ is an abelian scheme with zero section $[\pi_m]\circ[i]$, and $[\pi_m]$ is the fibre product over $M^{\pi_m(K)}(P_a,\mathfrak{X}_a)$ of total spaces of line bundles with their zero sections removed. $[i]$ defines a rigidification of every such line bundle, i.e., a trivialization along the zero section of $[\pi_a]$.

Proof: If $(P, \mathfrak{X})$ is irreducible, the claims, up to the last one, were shown in the course of the above discussion. For the rigidification of the line bundles, we apply base change by $\pi_m \circ i$ to the whole situation ([P1], 2.20) and thus may assume that $W = U$. But then $[\pi'_{m,j}] \circ [i]$ defines a section "one" of $[\pi''_{m,j}]$, which is a $\mathbb{G}_m$–torsor over $M^L(G, \mathcal{H})$. So we have defined an isomorphism of $[\pi''_{m,j}]$ and $\mathbb{G}_{m, M^L(G,\mathcal{H})}$.

In the general case, let P_1 be the smallest normal subgroup of P containing the images of all h_x. By [P1], 2.13, we have the equality $P = P_1 \cdot \pi^{-1}(Z(G))$: set $\mathbb{S}^1 := \ker(\mathrm{Norm} : \mathbb{S} \longrightarrow \mathbb{G}_{m,\mathbb{R}})$. For the proof of the equality, we may suppose $P = G = G^{\mathrm{ad}}$. By axiom v) and since $h_x(\mathbb{S}^1) \subset P_{1,\mathbb{R}}$, the group $(P/P_1)_{\mathbb{R}}$ is of compact type, hence equal to 1 by axiom vi). So axiom vii)' implies that any neat arithmetic subgroup of $P(\mathbb{Q})$ is already contained in $P_1(\mathbb{Q})$.

In [P1], 2.13 it is described how $(P, \mathfrak{X})$ can be "covered" by irreducible Shimura data $(P_1, \mathfrak{X}_{1,i})$ where the $\mathfrak{X}_{1,i}$ are simply $P_1(\mathbb{R}) \cdot U(\mathbb{C})$–orbits in $\mathfrak{X}$, which turn out to be unions of connected components of $\mathfrak{X}$. In particular, 1.2.a) shows that $M^K(P, \mathfrak{X})$ is the union of Shimura varieties associated to irreducible Shimura data $(P_1, \mathfrak{X}_{1,i})$, possibly identified by algebraic isomorphisms. q.e.d.

Observe that $M^L(G, \mathcal{H})$ won't in general be geometrically connected. In fact its finitely many geometrically connected components cannot be expected to be defined over $E(G, \mathcal{H})$ ($= E(P, \mathfrak{X})$ by [P1], 11.2.b)) but over some extension field $E^L(G, \mathcal{H})$, that depends on L.

Corollary 1.4: $[\pi]$ can be compactified in such a way that $M^K(P, \mathfrak{X})$ is the complement of a relative divisor with normal crossings in a smooth, projective $M^L(G, \mathcal{H})$–scheme.

Proof: $M^{\pi_m(K)}(P_a, \mathfrak{X}_a)$ and $M^L(G, \mathcal{H})$ are quasi–projective over $E(P, \mathfrak{X})$, hence ([Ha], II, Exercise 4.8) $[\pi_a]$ is quasi–projective, hence projective.

So it suffices to prove the claim for $[\pi_m]$ instead of $[\pi]$. As in [P1], 5.5 we may compactify each of the line bundles to give a $(\mathbb{P}^1)^l$–bundle over $M^{\pi_m(K)}(P_a, \mathfrak{X}_a)$. By [Ha], II, Exercise 7.10.c), this is a smooth, projective scheme over $M^{\pi_m(K)}(P_a, \mathfrak{X}_a)$. The complement of $M^K(P, \mathfrak{X})$ is a relative divisor with normal crossings. q.e.d.

So except for the geometrical connectedness of $M^L(G, \mathcal{H})$, we are in the situation of [W2], § 3.

Analyzing the proof of 1.3, it is not hard to see that 1.4 holds without the assumption on the special shape of the neat subgroup K of $P(\mathbb{A}_f)$. Also, it is true more generally for morphisms $[\varphi]$ corresponding to morphisms of Shimura data

$$(P, \mathfrak{X}) \longrightarrow (P, \mathfrak{X})/W_0$$

where $W_0 \leq W$ is a normal subgroup of P.

Definition:

a) Let $\overline{X}$ be a pathwise connected topological space, $\overline{x} \in \overline{X}$ such that $\pi_1(\overline{X}, \overline{x})$ is finitely generated, and denote by $W(\pi_1(\overline{X}, \overline{x}))$ the Tannakian dual of the category of unipotent representations of $\pi_1(\overline{X}, \overline{x})$ over $\mathbb{Q}$.
$\overline{X}$ is called a *unipotent* $K(\pi, 1)$ if the natural map

$$H^{\cdot}(W(\pi_1(\overline{X}, \overline{x})), \mathbb{Q}) \longrightarrow H^{\cdot}(\overline{X}, \mathbb{Q})$$

is an isomorphism.

b) Let $\overline{X}$ be a pathwise connected scheme ([SGA4,III], Exp. IX, Définition 2.12) over an algebraically closed field $\overline{k}$ of characteristic 0, $\overline{x}$ a geometric point such that $\pi_1(\overline{X}, \overline{x})$ is topologically finitely generated, and denote by $W_l(\pi_1(\overline{X}, \overline{x}))$ the Tannakian dual of the category of unipotent l–adic representations of $\pi_1(\overline{X}, \overline{x})$.
$\overline{X}$ is called a *unipotent* l–$K(\pi, 1)$ if the natural map

$$H^{\cdot}(W_l(\pi_1(\overline{X}, \overline{x})), \mathbb{Q}_l) \longrightarrow H^{\cdot}_{\acute{e}t}(\overline{X}, \mathbb{Q}_l)$$

is an isomorphism.

Example: If a nilpotent finitely generated group is the fundamental group of a topological space $\overline{X}$, and $\overline{X}$ is a $K(\pi, 1)$, then $\overline{X}$ is a unipotent $K(\pi, 1)$ (compare the remarks preceding and following [W2], Lemma 2.4).

We recall the following result:

Lemma 1.5: ([W2], Lemma 4.1.)
If $\overline{k}$ can be embedded into $\mathbb{C}$, and if $\overline{X}/\overline{k}$ is connected and of finite type, then for any embedding $\overline{k} \hookrightarrow \mathbb{C}$ such that $\pi_1(\overline{X}(\mathbb{C}), \overline{x})$ is finitely generated, $\overline{X}$ is a unipotent l–$K(\pi, 1)$ if and only if $\overline{X}(\mathbb{C})$ is a unipotent $K(\pi, 1)$.

Since we shall always be in the situation of the lemma, we shall always loosely speak of unipotent $K(\pi,1)$s when considering schemes, the condition being checked at any prime number l or over $\mathbb{C}$.

We shall occasionally write $\overline{[\pi]}$ for the morphism $[\pi]\otimes_{E(P,\mathfrak{X})}\overline{E(P,\mathfrak{X})}$ of varieties over the algebraic closure $\overline{E(P,\mathfrak{X})}$ of $E(P,\mathfrak{X})$, and $\overline{[\pi]_{\mathbb{C}}}$ for the morphism of topological spaces underlying $[\pi](\mathbb{C})$.

Lemma 1.6: The fibres of $\overline{[\pi]}$ are unipotent $K(\pi,1)$s.
Proof: By 1.5, this can be checked on the level of $\overline{[\pi]_{\mathbb{C}}}$. By [P1], 3.13, the fundamental groups of the fibres of $\overline{[\pi]_{\mathbb{C}}}$ are nilpotent. So it remains to show that the fibres are $K(\pi,1)$s.
But this follows from the very construction of $M^K(P,\mathfrak{X})(\mathbb{C})$ as a quotient of $\mathfrak{X}$: namely, as $\mathcal{H}=(W(\mathbb{R})\cdot U(\mathbb{C}))\backslash\mathfrak{X}$ and since $W(\mathbb{R})\cdot U(\mathbb{C})$ acts faithfully on $\mathfrak{X}$, the universal covering of the fibres of $\overline{[\pi]_{\mathbb{C}}}$ is the contractible space $W(\mathbb{R})\cdot U(\mathbb{C})$.
q.e.d.

§ 2 The canonical construction of mixed sheaves: Hodge version

As is apparent from the results recalled at the beginning of § 1, the desire for a Hodge theoretical version of the construction of mixed sheaves on Shimura varieties from representations of the underlying group P dictates almost all of the axioms one imposes on mixed Shimura data.
The main part of this paragraph will be taken up by proving that the variations of Hodge structure arising via representations satisfy admissibility in the sense of [Ka].

Let $(P,\mathfrak{X})$, $(G,\mathcal{H})$ be as in § 1, $\pi:(P,\mathfrak{X})\longrightarrow(G,\mathcal{H})$ the projection, $K\leq P(\mathbb{A}_f)$ an arbitrary neat open compact subgroup, $L:=\pi(K)$.
Our aim is to define a tensor functor

$$\mu_{K,\infty,\sigma_0}:\mathrm{Rep}_F(P)\longrightarrow[\pi]_{\mathbb{C}}\text{–}U\mathrm{Var}_F(M^K(P,\mathfrak{X})_{\mathbb{C}})$$

for any subfield F of $\mathbb{R}$.
Here, $\mathrm{Var}_F(M^L(G,\mathcal{H})_{\mathbb{C}})$ is defined to be the full subcategory of those objects of the category of graded–polarizable variations of F–MHS on $M^L(G,\mathcal{H})(\mathbb{C})$, that are admissible in the sense of [Ka]. Furthermore, $[\pi]_{\mathbb{C}}$–$U\mathrm{Var}_F(M^K(P,\mathfrak{X})_{\mathbb{C}})$ is the category of admissible variations on $M^K(P,\mathfrak{X})_{\mathbb{C}}$, which are $[\pi]_{\mathbb{C}}$–unipotent, i.e.,

admit a filtration, whose graded objects lie in $[\pi]^*_{\mathbb{C}}\mathrm{Var}_F(M^L(G,\mathcal{H})_{\mathbb{C}})$.

Each $\mathbf{V} \in \mathrm{Rep}_F(P)$ defines a local system

$$P(\mathbb{Q})\backslash(\mathfrak{X} \times (P(\mathbb{A}_f)/K) \times \mathbf{V}) \quad \text{on} \quad M^K(\mathbb{C}) = P(\mathbb{Q})\backslash(\mathfrak{X} \times (P(\mathbb{A}_f)/K))\,.$$

So on any connected component $\Gamma(p_f)\backslash\mathfrak{X}^0$ of $M^K(\mathbb{C})$, the local system is canonically isomorphic to $\Gamma(p_f)\backslash(\mathfrak{X}^0 \times \mathbf{V})$.
By definition, $p \in P(\mathbb{Q})$ transforms the MHS on $\mathbf{V}$ at $x \in \mathfrak{X}$ into the one at $p(x)$.
So the variations of F–Hodge structure given by 1.1.b) descend to $M^K(\mathbb{C})$, as does the property of graded–polarizability: by axiom vi), the representation of P on $\mathrm{Gr}^W_\cdot\mathbf{V}$ automatically factors through G. Since G is reductive, $\mathrm{Rep}_F(G)$ is semisimple, hence we may apply 1.1.c) to get a polarization of each $\mathrm{Gr}^W_n\mathbf{V}$ on $\mathfrak{X}$, which is $P(\mathbb{Q})$–equivariant, hence also descends to $M^K(\mathbb{C})$. Since W acts trivially on $\mathrm{Gr}^W_\cdot\mathbf{V}$, the variation associated to $\mathbf{V}$ is $[\pi](\mathbb{C})$–unipotent.
Observe that this construction of variations is well–behaved under the morphisms $[\varphi](\mathbb{C})$ and $[\cdot p_f](\mathbb{C})$:
if $\varphi : (P_1, \mathfrak{X}_1) \longrightarrow (P_2, \mathfrak{X}_2)$ is a morphism of Shimura data, and $\varphi(K_1) \le K_2$, then for any $\mathbf{V} \in \mathrm{Rep}_F(P_2)$ there is a canonical isomorphism

$$\mu_{K_1,\infty,\sigma_0}(\varphi^*\mathbf{V}) \xrightarrow{\sim} [\varphi](\mathbb{C})^*\mu_{K_2,\infty,\sigma_0}(\mathbf{V})$$

of variations of Hodge structure.
Similarly, if $p_f \in P(\mathbb{A}_f)$, and $K' \le p_f \cdot K \cdot p_f^{-1}$, then $[\cdot p_f](\mathbb{C})$ induces an isomorphism $\mu_{K',\infty,\sigma_0}(\mathbf{V}) \xrightarrow{\sim} [\cdot p_f](\mathbb{C})^*\mu_{K,\infty,\sigma_0}(\mathbf{V})$ of variations.

In order to prove admissibility it turns out to be necessary to show that the generic pro–variation of [W2], § 3 arises via the canonical construction.
So let $i : (G, \mathcal{H}) \longrightarrow (P, \mathfrak{X})$ be as before and assume temporarily that K is of the shape $K^W \rtimes L$. Then $[\pi]$ admits the section $[i]$.
We recall the definition of $\mathcal{G}en_{[i]_{\mathbb{C}}}$:
Fix $y \in M^L(\mathbb{C})$, and set $x := [i]_{\mathbb{C}}(y)$. Write $\overline{x} := x$ and $\overline{y} := y$ when considering x and y as elements of the topological space $M^K(\mathbb{C})$. The local system underlying the restriction of $\mathcal{G}en_{[i]_{\mathbb{C}}}$ to the connected component containing x is the completion of the group ring of $\pi_1(M^K(\mathbb{C})_{\overline{y}}, \overline{x})$ with respect to powers of the augmentation ideal, equipped with the action of

$$\pi_1(M^K(\mathbb{C}), \overline{x}) = \pi_1(M^K(\mathbb{C})_{\overline{y}}, \overline{x}) \rtimes \pi_1(M^L(\mathbb{C}), \overline{y})$$

given by multiplication of the first factor and conjugation of the second. The weight and Hodge filtration can be defined fibrewise by observing that the fibre over $\overline{y}' \in M^L(\mathbb{C})$ is canonically isomorphic to the local system underlying the "canonical variation with basepoint y'" of [HZ], § 1.
Since these data provide an example of the "path space variations" considered in [HZ], §§ 4 and 6, they define a pro–object $\mathcal{G}en_{[i]_\mathbb{C}}$ of $[\pi]_\mathbb{C}$–$U\mathrm{Var}_\mathbb{Q}(M^K_\mathbb{C})$.
For details see [W2], § 3.

Definition: $\mathcal{L}og_{\infty,\sigma_0} := \mathcal{L}og(i,K)_{\infty,\sigma_0} := \mathcal{G}en_{[i]_\mathbb{C}}$ is called the *logarithmic pro–variation* on $M^K(P,\mathfrak{X})_\mathbb{C}$.

$\mathcal{L}og_{\infty,\sigma_0}$ comes equipped with a section

$$1 \in (W_0 \cap F^0)(\Gamma(M^L(G,\mathcal{H})_\mathbb{C}, [i]^*_\mathbb{C}\mathcal{L}og_{\infty,\sigma_0})).$$

The pair $(\mathcal{L}og_{\infty,\sigma_0}, 1)$ is rigid, i.e., it admits no non–trivial automorphisms. In fact, this already holds on the level of underlying pro–local systems ([W2], Theorem 3.5.iii)). $\mathcal{L}og_{\infty,\sigma_0}$ carries the structure of cocommutative coalgebra. It induces on $[i]^*_\mathbb{C}\mathcal{L}og_{\infty,\sigma_0}$ a Hopf algebra structure. Furthermore, the pair $(\mathcal{L}og_{\infty,\sigma_0}, 1)$ has the universal property of [W2], Theorem 3.5.i).

Now consider the following pro–object of $\mathrm{Rep}_\mathbb{Q}(P)$:
let $\hat{\mathfrak{U}}(\mathrm{Lie}\, W)$ be the completed universal envelope of $\mathrm{Lie}\, W$.
W acts by multiplication, and $i(G)$ acts by conjugation. In addition, there is an element $1 \in H^0(i(G), \hat{\mathfrak{U}}(\mathrm{Lie}\, W))$. After applying μ_{K,∞,σ_0}, we get a graded–polarizable pro–variation of Hodge structure $\mathcal{G} := \mu_{K,\infty,\sigma_0}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$ on $M^K(\mathbb{C})$ and a section $1 \in (W_0 \cap F^0)(\Gamma(M^L(\mathbb{C}), [i](\mathbb{C})^*\mathcal{G}))$. $\mathcal{G}$ carries the structure of cocommutative coalgebra since this is already true on the level of representations. Similarly, $[i](\mathbb{C})^*\mathcal{G}$ is equipped with a Hopf algebra structure.
We don't know a priori that $\mathcal{G}$ is admissible.
However, this follows from the next result:

Theorem 2.1: There is a unique morphism

$$\varphi : \mathcal{L}og(i,K)_{\infty,\sigma_0} \longrightarrow \mu_{K,\infty,\sigma_0}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$$

of pro–variations of Hodge structure sending 1 to 1. It is an isomorphism of cocommutative coalgebras, and $[i](\mathbb{C})^*(\varphi)$ respects the multiplicative structure of both sides.

Before giving the proof, which is somewhat involved, we show that 2.1 implies admissibility of every variation of Hodge structure arising via the canonical construction.

Theorem 2.2: The construction described further above defines a tensor functor

$$\mu_{K,\infty,\sigma_0} : \mathrm{Rep}_F(P) \longrightarrow [\pi]_{\mathbb{C}}\text{–}U\mathrm{Var}_F(M^K(P,\mathfrak{X})_{\mathbb{C}})$$

for any neat open compact subgroup K of $P(\mathbb{A}_f)$.

Proof: First suppose $P = G$ is reductive.

Let $\mathbf{V} \in \mathrm{Rep}_{\mathbb{Q}}(P)$. It is the direct sum of its pure constituents. So we may suppose $\mathbf{V}$ is pure. The associated variation is pure, and since $\Gamma(p_f)$ is an arithmetic subgroup, it stabilizes a $\mathbb{Z}$–lattice in any algebraic representation of P over $\mathbb{Q}$. By Schmid's Nilpotent Orbit Theorem ([Sch], Theorem 4.9; recall ([Ka], § 0) that we check admissibility via the curve test), the variation associated to any such representation is admissible. In particular, we may take a faithful representation of P over $\mathbb{Q}$. After scalar extension to F, it generates $\mathrm{Rep}_F(P)$ as a tensor category ([DM], proof of Proposition 2.20.b)).

This proves the claim if $P = G$ is reductive.

For the general case, we recall that admissibility can be checked after base change with a finite covering of $M^L(G,\mathcal{H})$ ([Ka], Lemma 1.9.1).

So we may assume K is of the shape

$$K = K^W \rtimes L$$

and get a section

$$[i] : M^L \longrightarrow M^K$$

of $[\pi]$.

Now the finite–dimensional subquotients of $\hat{\mathfrak{U}}(\mathrm{Lie}\, W)$, together with $\pi^*\mathrm{Rep}_F(G)$, generate $\mathrm{Rep}_F(P)$ as a full Tannakian subcategory, that is closed under formation of subobjects. So the assertion follows from the pure case and 2.1. q.e.d.

In Tannakian terms, Theorems 2.1 and 2.2 admit the following reformulation: fix $y \in M^L(G,\mathcal{H})(\mathbb{C})$, and let $x := [i]_{\mathbb{C}}(y)$.

Then if $P_{\overline{x}}$ and $G_{\overline{y}}$ denote the Tannaka duals of $[\pi]_{\mathbb{C}}\text{–}U\mathrm{Var}_F(M^K(P,\mathfrak{X})^0_{\mathbb{C}})$ and $\mathrm{Var}_F(M^L(G,\mathcal{H})^0_{\mathbb{C}})$, where ^{0} denotes the connected components containing x and y respectively, μ_{K,∞,σ_0} and μ_{L,∞,σ_0} define morphisms

$$P_{\overline{x}} \longrightarrow P\,, \quad G_{\overline{y}} \longrightarrow G\,,$$

and the diagram

$(*)$

$$\begin{tikzcd}
1 \arrow[r] & \overline{W}_{\overline{x}} \arrow[r] \arrow[d, "\wr"'] & P_{\overline{x}} \arrow[r, shift left, "{[\pi]}"] \arrow[d] & G_{\overline{y}} \arrow[l, shift left, "{[i]}"] \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & W \arrow[r] & P \arrow[r, shift left, "\pi"] & G \arrow[l, shift left, "i"] \arrow[r] & 1
\end{tikzcd}$$

commutes. Here, the isomorphism

$$\overline{W}_{\overline{x}} \xrightarrow{\sim} W$$

is induced by [W2], Corollary 3.4.i) and the identification of the fundamental group of the fibre of $[\pi]_{\mathbb{C}}$ over $\overline{y}$ with an arithmetic subgroup of $W(\mathbb{Q})$ (see the proof of 2.4).

Theorem 2.3: Let d be the relative dimension of $[\pi]$, and consider μ_{K,∞,σ_0} and μ_{L,∞,σ_0} as functors into the categories $MHM_F(M^K_{\mathbb{C}})$ and $MHM_F(M^L_{\mathbb{C}})$ of algebraic mixed F–Hodge modules ([S1], [S2], in particular [S2], Theorem 3.27) on $M^K(P,\mathfrak{X})_{\mathbb{C}}$ and $M^L(G,\mathcal{H})_{\mathbb{C}}$ respectively.
Then for any q, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P) \arrow[r, "\mu_{K,\infty,\sigma_0}"] \arrow[d, "{H^q(W,\,\_)}"'] & MHM_F(M^K_{\mathbb{C}}) \arrow[d, "{\mathcal{H}^{q-d}([\pi]_{\mathbb{C}})_*}"] \\
\mathrm{Rep}_F(G) \arrow[r, "\mu_{L,\infty,\sigma_0}"] & MHM_F(M^L_{\mathbb{C}})
\end{tikzcd}$$

of functors.
Proof: Corollary 1.4 and Lemma 1.6 show that the hypotheses of [W2], Theorem 4.3 are met for $q_0 := \infty$. Our claim follows from the commutativity of the diagram $(*)$. q.e.d.

We prepare the proof of Theorem 2.1:

Lemma 2.4: The statement of 2.1 holds on the level of underlying pro–local systems, i.e., there is a unique morphism

$$\overline{\varphi} : \mathrm{For}(\mathcal{L}og(i,K)_{\infty,\sigma_0}) \longrightarrow \mathrm{For}(\mu_{K,\infty,\sigma_0}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W)))$$

of pro–local systems sending 1 to 1. It is an isomorphism of cocommutative coalgebras, and $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ respects the multiplicative structure of both sides.

Proof: Choose a base point $y \in M^L(\mathbb{C})$ and set $x := [i]_{\mathbb{C}}(y) \in M^K(\mathbb{C})$. These choices give an identification of the split exact sequence

$$1 \longrightarrow \Gamma(p_f) \cap W(\mathbb{Q}) \longrightarrow \Gamma(p_f) \underset{i}{\overset{\pi}{\rightleftarrows}} \Gamma(g_f) \longrightarrow 1$$

and the split exact sequence of fundamental groups

$$1 \longrightarrow \pi_1(M^K(\mathbb{C})_{\overline{y}}, \overline{x}) \longrightarrow \pi_1(M^K(\mathbb{C}), \overline{x}) \underset{\overline{[i]_{\mathbb{C}}}}{\overset{\overline{[\pi]_{\mathbb{C}}}}{\rightleftarrows}} \pi_1(M^L(\mathbb{C}), \overline{y}) \longrightarrow 1\,.$$

In particular, by [D1], 9.5, we get an identification of W and the pro–unipotent envelope of the group

$$\pi_1(M^K(\mathbb{C})_{\overline{y}}, \overline{x})\,.$$

So we obtain an isomorphism of $\mathrm{For}(\mathcal{L}og(i,K)_{\infty,\sigma_0})$ and $\mathrm{For}(\mathcal{G})$ respecting the multiplicative structure. By [W2], Theorem 3.5.ii) this is the only morphism sending 1 to 1. q.e.d.

We have to show that $\overline{\varphi}$ respects the weight and Hodge filtrations. In the course of the proof of 2.1, we shall frequently apply base change to and from other Shimura data and compose $\overline{\varphi}$ with isomorphisms of variations of Hodge structure.

The rigidity assertion of 2.4 will make it easy to keep track of the identifications made.

For example, if $g_f \in G(\mathbb{A}_f)$ and $L' = g_f \cdot L \cdot g_f^{-1}$, $K' = i(g_f) \cdot K \cdot i(g_f)^{-1}$, then we have an isomorphism

$$\mathcal{G}' := \mu_{K',\infty,\sigma_0}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W)) \xrightarrow{\sim} [\cdot i(g_f)](\mathbb{C})^* \mathcal{G}$$

of variations of Hodge structure sending $1' \in (W_0 \cap F^0)(\Gamma(M^{L'}(\mathbb{C}), [i](\mathbb{C})^* \mathcal{G}'))$ to $[\cdot g_f]^* 1$.

On the other hand, by [W2], Theorem 3.5.i), there is an isomorphism

$$\mathcal{L}og(i,K')_{\infty,\sigma_0} \xrightarrow{\sim} [\cdot i(g_f)]^*_{\mathbb{C}} \mathcal{L}og(i,K)_{\infty,\sigma_0}$$

of variations respecting the unit sections.

2.4 tells us that under these isomorphisms, $\overline{\varphi'}$ is transformed into $[\cdot i(g_f)](\mathbb{C})^* \overline{\varphi}$. In particular, if $\overline{\varphi'}$ respects the weight and Hodge filtrations at some point $x' \in M^{K'}(\mathbb{C})$, then $\overline{\varphi}$ respects the filtrations at $[\cdot i(g_f)](x') \in M^K(\mathbb{C})$.

Lemma 2.5: The statement of 2.1 holds if $U = 0$, i.e., if $[\pi]_{\mathbb{C}}$ is an abelian scheme.

Proof: First assume that G is a torus. Then $M^L_{\mathbb{C}}$ is just a finite set of points ([P1], Example 2.6), and $M^K_{\mathbb{C}}$ is projective over $\operatorname{Spec}(\mathbb{C})$. But then the admissibility condition is empty, and $\mathcal{G} \in [\pi]_{\mathbb{C}}\text{–}U\mathrm{Var}_{\mathbb{Q}}(M^K(P, \mathfrak{X})_{\mathbb{C}})$. By [W2], Theorem 3.5.i), there is a unique morphism

$$\varphi : \mathcal{L}og(i, K)_{\infty,\sigma_0} \longrightarrow \mathcal{G}$$

sending 1 to 1.

2.4 tells us that the underlying morphism of pro–local systems is $\overline{\varphi}$. So we get the desired assertion in this case.

Now let G be arbitrary. Since the weight and Hodge filtrations are sub–vector bundles it will be sufficient to show that $\overline{\varphi}$ respects them on a subset of $M^K(\mathbb{C})$ not contained in any proper closed analytic subset.

By [P1], Lemma 11.6, there is an embedding of Shimura data

$$k : (T, \mathcal{Y}) \longrightarrow (G, \mathcal{H})$$

with a torus T, and by [P1], Lemma 11.7, the union of the images of the maps

$$[\cdot g_f](\mathbb{C}) \circ [k](\mathbb{C}) : M^{L'}(T, \mathcal{Y})(\mathbb{C}) \longrightarrow M^L(G, \mathcal{H})(\mathbb{C})\,,$$

for all $L' \leq T(\mathbb{A}_f)$ open compact, and $g_f \in G(\mathbb{A}_f)$ such that this map is defined, is not contained in any proper closed analytic subset of $M^L(\mathbb{C})$.

Observe that we may suppose $(T, \mathcal{Y})$ to be irreducible. Since the weight cocharacter $\mathbb{G}_{m,\mathbb{R}} \longrightarrow T_{\mathbb{R}}$ is trivial and $\mathbb{S}/w(\mathbb{G}_{m,\mathbb{Q}})$ is of compact type, T itself is of compact type, and hence axiom vii)' is satisfied.

By the remark preceding the lemma, is suffices to show the statement after base change with $[k](\mathbb{C})$. But by 2.4, this yields the morphism $\overline{\varphi}$ for the base changed situation, where we already know the statement is true. q.e.d.

In particular, Lemma 2.5 gives a canonical identification of the variations

$$(\mathcal{H}^{-d+1}([\pi]_{\mathbb{C}})_*\mathbb{Q}(0))^\vee \quad \text{and} \quad \mu_{L,\infty,\sigma_0}(\operatorname{Lie} W)$$

if $U = 0$, where $d = \frac{1}{2}\dim V$ is the relative dimension of $[\pi]$.[†]

[†] We follow the notation of [W2], § 4, i.e., use the "perverse" rather than the "classical" numbering of the higher direct images. So $\mathcal{H}^{-d+1}([\pi]_{\mathbb{C}})_*\mathbb{Q}(0)$ is a variation of Hodge structure whose underlying local system is $R^1(\overline{[\pi]_{\mathbb{C}}})_*\mathbb{Q}$.

Indeed, since $[i]^*_{\mathbb{C}}\varphi$ respects the multiplicative structure, it induces an isomorphism of $\mathfrak{a}/\mathfrak{a}^2$ and $\mu_{L,\infty,\sigma_0}(\mathfrak{b}/\mathfrak{b}^2)$, where $\mathfrak{a}$ and $\mathfrak{b}$ denote the respective augmentation ideals of $[i]^*_{\mathbb{C}}\mathcal{L}og(i,K)_{\infty,\sigma_0}$ and $\hat{\mathfrak{U}}(\mathrm{Lie}\,W)$.
But these quotients are canonically isomorphic to the above variations respectively.

Lemma 2.6: Let $i : (G,\mathcal{H}) \longrightarrow (P,\mathfrak{X})$ be the fixed splitting of π. For each $v \in W(\mathbb{Q})$, let

$$i_v : (G,\mathcal{H}) \longrightarrow (P,\mathfrak{X})$$

be the splitting covering

$$i_v := \mathrm{int}(v) \circ i : G \longrightarrow P\,,$$

and

$$[i_v] : M^{L_v}(G,\mathcal{H}) \hookrightarrow M^K(P,\mathfrak{X})\,,\ \text{where } L_v := i_v^{-1}(K).$$

Then

$$\bigcup_{v\in W(\mathbb{Q})} [i_v]_{\mathbb{C}}(M^{L_v}(\mathbb{C})) \subset M^K(\mathbb{C})$$

is not contained in any proper closed analytic subset of $M^K(\mathbb{C})$.
Proof: The union equals

$$\bigcup_{v\in W(\mathbb{Q})} P(\mathbb{Q})\backslash\Big(P(\mathbb{Q})(i(\mathcal{H}) \times (i(G)(\mathbb{A}_f)v^{-1}K/K))\Big)\,.$$

Since W is unipotent, $W(\mathbb{A}_f) = W(\mathbb{Q})\cdot K^W$, and the union is equal to

$$P(\mathbb{Q})\backslash\Big((P(\mathbb{Q})\cdot i(\mathcal{H})) \times P(\mathbb{A}_f)/K\Big)\,.$$

Now copy the proof of [P1], Lemma 11.7. q.e.d.

Lemma 2.7: Let $K = K^W \rtimes L$ and $v \in W(\mathbb{A}_f)$.
Then K^W is contained in an open compact subgroup $\widetilde{K^W}$ of $W(\mathbb{A}_f)$ containing v and stable under $i(L)$.
Proof: The group generated by K^W, v and its $i(L)$–translates is still open and compact. This claim is quite obvious in the case where W is abelian. The general case follows since W is unipotent. q.e.d.

Lemma 2.8: The statement of 2.1 holds if $W = U$, i.e., if

$$M^K(P,\mathfrak{X})_{\mathbb{C}} = \mathbb{G}^l_{m,\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} M^L(G,\mathcal{H})_{\mathbb{C}} \,.$$

Proof: By 2.6, it suffices to show that $\overline{[i_v]}^*_{\mathbb{C}}(\overline{\varphi})$ respects the weight and Hodge filtrations, for any $v \in W(\mathbb{Q})$.
Let $\widetilde{K} := \widetilde{K}^W \rtimes L$ as in 2.7. $\widetilde{K}$ is still neat, and $\mu_{K,\infty,\sigma_0} = [\cdot 1]^*_{\mathbb{C}} \circ \mu_{\widetilde{K},\infty,\sigma_0}$, where

$$[\cdot 1] : M^K(P,\mathfrak{X}) \longrightarrow M^{\widetilde{K}}(P,\mathfrak{X}) \,.$$

On the other hand, $\mathcal{L}og(i,K)_{\infty,\sigma_0} = [\cdot 1]^*_{\mathbb{C}}(\mathcal{L}og(i,\widetilde{K})_{\infty,\sigma_0})$ as follows from [W2], Theorem 3.5.i), so we may assume v is contained in K.
But then, the morphisms on the level of varieties $[i]$ and $[i_v]$ coincide, and we are reduced to the case $v = 1$.
$[i]^*_{\mathbb{C}}\mathcal{L}og(i,K)_{\infty,\sigma_0}$ is the completed symmetric algebra in

$$(\mathcal{H}^{-l+1}([\pi]_{\mathbb{C}})_*\mathbb{Q}(0))^\vee = \mathbb{Q}(1)^{\dim U} \,,$$

and $\mu_{L,\infty,\sigma_0}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$ is the completed symmetric algebra in $\mu_{L,\infty,\sigma_0}(\mathrm{Lie}\, W)$, which is also $\mathbb{Q}(1)^{\dim U}$. In order to prove this last claim, observe that μ_{K,∞,σ_0} factors through $\mathrm{Rep}_{\mathbb{Q}}(P_1)$, where P_1 is as in the proof of 1.3. By [P1], Proposition 2.14, P_1 acts on $\mathrm{Lie}\, W$ through a scalar character $P_1 \longrightarrow \mathbb{G}_m$, and the claim follows because of weight reasons.
Since $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ respects the embeddings of the Lie algebras in their universal envelopes, and since any morphism

$$\mathrm{For}(\mathbb{Q}(1)) \longrightarrow \mathrm{For}(\mathbb{Q}(1))$$

respects the weight and Hodge filtrations, we get the desired conclusion. q.e.d.

We recall the definition of the Shimura data $(CSp_{2g,\mathbb{Q}}, \mathcal{H}_{2g})$, $(V_{2g} \rtimes CSp_{2g,\mathbb{Q}}, \mathcal{H}'_{2g})$ and $(P_{2g}, \mathfrak{X}_{2g})$ for $g \in \mathbb{N}$ ([P1], Examples 2.7 and 2.25)†:
let $\dim_{\mathbb{Q}}(V_{2g}) = 2g$,

$$\Psi : V_{2g} \times V_{2g} \longrightarrow \mathbb{Q}$$

a nondegenerate alternating form,
$CSp_{2g,\mathbb{Q}}$ the group of all $f \in GL(V_{2g})$ such that

$$\Psi(f(v), f(v')) = \lambda(f) \cdot \Psi(v, v')$$

for some $\lambda(f) \in \mathbb{G}_m$,

---

†By definition, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non–negative integers.

$\mathcal{H}_{2g}$ the set of all homomorphisms

$$k : \mathbb{S} \longrightarrow CSp_{2g,\mathbb{R}}$$

which induce a pure Hodge structure of type $\{(-1,0),(0,-1)\}$ on V_{2g} and for which either Ψ or $-\Psi$ defines a polarization.

$CSp_{2g,\mathbb{Q}}$ is known to be reductive.

By the construction of "unipotent extension" explained in [P1], 2.16 and Proposition 2.17, there exist mixed Shimura data $(P_{2g,a},\mathfrak{X}_{2g,a}) := (V_{2g} \rtimes CSp_{2g,\mathbb{Q}}, \mathcal{H}'_{2g})$ and $(P_{2g},\mathfrak{X}_{2g})$, P_{2g} denoting the group $W_{2g} \rtimes CSp_{2g,\mathbb{Q}}$, where

$$1 \longrightarrow U_{2g} \longrightarrow W_{2g} \longrightarrow V_{2g} \longrightarrow 1$$

is the central extension by $U_{2g} := \mathbb{G}_{a,\mathbb{Q}}$ defined by Ψ.

We have morphisms

$$(P_{2g},\mathfrak{X}_{2g}) \xrightarrow{\pi_m} (P_{2g,a},\mathfrak{X}_{2g,a}) \xrightarrow{\pi_a} (CSp_{2g,\mathbb{Q}},\mathcal{H}_{2g})$$

of Shimura data.

By Lemma 2.5, the statement of 2.1 holds for $(P_{2g,a},\mathfrak{X}_{2g,a})$.

In particular, we have the canonical identification of admissible variations

$$(\mathcal{H}^{-g+1}([\pi_a]_{\mathbb{C}})_*\mathbb{Q}(0))^\vee \xrightarrow{\sim} \mu_{L,\infty,\sigma_0}(V_{2g})$$

for any neat, open and compact $L \leq CSp_{2g}(\mathbb{A}_f)$.

Lemma 2.9: Let $(P,\mathfrak{X})$ be irreducible Shimura data.

Let $2g := \dim_{\mathbb{Q}}(V) > 0$ and $l-1 := \dim_{\mathbb{Q}}(U)$.

Then there exist morphisms

$$(P',\mathfrak{X}') \longrightarrow (P,\mathfrak{X})$$
$$\text{and } (P',\mathfrak{X}') \longrightarrow \prod_{i=1}^{l}(P_{2g},\mathfrak{X}_{2g})$$

of Shimura data with the following properties:

a) $P' \longrightarrow P$ is an epimorphism. Its kernel is of dimension 1 and of weight -2.

So the pure Shimura data underlying $(P',\mathfrak{X}')$ coincide with $(G,\mathcal{H})$.

b) the induced map

$$(P',\mathfrak{X}') \longrightarrow (G,\mathcal{H}) \times_{\prod_{i=1}^{l}(CSp_{2g,\mathbb{Q}},\mathcal{H}_{2g})} \prod_{i=1}^{l}(P_{2g},\mathfrak{X}_{2g})$$

is an embedding.

Proof: In the proof of [P1], 2.26.b), the above morphisms are constructed. a) follows directly from 2.26.b), while b) follows from the last statement on page 45 in [P1]. q.e.d.

For the proof of Theorem 2.1, we essentially reduce ourselves to this case.

We need one last preparatory result:

Theorem 2.10: Let $L \leq CSp_{2g}(\mathbb{A}_f)$ be neat, open and compact.

a) In $\mathrm{Var}_{\mathbb{Q}}(M^L(CSp_{2g,\mathbb{Q}}, \mathcal{H}_{2g})_{\mathbb{C}})$, every extension

$$0 \longrightarrow \mathbb{Q}(1) \longrightarrow \mathbf{E} \longrightarrow (\mathcal{H}^{-g+1}([\pi_a]_{\mathbb{C}})_*\mathbb{Q}(0))^{\vee} \longrightarrow 0$$

splits.

b) Any splitting of the underlying extension of local systems in a) is compatible with the weight and Hodge filtrations.

Proof: a) On a complex manifold, to give a one–extension of local systems in the category of perverse sheaves is equivalent to giving a one–extension in the category of local systems. By [S2], Theorem 3.27, we have to show that

$$\begin{aligned}&\mathrm{Ext}^1_{MHM_{\mathbb{Q}}(M^L_{\mathbb{C}})}(\mathbb{Q}(-1), \mathcal{H}^{-g+1}([\pi_a]_{\mathbb{C}})_*\mathbb{Q}(0)) = \\ &\mathrm{Ext}^1_{MHM_{\mathbb{Q}}(M^L_{\mathbb{C}})}(\mathbb{Q}(0), \mathcal{H}^{-g+1}([\pi_a]_{\mathbb{C}})_*\mathbb{Q}(0)(1))\end{aligned}$$

is zero.

We want to write down the Leray spectral sequence on the level of algebraic mixed Hodge modules for the map

$$a : M^L_{\mathbb{C}} \longrightarrow \mathrm{Spec}(\mathbb{C})\,.$$

This is possible because $\mathcal{H}^q a_*$ and $\mathcal{H}^q a^*$ are defined not only as cohomological functors but as cohomology objects of functors a_* and a^* defined on the level of derived categories. a_* and a^* are adjoint, and although they don't in general appear as right or left derived functors, it is possible, using the theory of exact couples ([Hu], VIII, § 6) to construct the Leray spectral sequence.

We get an exact sequence

$$0 \to \mathrm{Ext}^1_{MHS_{\mathbb{Q}}}(\mathbb{Q}(0), H^0(\mathbf{V})) \to \mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{Q}}(M^L_{\mathbb{C}})}(\mathbb{Q}(0), \mathbf{V}) \to \mathrm{Hom}_{MHS_{\mathbb{Q}}}(\mathbb{Q}(0), H^1(\mathbf{V}))\,,$$

where we set $\mathbf{V} := \mathcal{H}^{-g+1}([\pi_a]_{\mathbb{C}})_* \mathbb{Q}(0)(1)$.[†] Also, we write $H^q(\mathbf{V})$ for the group $H^q(M^L(\mathbb{C}), \mathbf{V})$.

$H^0(\mathbf{V})$ and $H^1(\mathbf{V})$ can be calculated via cohomology of the fundamental group $\Gamma(g_f) \leq CSp_{2g}(\mathbb{Q})$. Observe that $\mathbf{V}(-1)$ corresponds to the representation V_{2g} of CSp_{2g}.

So there are no non–trivial invariants under $\Gamma(g_f)$, and

$$H^0(\mathbf{V}) = 0 \, .$$

If $g > 1$, then by [R], § 3, Theorem 2,

$$H^1(\mathbf{V}) = 0 \, .$$

For $g = 1$, we have to analyze the Hodge type of $H^1(\mathbf{V})$ more closely.

Its weight zero part is $H^1_!(V_2)(1)$, which by [Z], § 12, is of Hodge type $\{(-1, 1), (1, -1)\}$. By [S1], 5.3.10, this is the same Hodge structure as the one given by Saito's formalism.

Hence there are no non–trivial morphisms of MHS

$$\mathbb{Q}(0) \longrightarrow H^1(\mathbf{V}) \, .$$

Alternatively, it is possible to prove

$$\mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{Q}}(M^L_{\mathbb{C}})}(\mathbb{Q}(0), \mathbf{V}) = 0$$

for $g = 1$ without making use of Saito's formalism by directly writing down the extension data in terms of multi–valued functions on $M^L(\mathbb{C})$. Unsurprisingly, these functions turn out to be connected to cusp forms of weight 3 with rational Eichler–Shimura cocycle. By [Z], § 12, such cusp forms are 0.

b) Any two splittings of the underlying extension of local systems, or rather the dual extension, differ by a morphism

$$\mathbb{Q} \longrightarrow \mathrm{For}(\mathbf{V})$$

of local systems.

Since $H^0(\mathbf{V}) = 0$, there is at most one such splitting. But by a), at least one splitting exists on the level of variations. q.e.d.

[†] Actually, the last map is surjective because in the category $MHS_{\mathbb{Q}}$ of mixed graded–polarizable Hodge structures, all Ext^qs vanish for $q \geq 2$ ([J], Remark 9.3.).

Remark: We note that the vanishing of $\operatorname{Ext}^1_{\mathrm{Var}_{\mathbb{Q}}(M^L_{\mathbb{C}})}(\mathbb{Q}(0), \mathbf{V})$, for $g = 1$, implies Shioda's theorem: the group of sections of the universal elliptic curve $M^K_{\mathbb{C}}$ over $M^L_{\mathbb{C}}$ injects, modulo torsion, into this group of extensions via the Abel–Jacobi map. Hence the Mordell–Weil group of $M^K_{\mathbb{C}}$ consists only of torsion.

Proof of Theorem 2.1: Recall that we have to prove that the isomorphism

$$\overline{\varphi} : \mathrm{For}(\mathcal{L}og(i, K)_{\infty,\sigma_0}) \xrightarrow{\sim} \mathrm{For}(\mathcal{G})$$

of Lemma 2.4 respects the weight and Hodge filtrations.

Without loss of generality, assume $W \neq 0$. By 2.8, we may also assume that $W \neq U$, i.e., that $V \neq 0$. By the same argument as in the proof of 2.9, it suffices to show that $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ respects the filtrations.

Since G is reductive and because $\mathrm{Lie}\, U$ and $\mathrm{Lie}\, V$ have different weights, the sequence of G–modules

$$0 \longrightarrow \mathrm{Lie}\, U \longrightarrow \mathrm{Lie}\, W \longrightarrow \mathrm{Lie}\, V \longrightarrow 0$$

splits in a unique way:

$$\mathrm{Lie}\, W = \mathrm{Lie}\, U \oplus \mathrm{Lie}\, V \, ,$$

so by the Poincaré–Birkhoff–Witt Theorem ([Hum], 17.3, Corollary C), we get a non–multiplicative isomorphism of G–modules

$$\hat{\mathfrak{U}}(\mathrm{Lie}\, W) \longrightarrow \hat{\mathfrak{U}}(\mathrm{Lie}\, U) \hat{\otimes}_{\mathbb{Q}} \hat{\mathfrak{U}}(\mathrm{Lie}\, V)$$

sending 1 to $1 \hat{\otimes} 1$.

We need to show

i) the same statement for the sequence

$$0 \longrightarrow \mathrm{Lie}(\overline{U}_{\overline{x}}) \longrightarrow \mathrm{Lie}(\overline{W}_{\overline{x}}) \xrightarrow{[\pi_m]_{\mathbb{C}}} \mathrm{Lie}(\overline{V}_{\overline{x}_a}) \longrightarrow 0$$

of variations on $M^L_{\mathbb{C}}$.

Here, $[\pi_m]_{\mathbb{C}}$ comes from the projection

$$\pi_m : (P, \mathfrak{X}) \longrightarrow (P, \mathfrak{X})/U = (P_a, \mathfrak{X}_a) \, ,$$

$\overline{x}_a := [\pi_m](\mathbb{C})(\overline{x})$,

$\overline{U}_{\overline{x}}$ is the pro–unipotent envelope of $\pi_1(M^K(\mathbb{C})_{\overline{x}_a}, \overline{x})$, and

$\overline{V}_{\overline{x}_a}$ is the pro–unipotent envelope of $\pi_1(M^{\pi_m(K)}(\mathbb{C})_{\overline{y}}, \overline{x}_a)$.

ii) that under the isomorphism $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$, which respects the underlying sequences of local systems, the unique splittings correspond.

Let us first see why i) and ii) conclude the proof:
first observe that because of the multiplicative structure of $[i]^*_{\mathbb{C}}\mathcal{L}og(i,K)_{\infty,\sigma_0}$, the variation $\mathrm{Lie}(\overline{W}_{\overline{x}})$ on $M^L_{\mathbb{C}}$ determines the variation $[i]^*_{\mathbb{C}}\mathcal{L}og(i,K)_{\infty,\sigma_0}$ uniquely.
The isomorphism

$$\mathrm{For}(\mathrm{Lie}(\overline{V}_{\overline{x}_a})) \xrightarrow{\sim} \mathrm{For}(\mu_{L,\infty,\sigma_0}(\mathrm{Lie}\, V))$$

induced by $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ is exactly the one coming from the relative situation $(P_a,\mathfrak{X}_a) \longrightarrow (G,\mathcal{H})$ where we know it is compatible with the filtrations by the remark following 2.5.
The isomorphism

$$\mathrm{For}(\mathrm{Lie}(\overline{U}_{\overline{x}})) \xrightarrow{\sim} \mathrm{For}(\mu_{L,\infty,\sigma_0}(\mathrm{Lie}\, U))$$

induced by $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ is automatically compatible with the filtrations since $\mathrm{Lie}(\overline{U}_{\overline{x}})$ and $\mu_{L,\infty,\sigma_0}(\mathrm{Lie}\, U)$ are isomorphic to $\mathbb{Q}(1)^{\dim U}$.
So i) and ii) are necessary and sufficient for $\overline{[i]}^*_{\mathbb{C}}(\overline{\varphi})$ itself to respect the filtrations.

Now for the proof of i) and ii):
we may replace $(P,\mathfrak{X})$ by the Shimura data $(P',\mathfrak{X}')$ of 2.9. This has the effect of enlarging U. The sheaves and the morphism on the level of $(P,\mathfrak{X})$ are obtained by those on the level of $(P',\mathfrak{X}')$ by taking $\ker(P' \longrightarrow P)$–coinvariants. Similarly, since $(P',\mathfrak{X}')$ has the same underlying pure Shimura data as $(\widetilde{P},\widetilde{\mathfrak{X}}) := (G,\mathcal{H}) \times_{\prod_{i=1}^l (CSp_{2g,\mathbb{Q}},\mathcal{H}_{2g})} \prod_{i=1}^l (P_{2g},\mathfrak{X}_{2g})$, we may replace $(P',\mathfrak{X}')$ by $(\widetilde{P},\widetilde{\mathfrak{X}})$. This has the effect of enlarging V. The sheaves on the level of $(P',\mathfrak{X}')$ are subsheaves of those on the level of $(\widetilde{P},\widetilde{\mathfrak{X}})$, pulled back via the embedding, and an analogous statement is true for the morphism $\overline{\varphi}$.
So we may finally assume $(P,\mathfrak{X}) = (P_{2g},\mathfrak{X}_{2g})$.
Then i) follows from 2.10.a) while ii) is a direct consequence of 2.10.b). q.e.d.

§ 3 The canonical construction of mixed sheaves: de Rham version

The aim of this paragraph is to show that the flat vector bundles together with their connection, weight and Hodge filtrations defined by μ_{K,∞,σ_0} descend canonically to the reflex field $E := E(P, \mathfrak{X})$ of our mixed Shimura data.
More precisely, let F be a subfield of $\mathbb{R}$. Via $\overline{\sigma_0}$, we consider $\overline{E}$ as a subfield of $\mathbb{C}$. In particular, the composite FE is defined as a subfield of $\mathbb{C}$. Recall ([D3], II, Théorème 5.9) that any flat analytic vector bundle on the set of $\mathbb{C}$–valued points of a smooth complex variety carries a canonical algebraic structure. The algebraic connection is regular at infinity. If the vector bundle underlies an admissible variation of Hodge structure, then the Hodge filtration is a filtration by subbundles that are algebraic with respect to this canonical algebraic structure ([Ka], Proposition 1.11.3).
So if K is a neat open compact subgroup of $P(\mathbb{A}_f)$, the functor μ_{K,∞,σ_0} induces a functor

$$\mu_{K,DR,\mathbb{C}} : \mathrm{Rep}_F(P) \longrightarrow [\pi]_{\mathbb{C}}\text{–}U\mathrm{BiF}(M^K(P,\mathfrak{X})_{\mathbb{C}}) \, ,$$

where the right hand side denotes the category of $[\pi]_{\mathbb{C}}$–unipotent objects in the category $\mathrm{BiF}(M^K(P,\mathfrak{X})_{\mathbb{C}})$ of vector bundles on $M^K(P,\mathfrak{X})_{\mathbb{C}}$, together with a flat connection, which is regular at infinity, a finite ascending weight filtration $W.$ by flat subbundles and a descending Hodge filtration $\mathcal{F}^{\cdot}$ by subbundles.
We want to define a functor

$$\mu_{K,DR} : \mathrm{Rep}_F(P) \longrightarrow [\pi]_{FE}\text{–}U\mathrm{BiF}(M^K(P,\mathfrak{X})_{FE}) \, ,$$

where the right hand side is defined in a manner analogous to the category above.[†] We require that $\mu_{K,DR} \otimes_{FE} \mathbb{C} = \mu_{K,DR,\mathbb{C}}$ and that $\mu_{K,DR}$ is well behaved under pull–back via morphisms induced by the action of $P(\mathbb{A}_f)$ or by morphisms of Shimura data.

In the case of pure Shimura data, the existence of $\mu_{K,DR}$ is due to Milne:

Theorem 3.1: Let $(G, \mathcal{H})$ be pure Shimura data, $L \leq G(\mathbb{A}_f)$ neat, open and compact. Then there is a functor

$$\mu_{L,DR} : \mathrm{Rep}_F(G) \longrightarrow \mathrm{BiF}(M^L(G,\mathcal{H})_{FE})$$

[†] For the definition of regularity at infinity of a connection over an arbitrary field of characteristic 0, see [D3], II, remark following Définition 4.5.

with the required properties. Furthermore, if $g \geq 1$ and $(G, \mathcal{H}) = (CSp_{2g,\mathbb{Q}}, \mathcal{H}_{2g})$, then $\mu_{L,DR}$ maps the representation V_{2g} of $CSp_{2g,\mathbb{Q}}$ over $\mathbb{Q}$ to the vector bundle $R^1[\pi]_* \Omega^{\cdot}_{M^K/M^L}$, equipped with the Gauß–Manin connection and the natural weight and Hodge filtrations. Here we choose a section i of the morphism $\pi : (V_{2g} \rtimes CSp_{2g,\mathbb{Q}}, \mathcal{H}'_{2g}) \longrightarrow (CSp_{2g,\mathbb{Q}}, \mathcal{H}_{2g})$ and let $K := K^{V_{2g}} \rtimes L$, where $K^{V_{2g}}$ is any open compact subgroup of $V_{2g}(\mathbb{A}_f)$ stable under conjugation by L, so $[\pi] : M^K \longrightarrow M^L$ is an abelian scheme of relative dimension g.

Proof: Up to the FE–rationality of the Hodge filtration, the claims follow from [M], III, Theorem 5.1.a), Lemma 3.1.b) and Example 4.2.a).[†] Note that we may replace $(G, \mathcal{H})$ by $(G, h(\mathcal{H}))$. Because of our axiom vii)', the pure Shimura data $(G, \mathcal{H})$ satisfy (2.1*) of [M], II, and hence ([M], page 347) $G = G^c$ in the notation of [M], III.

In order to show that the Hodge filtration is defined over FE, we need to examine the proof of [M], III, Theorem 5.1.a) more closely.

Let $\beta : \mathcal{H} \hookrightarrow \check{\mathcal{H}}(\mathbb{C})$ be the Borel embedding of $\mathcal{H} = h(\mathcal{H})$ ([M], III, § 1 or [P1], proof of Proposition 1.7), and let $\mathbf{V} \in \mathrm{Rep}_F(G)$.

Define the standard principal bundle

$$P(G, \mathcal{H})(\mathbb{C}) := G(\mathbb{Q}) \setminus (\mathcal{H} \times G(\mathbb{C}) \times G(\mathbb{A}_f)) \, .$$

By [M], III, Propositions 3.4 and 3.5, there is a $G(\mathbb{C})$–equivariant map

$$\gamma(\mathbb{C}) : P(G, \mathcal{H})(\mathbb{C}) \longrightarrow \check{\mathcal{H}}(\mathbb{C}) \, ,$$

and if we consider the following diagram:

$$\begin{tikzcd}
 & P(G, \mathcal{H})(\mathbb{C}) \arrow[dl, "\gamma(\mathbb{C})"'] \arrow[dr, "pr_{13}(\mathbb{C})"] & \\
\check{\mathcal{H}}(\mathbb{C}) & & M^L(G, \mathcal{H})(\mathbb{C}) \\
 & \mathcal{H} \times G(\mathbb{A}_f) \arrow[ul, "\beta \circ pr_1"] \arrow[ur] &
\end{tikzcd}$$

then the analytic vector bundle $F_{\mathcal{O}}(\mu_{L,\infty,\sigma_0}(\mathbf{V}))$ underlying $\mu_{L,\infty,\sigma_0}(\mathbf{V})$ can also be obtained as follows: $\mathbf{V}_{\mathbb{C}}$ defines a vector bundle on $\check{\mathcal{H}}(\mathbb{C})$, and $F_{\mathcal{O}}(\mu_{L,\infty,\sigma_0}(\mathbf{V}))$

[†] As can be seen from the proof of [M], III, Theorem 5.1.a), the assumption that FE be a number field is not necessary for our purposes.

is uniquely determined by requiring that $\gamma(\mathbb{C})^*(\mathbf{V}_{\mathbb{C}}) = pr_{13}(\mathbb{C})^*(F_{\mathcal{O}}(\mu_{L,\infty,\sigma_0}(\mathbf{V})))$. Note that we defined $\mu_{L,\infty,\sigma_0}(\mathbf{V})$ via the lower branch of the above diagram. Now $\check{\mathcal{H}}(\mathbb{C})$ and $P(G,\mathcal{H})(\mathbb{C})$ are the sets of $\mathbb{C}$–valued points of schemes $\check{\mathcal{H}}_{\mathbb{C}}$ and $P(G,\mathcal{H})_{\mathbb{C}}$ over $\mathbb{C}$. The main results of [M], III, § 4 state that $P(G,\mathcal{H})_{\mathbb{C}}$ has a model $P(G,\mathcal{H})$ over E and that $pr_{13}(\mathbb{C})$ descends to an algebraic morphism pr_{13} over E ([M], III, Theorems 4.3.a) and 4.1.a)), which is faithfully flat. Furthermore ([M], III, Theorem 4.6.a)), $\gamma(\mathbb{C})$ descends to an algebraic morphism γ over E. Here, the model $\check{\mathcal{H}}$ of $\check{\mathcal{H}}_{\mathbb{C}}$ over E is given by the fact that E is the field of definition of the $G(\mathbb{C})$–conjugacy class of the μ_h, $h \in \mathcal{H}$.

Now the model of $F_{\mathcal{O}}(\mu_{L,\infty,\sigma_0}(\mathbf{V}))$ is defined by the same rule as above, applied to the algebraic morphisms

$$\begin{tikzcd}\check{\mathcal{H}} & P(G,\mathcal{H}) \arrow[l, "\gamma"'] \arrow[r, two heads, "pr_{13}"] & M^L(G,\mathcal{H})\end{tikzcd} .$$

So what we have to show is that the Hodge filtration of the vector bundle $\mathbf{V}_{\mathbb{C}}$ on $\check{\mathcal{H}}(\mathbb{C})$ is defined over FE. In order to prove this, we may replace $\mathbf{V}$ by a faithful representation ([DM], proof of Proposition 2.20.b)). But then $\check{\mathcal{H}}_{\mathbb{C}}$ is realized as a subvariety of a Grassmannian $GL(\mathbf{V})_{\mathbb{C}}/Q$, and our claim follows from the definition of $E = E(G,\mathcal{H})$. q.e.d.

Corollary 3.2: Let $(P,\mathfrak{X})$ be mixed Shimura data, and let π denote the projection $(P,\mathfrak{X}) \longrightarrow (G,\mathcal{H}) = (P,\mathfrak{X})/W$.

Choose a section i of π to write $P = W \rtimes G$.

If $\operatorname{Lie} W$ is pure, then $\mu_{L,DR}(\operatorname{Lie} W) = R^1[\pi]_* \Omega^{\cdot}_{M^K/M^L}$, the latter being equipped with the Gauß–Manin connection and the natural weight and Hodge filtrations. Here we let $K := K^W \rtimes L$, where K^W is any open compact subgroup of $W(\mathbb{A}_f)$ stable under conjugation by L.

Proof: If $U = 0$, we use 1.1.c) to reduce ourselves to the situation considered in 3.1.

So let $W = U$. We may assume that $(P,\mathfrak{X})$ is irreducible. Using [P1], Proposition 2.14, we see that we may replace $(P,\mathfrak{X})$ by the mixed Shimura data $(P_0,\mathfrak{X}_0)$ of [P1], 2.24. The underlying pure Shimura data is $(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$, and there is a morphism $\varphi : (CSp_{2,\mathbb{Q}},\mathcal{H}_2) \longrightarrow (\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$. We replace $(P_0,\mathfrak{X}_0)$ by its base change by φ, observing that the representation U_0 of $CSp_{2,\mathbb{Q}}$ is the determinant of the representation V_2. Since $\mu_{L,DR}$ respects the tensor structures, our claim is proven. q.e.d.

One way of extending the de Rham version of the canonical construction to the mixed case would be to generalize the results of [M], III, §§ 1–5 in this direction. We prefer another approach, which uses the results of [W2], § 3: fix a Levi section i of $\pi : (P, \mathfrak{X}) \longrightarrow (G, \mathcal{H})$, and let $K = K^W \rtimes L$ be as before. By the construction described before [W2], Theorem 3.6, the bifiltered flat pro–vector bundle underlying the logarithmic pro–variation $\mathcal{L}og(i, K)_{\infty, \sigma_0}$ descends canonically to E, giving rise to a pro–object $\mathcal{L}og(i, K)_{DR}$ of $[\pi]$–$U\mathrm{BiF}(M^K(P, \mathfrak{X}))$.

Definition: $\mathcal{L}og_{DR} := \mathcal{L}og(i, K)_{DR}$ is called the *logarithmic bifiltered flat pro–vector bundle* on $M^K(P, \mathfrak{X})$.

$\mathcal{L}og_{DR}$ comes equipped with a flat section

$$1 \in (W_0 \cap F^0)(\Gamma(M^L(G, \mathcal{H}), [i]^* \mathcal{L}og_{DR})).$$

The pair $(\mathcal{L}og_{DR}, 1)$ is rigid, i.e., it admits no non–trivial automorphisms. In fact, this already holds on the level of underlying flat vector bundles. $\mathcal{L}og_{DR}$ carries the structure of cocommutative coalgebra. It induces on $[i]^* \mathcal{L}og_{DR}$ a Hopf algebra structure. These claims all follow from

Theorem 3.3: Let $\mathrm{VB}(M^L)$ and $[\pi]$–$U\mathrm{VB}(M^K)$ denote the Tannakian categories of flat vector bundles on M^L and $[\pi]$–unipotent flat vector bundles on M^K, whose connection is regular at infinity. Let $\mathrm{For}(\mathcal{L}og_{DR})$ be the pro–object of $[\pi]$–$U\mathrm{VB}(M^K)$ underlying $\mathcal{L}og_{DR}$.

i) The natural transformation of functors from $[\pi]$–$U\mathrm{VB}(M^K)$ to $\mathrm{VB}(M^L)$

$$ev : [\pi]_* \underline{\mathrm{Hom}}(\mathrm{For}(\mathcal{L}og_{DR}), \_) \longrightarrow i^* \, ,$$
$$\varphi \longmapsto (i^* \varphi)(1)$$

is an isomorphism.

ii) In particular, the natural transformation of functors from $[\pi]$–$U\mathrm{VB}(M^K)$ to Vec_E

$$\Gamma(M^L, ev)^{\nabla = 0} : \mathrm{Hom}(\mathrm{For}(\mathcal{L}og_{DR}), \_) \longrightarrow \Gamma(M^L, i^* \_)^{\nabla = 0}$$

is an isomorphism.

Proof: This is the remark preceding [W2], Theorem 3.6. q.e.d.

It follows that for K of the shape above, there is at most one way of defining $\mu_{K,DR}$ in a way compatible with both $\mu_{L,DR}$ and μ_{K,∞,σ_0}:
let $\hat{\mathfrak{U}}(\operatorname{Lie} W)$ be the completed universal envelope of $\operatorname{Lie} W$ over $\mathbb{Q}$, equipped with the usual action of P. By functoriality, the pro–object $\mathcal{G} := \mu_{K,DR}(\hat{\mathfrak{U}}(\operatorname{Lie} W))$ of $[\pi]$–$U\mathrm{BiF}(M^K(P,\mathfrak{X}))$, that we want to define, admits a flat section

$$1 \in (W_0 \cap F^0)(\Gamma(M^L(G,\mathcal{H}), [i]^*\mathcal{G})).$$

By 3.3.ii) there is a unique isomorphism $\varphi : \mathrm{For}(\mathcal{L}og_{DR}) \xrightarrow{\sim} \mathrm{For}(\mathcal{G})$ of flat vector bundles sending 1 to 1. But since $\mu_{K,DR} \otimes_E \mathbb{C} = \mu_{K,DR,\mathbb{C}}$, the pro–object $\mathcal{G}\hat{\otimes}_E\mathbb{C}$ is the flat bifiltered pro–vector bundle underlying $\mathcal{L}og_{\infty,\sigma_0} = \mu_{K,\infty,\sigma_0}(\hat{\mathfrak{U}}(\operatorname{Lie} W))$. We conclude from [W2], Theorem 3.5.ii) and 3.5.i) that φ must necessarily respect $W_\cdot$ and $\mathcal{F}^\cdot$, and so we have a unique isomorphism $\mathcal{L}og_{DR} \xrightarrow{\sim} \mu_{K,DR}(\hat{\mathfrak{U}}(\operatorname{Lie} W))$ sending 1 to 1.
Note that $\pi^*(\mathrm{Rep}_F(G))$ and the finite–dimensional subquotients of $\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}}F$ generate $\mathrm{Rep}_F(P)$ as a Tannakian category. So the extension of $\mu_{L,DR}$ to $\mathrm{Rep}_F(P)$ is uniquely determined by its value on $\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}}F$, and we must define this to be $\mathcal{L}og_{DR}\hat{\otimes}_E FE$. We have to show that this actually defines a functor $\mu_{K,DR} : \mathrm{Rep}_F(P) \longrightarrow [\pi]_{FE}$–$U\mathrm{BiF}(M^K(P,\mathfrak{X})_{FE})$, i.e., an FE–structure on the functor $\mu_{K,DR,\mathbb{C}}$.
Observe that the weight–graded parts of $\hat{\mathfrak{U}}(\operatorname{Lie} W)$ are subquotients of finite sums of tensor powers of a/a^2, where a denotes the augmentation ideal of $\hat{\mathfrak{U}}(\operatorname{Lie} W)$. This together with Corollary 3.2 shows that for any finite–dimensional subquotient $\mathbf{V}$ of $\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}}F$, on which W acts trivially, the corresponding subquotient $\mu_{K,DR,\mathbb{C}}(\mathbf{V}_{\mathbb{C}})$ of $\mathcal{L}og_{DR}\hat{\otimes}_E\mathbb{C}$ carries an FE–structure, which coincides with that given by $[\pi]^*_{FE}\mu_{L,DR}(\mathbf{V})$.[†] Given an arbitrary subquotient $\mathbf{V}$ of $\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}}F$, the flat vector bundle underlying $\mu_{K,DR,\mathbb{C}}(\mathbf{V}_{\mathbb{C}})$ carries an FE–structure because the same is true for $[i]^*_{\mathbb{C}}\mu_{K,DR,\mathbb{C}}(\mathbf{V}_{\mathbb{C}})$: use e.g. [W2], Lemma 2.10. Lemma 2.6 then shows that the bifiltered flat vector bundle $\mu_{K,DR,\mathbb{C}}(V_{\mathbb{C}})$ descends to FE. This is defined to be $\mu_{K,DR}(\mathbf{V})$.
It remains to define the value of $\mu_{K,DR}$ on morphisms. We claim that the morphisms on the level of $\mu_{K,DR,\mathbb{C}}$ descend to FE. This is certainly true for morphisms of representations of G. But a morphism $\mathbf{V}\longrightarrow\mathbf{W}$ can be interpreted as an element of $\mathbf{V}^\vee\otimes_F\mathbf{W}$ invariant under P. It gives rise to a flat section of

---

[†] The reader should observe that the image of $\mu_{K,DR,\mathbb{C}}$ is a subcategory of $[\pi]_{\mathbb{C}}$–$U\mathrm{BiF}(M^K(P,\mathfrak{X})_{\mathbb{C}})$, that is actually abelian.

$\mu_{K,DR,\mathbb{C}}(\mathbf{V}^\vee \otimes_F \mathbf{W})$, whose FE–rationality can be tested over $[i]_{FE}(M^L(G,\mathcal{H})_{FE})$.

Observe that any open and compact $K \leq P(\mathbb{A}_f)$ is contained in a subgroup of the form $K^W \rtimes L$: use Lemma 2.7. So we arrive at the following

Theorem 3.4: There is a functor

$$\mu_{K,DR} : \mathrm{Rep}_F(P) \longrightarrow [\pi]_{FE}\text{–}U\mathrm{BiF}(M^K(P,\mathfrak{X})_{FE})\,,$$

which is uniquely determined by the following properties i)–iii):

i) $\mu_{K,DR} \otimes_{FE} \mathbb{C} = \mu_{K,DR,\mathbb{C}}$.

ii) If $L := \pi(K)$, then the diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P) \arrow[r, "\mu_{K,DR}"] & {[\pi]_{FE}\text{–}U\mathrm{BiF}(M^K(P,\mathfrak{X})_{FE})} \\
\mathrm{Rep}_F(G) \arrow[u, "\pi^*"] \arrow[r, "\mu_{L,DR}"] & \mathrm{BiF}(M^L(G,\mathcal{H})_{FE}) \arrow[u, "{[\pi]^*_{FE}}"']
\end{tikzcd}$$

commutes.

iii) For any neat open compact subgroup K' of K, the diagram

$$\begin{tikzcd}
& {[\pi]_{FE}\text{–}U\mathrm{BiF}(M^{K'}(P,\mathfrak{X})_{FE})} \\
\mathrm{Rep}_F(P) \arrow[ur, "\mu_{K',DR}"] \arrow[dr, "\mu_{K,DR}"'] & \\
& {[\pi]_{FE}\text{–}U\mathrm{BiF}(M^{K}(P,\mathfrak{X})_{FE})} \arrow[uu, "{[\cdot 1]^*_{FE}}"']
\end{tikzcd}$$

commutes.

If K is of the shape $K^W \rtimes L$, then the pro–object $\mu_{K,DR}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W)\hat{\otimes}_{\mathbb{Q}} F)$ is mapped to $\mathcal{L}og(i,K)_{DR}\hat{\otimes}_E FE$. Furthermore, $\mu_{K,DR}$ has the following properties:

iv) For any morphism $\varphi : (P', \mathfrak{X}') \longrightarrow (P, \mathfrak{X})$ of Shimura data and any neat open compact subgroup $K' \leq P'(\mathbb{A}_f)$ such that $\varphi(K') \leq K$, the diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P') \arrow[r, "\mu_{K',DR}"] & {[\pi']_{FE'}\text{–}U\mathrm{BiF}(M^{K'}(P,\mathfrak{X})_{FE'})} \\
\mathrm{Rep}_F(P) \arrow[u, "\varphi^*"] \arrow[r, "\mu_{K,DR} \otimes_{FE} FE'"] & {[\pi]_{FE'}\text{–}U\mathrm{BiF}(M^{K}(G,\mathfrak{X})_{FE'})} \arrow[u, "{[\varphi]^*_{FE'}}"']
\end{tikzcd}$$

commutes, where we let $E' := E(P', \mathfrak{X}')$.

v) For any $p_f \in P(\mathbb{A}_f)$, the diagram

$$\begin{tikzcd}
& {[\pi]_{FE}\text{–}U\mathrm{BiF}(M^{p_f K p_f^{-1}}(P,\mathfrak{X})_{FE})} \\
\mathrm{Rep}_F(P) \arrow[ur, "\mu_{p_f K p_f^{-1},DR}"] \arrow[dr, "\mu_{K,DR}"'] & \\
& {[\pi]_{FE}\text{–}U\mathrm{BiF}(M^{K}(P,\mathfrak{X})_{FE})} \arrow[uu, "{[\cdot p_f]^*_{FE}}"']
\end{tikzcd}$$

commutes.

Proof: left to the reader. q.e.d.

If we consider the tensor categories $\mathrm{VB}(M^L_{FE})$ and $[\pi]_{FE}$–$U\mathrm{VB}(M^K_{FE})$ of flat vector bundles on M^L_{FE} and $[\pi]_{FE}$–unipotent flat vector bundles on M^K_{FE}, whose connection is regular at infinity, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P) \arrow[r, "\mathrm{For}(\mu_{K,DR})"] & {[\pi]_{FE}\text{–}U\mathrm{VB}(M^K_{FE})} \\
\mathrm{Rep}_F(G) \arrow[u, "\pi^*"] \arrow[r, "\mathrm{For}(\mu_{L,DR})"] & \mathrm{VB}(M^L_{FE})\,. \arrow[u, "{[\pi]^*_{FE}}"']
\end{tikzcd}$$

The categories on the right are naturally contained in the respective categories $RH(M^K_{FE})$ and $RH(M^L_{FE})$ of regular holonomic $\mathcal{D}$–modules. For the definition and basic properties of these categories see [Bo], V–VIII. Observe that the results of [Bo], VI, VII are valid over arbitrary base fields of characteristic zero. Furthermore, the functors "inverse image" and "direct image" are compatible with base change of the field. By [Bo], VII, Proposition 10.4.i) and [Bo], VI, Proposition 1.7, the regular holonomic $\mathcal{D}$–modules, which are coherent as modules over the structure sheaf, are precisely the flat vector bundles, whose connection is regular at infinity.

Theorem 3.5: Let d be the relative dimension of $[\pi]$, and consider $\mathrm{For}(\mu_{K,DR})$ and $\mathrm{For}(\mu_{L,DR})$ as functors into the categories $RH(M^K_{FE})$ and $RH(M^L_{FE})$. Then for any q, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P) \arrow[r, "\mathrm{For}(\mu_{K,DR})"] \arrow[d, "H^q(W,\,\_)"'] & RH(M^K_{FE}) \arrow[d, "\mathcal{H}^{q-d}([\pi]_{FE})_*"] \\
\mathrm{Rep}_F(G) \arrow[r, "\mathrm{For}(\mu_{L,DR})"] & RH(M^L_{FE})
\end{tikzcd}$$

Proof: The existence of a natural transformation

$$\mathrm{For}(\mu_{L,DR}) \circ H^q(W,\,\_) \longrightarrow \mathcal{H}^{q-d}([\pi]_{FE})_* \circ \mathrm{For}(\mu_{K,DR})$$

is guaranteed by [Hub], Theorem 2.6.
That it is an isomorphism can be checked over $\mathbb{C}$. There, it follows from [W2], Theorem 4.3 and the compatibility of higher direct images under the Riemann–Hilbert correspondence ([Bo], VIII, Theorems 14.4.i) and 22.4). q.e.d.

§ 4 The canonical construction of mixed sheaves: λ–adic version

We now turn to the λ–adic situation. The picture is incomplete, but at least our results show that the problem occurring is invariant under unipotent extensions.

Let $F/\mathbb{Q}_l$ be finite, $K \leq P(\mathbb{A}_f)$ neat, open and compact, and $\lambda \in F$ a prime element. For any open normal subgroup $K' \trianglelefteq K$, we have the morphism

$$[\cdot 1] = [\cdot 1]_{K',K} : M^{K'}(P,\mathfrak{X}) \longrightarrow M^K(P,\mathfrak{X})$$

which identifies M^K with $M^{K'}/K$.
By [P2], Proposition 3.3.3, whose proof works equally well in the case of mixed Shimura varieties, $[\cdot 1]$ is a Galois covering with Galois group K/K'. The more natural thing is to view this as an action from the right. But we consider the associated action from the left since this will give us the correct comparison statement:

$$k \in K \text{ acts via } [\cdot k^{-1}]_{K',K'} \, .$$

So K is the Galois group of the pro–covering

$$\varprojlim_{K' \le K} M^{K'}(P, \mathfrak{X}) \longrightarrow M^K(P, \mathfrak{X}) \,.$$

Via the continuous morphism

$$K \hookrightarrow P(\mathbb{A}_f) \longrightarrow\!\!\!\!\rightarrow P(\mathbb{Q}_l) \hookrightarrow P(F) \,,$$

each $\mathbf{V} \in \mathrm{Rep}_F(P)$ defines a lisse λ–adic sheaf on $M^K(P, \mathfrak{X})$.
So we constructed a tensor functor

$$\mu_{K,\lambda} : \mathrm{Rep}_F(P) \longrightarrow [\pi]\text{–}U\mathrm{Et}^l_F(M^K(P, \mathfrak{X})) \,,$$

the right hand side denoting the category of those lisse étale F–sheaves on $M^K(P, \mathfrak{X})$ admitting a filtration, whose graded objects lie in the image of $[\pi]^*$.

Lemma 4.1: Let $\mathbf{V} \in \mathrm{Rep}_{\mathbb{Q}}(P)$.
Then via the canonical embedding $\overline{E(P, \mathfrak{X})} \xrightarrow{\sigma_0} \mathbb{C}$, the local system underlying $\mu_{K,\infty}(\mathbf{V})$ on $M^K(\mathbb{C})$ corresponds to the l–adic sheaf $(\mu_{K,\lambda}(\mathbf{V}))_{\sigma_0}$ on $M^K_{\mathbb{C}}$.
Proof: straightforward. q.e.d.

In the case of pure Shimura varieties, it is conjectured ([LR], § 6; [P2], Conjecture 5.4.1) that the image of $\mu_{K,\lambda}$ is contained in the category of mixed sheaves and that the weights are "the right ones". It seems reasonable to do the same in the mixed case.
We start by defining mixedness for representations of P:

Definition: Let F be any field of characteristic zero, $\mathbf{V} \in \mathrm{Rep}_F(P)$.

a) $\mathbf{V}$ is called *pure of weight* $n \in \mathbb{Z}$ if W acts trivally on $\mathbf{V}$ and if for one (hence all) $x \in \mathfrak{X}$, the rational cocharacter

$$\pi \circ h_x \circ w : \mathbb{G}_m \longrightarrow Z(G)$$

acts on M by

$$z \longmapsto (\text{multiplication by } z^{-n}) \,.$$

b) $\mathbf{V}$ is called *mixed of weights* $n_1 \le \ldots \le n_r$ if $\mathbf{V}$ has an ascending filtration $W_\cdot(\mathbf{V})$ by subrepresentations such that

$$\mathrm{Gr}^W_n \mathbf{V} \begin{cases} = 0 & \text{for } n \notin \{n_i \,|\, i = 1, \ldots, r\} \\ \text{is pure of weight } n_i & \text{for } n = n_i \end{cases} \,.$$

Using the fact that $\mathrm{Lie}\, W$ has weights < 0, it is not hard to see that every $\mathbf{V} \in \mathrm{Rep}_F(P)$ is mixed. Also, the definition of purity coincides with the one given in 1.1.c) for $F \subset \mathbb{R}$.

Conjecture 4.2: Let $F/\mathbb{Q}_l$ be finite.
If $\mathbf{V} \in \mathrm{Rep}_F(P)$ has weights $n_1 \le \ldots \le n_r$, then $\mu_{K,\lambda}(\mathbf{V})$ is a mixed sheaf in the sense of [D2], VI, and the weight filtration of $\mu_{K,\lambda}(\mathbf{V})$ corresponds via $\mu_{K,\lambda}$ to the weight filtration of $\mathbf{V}$.

This conjecture is independent of K. It implies that the image of $\mu_{K,\lambda}$ is contained in $[\pi]$–$U\mathrm{Et}_F^{l,m}(M^K(P,\mathfrak{X}))$, the full subcategory of $[\pi]$–$U\mathrm{Et}_F^{l}(M^K(P,\mathfrak{X}))$ of those objects, that are mixed.
We summarize results of Pink:

Theorem 4.3: ([P2], Proposition 5.5.4, Proposition 5.6.2, Proposition 5.6.1.)

a) Conjecture 4.2 is true when $P = G$ is a torus.

b) Conjecture 4.2 is true when $P = G$ and every $\mathbb{Q}$–simple factor of G^{ad} is of abelian type.[†]

c) If $P = G$ then Conjecture 4.2 is equivalent to

$$\mathrm{im}\,(\mu_{K,\lambda}) \subset \mathrm{Et}_F^{l,m}(M^K(P,\mathfrak{X}))\,,$$

the latter denoting the category of mixed lisse l–adic sheaves on $M^K(P,\mathfrak{X})$.

In order to study the mixed situation, we need to show an analogue of Theorem 2.1 in the λ–adic setting.
Again let $i : (G,\mathcal{H}) \longrightarrow (P,\mathfrak{X})$ be as usual and let K be of the shape $K^W \rtimes L$, so $[\pi]$ admits the section $[i]$. We recall the definition of the restriction of the generic pro–sheaf $\mathcal{G}en_{[i]}$ to the preimage $(M^K)^0$ of a connected component $(M^L)^0$ of M^L: Fix $y \in M^L(\overline{E(G,\mathcal{H})})^0$ and set $x := [i](y)$.

Let $P_{\overline{x}}$ denote the Tannaka dual of the category $[\pi]$–$U\mathrm{Et}_{\mathbb{Q}_l}^{l,m}((M^K)^0)$ with respect to the functor “fibre at $\overline{x}$”, $G_{\overline{y}}$ the Tannaka dual of the category $\mathrm{Et}_{\mathbb{Q}_l}^{l,m}((M^L)^0)$ with respect to the functor “fibre at $\overline{y}$”, and define $W_{\overline{x}}$ via the exact sequence

$$1 \longrightarrow W_{\overline{x}} \longrightarrow P_{\overline{x}} \xrightarrow{[\pi]} G_{\overline{y}} \longrightarrow 1\ .$$

[†] For a definition of the term “of abelian type”, see [P2], proof of Proposition 5.6.2. It covers the groups, whose absolute root system is of type A, B or C.

$W_{\overline{x}}$ is pro–unipotent, and $[i]$ defines a splitting of this sequence. We define a representation of

$$P_{\overline{x}} = W_{\overline{x}} \rtimes G_{\overline{y}}$$

on the completion of the universal envelope of $\mathrm{Lie}(W_{\overline{x}})$ with respect to powers of the augmentation ideal: $W_{\overline{x}}$ acts by multiplication, and $G_{\overline{y}}$ acts by conjugation. Recall (proof of Lemma 2.4, [W2], Corollary 3.2.i)) that $W_{\overline{x}}$ is canonically isomorphic to $W \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

Definition: $\mathcal{L}og_l := \mathcal{L}og(i,K)_l := \mathcal{G}en_{[i]}$ is called the *logarithmic l–adic pro–sheaf* on $M^K(P,\mathfrak{X})$.

$\mathcal{L}og_l$ comes equipped with a section

$$1 \in \Gamma(M^L(G,\mathcal{H}), [i]^*\mathcal{L}og_l).$$

The pair $(\mathcal{L}og_l, 1)$ is rigid, i.e., it admits no non–trivial automorphisms. In fact, this already holds on the level of underlying pro–sheaves on $M^L(G,\mathcal{H}) \otimes_{E(G,\mathcal{H})} \overline{E(G,\mathcal{H})}$ ([W2], Theorem 3.5.iii)). $\mathcal{L}og_l$ carries the structure of cocommutative coalgebra. It induces on $[i]^*\mathcal{L}og_l$ a Hopf algebra structure. Furthermore, the pair $(\mathcal{L}og_l, 1)$ has the universal property of [W2], Theorem 3.5.i).

Let $\hat{\mathfrak{U}}(\mathrm{Lie}\, W) \in \text{pro–}\mathrm{Rep}_{\mathbb{Q}_l}(P)$ be the completed universal envelope of $\mathrm{Lie} W$ over $\mathbb{Q}_l$, equipped with the usual action of P.

Theorem 4.4: There is a unique morphism

$$\varphi : \mathcal{L}og(i,K)_l \longrightarrow \mu_{K,l}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$$

of l–adic pro–sheaves sending 1 to 1. It is an isomorphism of cocommutative coalgebras, and $[i]^*(\varphi)$ respects the multiplicative structure of both sides.
In particular, $\mu_{K,l}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$ is mixed.

Proof: The vital observation, that makes the proof much easier than the one of 2.1 was indicated in the remark following [W2], Theorem 3.5:
namely, the universal property of $\mathcal{L}og(i,K)_l$ actually holds in the larger category of $[\pi]$–unipotent lisse l–adic sheaves, i.e., with the mixedness assumption removed. This gives the existence and uniqueness of a morphism φ sending 1 to 1. Applying [W2], Theorem 3.5.i) to $\mathcal{G}\hat{\otimes}_F\mathcal{G}$ and the section $1\hat{\otimes}_F 1$, where $\mathcal{G} := \mu_{K,l}(\hat{\mathfrak{U}}(\mathrm{Lie}\, W))$, we see that φ respects the coalgebra structure. Alternatively, we see that the statement analogous to 2.4 holds. This is also how we prove the last claim. q.e.d.

As a consequence, we get the following result:

Corollary 4.5: Assume the pure Shimura data $(G, \mathcal{H})$ admit a unipotent extension

$$\pi : (P, \mathfrak{X}) \longrightarrow (G, \mathcal{H})$$

such that for a given splitting

$$i : G \longrightarrow P$$

of π, the representation $\operatorname{Lie} W$ of G is faithful.
Then Conjecture 4.2 holds for $(G, \mathcal{H})$.
Proof: $\operatorname{Rep}_F(G)$ is generated by the finite–dimensional subquotients of $i^*(\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}_l} F)$. Now apply 4.4 and 4.3.c). q.e.d.

Using 2.9, it is not difficult to see that 4.5 actually also follows from 4.3.b).

It is now a formal matter to deduce

Theorem 4.6: a) Conjecture 4.2 holds for $(P, \mathfrak{X})$ if and only if it holds for $(G, \mathcal{H})$.
b) Conjecture 4.2 is equivalent to

$$\operatorname{im}(\mu_{K,\lambda}) \subset [\pi]\text{–}U\mathrm{Et}_F^{l,m}(M^K(P, \mathfrak{X}))\,.$$

Proof: To prove b), take $\mathbf{V} \in \operatorname{Rep}_F(P)$ and assume $\mu_{K,\lambda}(\mathbf{V})$ is mixed. As usual, we may assume $K = K^W \rtimes L$ as above. Then 4.3.b) shows that after applying $[i]^*$, the weight filtrations on $\mathbf{V}$ and on $\mu_{K,\lambda}(\mathbf{V})$ correspond. So they correspond altogether.
For a), use the mixedness of $\mu_{K,l}(\hat{\mathfrak{U}}(\operatorname{Lie} W))$. $\operatorname{Rep}_F(P)$ is generated by the finite–dimensional subquotients of $\hat{\mathfrak{U}}(\operatorname{Lie} W)\hat{\otimes}_{\mathbb{Q}_l} F$ together with $\operatorname{Rep}_F(G)$. q.e.d.

We conclude by stating the result analogous to 2.3:

Theorem 4.7: Assume that Conjecture 4.2 holds. Let d be the relative dimension of $[\pi]$, and consider $\mu_{K,\lambda}$ and $\mu_{L,\lambda}$ as functors into the categories $\operatorname{Perv}_F^m(M^K)$ and $\operatorname{Perv}_F^m(M^L)$ of mixed perverse λ–adic sheaves (compare [W2], § 4) on $M^K(P, \mathfrak{X})$ and $M^L(G, \mathcal{H})$ respectively.

Then for any q, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_F(P) \arrow[r, "\mu_{K,\lambda}"] \arrow[d, "H^q(W,\,\_)"'] & \mathrm{Perv}^m_F(M^K) \arrow[d, "\mathcal{H}^{q-d}([\pi])_*"] \\
\mathrm{Rep}_F(G) \arrow[r, "\mu_{L,\lambda}"] & \mathrm{Perv}^m_F(M^L)
\end{tikzcd}$$

of functors.

Proof: Corollary 1.4 and Lemmata 1.5 and 1.6 show that the hypotheses of [W2], Theorem 4.3 are met for $q_0 := \infty$. q.e.d.

Using the remark following [W2], Theorem 3.5, one sees that it is possible to prove a version of Theorem 4.7, which is independent of Conjecture 4.2 just by considering perverse λ–adic sheaves, i.e., removing the mixedness assumption. Alternatively, one might restrict one's attention to those representations of P, whose canonical construction is mixed.

§ 5 Conjugates of mixed Shimura varieties

In order to define a "mixed systems" version of the canonical construction, it turns out to be necessary to extend Milne's and Shih's results on conjugates of pure Shimura varieties ([M], II, §§ 4, 5) to the mixed case. As in [M], II, § 4, the Shimura data conjugate to given ones under an automorphism of $\mathbb{C}$ are defined using special points. A central observation, which will be of great help when deducing our results from those in the pure setting, states that special points on mixed Shimura data are precisely the images under Levi sections of special points of the underlying pure Shimura data (Lemma 5.2). The reader will notice that the main results of this paragraph do not require axiom vii)' of § 1. It suffices to assume vii), that $Z(G)^0$ splits over a CM–field and that the cocharacter $\pi \circ h_x \circ w$ of $Z(G)_{\mathbb{R}}$ is defined over $\mathbb{Q}$ for some (hence all) $x \in \mathfrak{X}$.

We start by recalling the definition and basic properties of the Serre and Taniyama groups: fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and let $\mathcal{S}$ denote the *Serre group*, i.e., the pro–torus over $\mathbb{Q}$, which is the Tannakian dual of the category $CM/\mathbb{C}$ of $\mathbb{Q}$–Hodge structures of CM–type. Our fixed embedding identifies $\mathcal{S}$ with the Tannakian dual of the category $CM/\overline{\mathbb{Q}}$ of CM–motives over $\overline{\mathbb{Q}}$ with respect to the Betti fibre functor H_B.

Furthermore, $\mathcal{T}$ is defined to be the *Taniyama group*, i.e., the Tannakian dual

of the category $CM/\mathbb{Q}$ of CM–motives over $\mathbb{Q}$.
There is a natural morphism $(\mathrm{res}^{\mathbb{C}}_{\mathbb{Q}})^* : \mathcal{S} \longrightarrow \mathcal{T}$.
Next, the natural inclusion of the category $Art/\mathbb{Q}$ of Artin motives over $\mathbb{Q}$ into $CM/\mathbb{Q}$ defines a morphism $q : \mathcal{T} \longrightarrow \mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$. The action of $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}}$ on $H_B \otimes_{\mathbb{Q}} \mathbb{A}_f$ defines a splitting $\mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}} \longrightarrow \mathcal{T}(\mathbb{A}_f)$ of $q \otimes_{\mathbb{Q}} \mathbb{A}_f$, which is denoted by *sp*.
The following is the content of [D4], Lemme 1:

Theorem 5.1: The sequence

$$1 \longrightarrow \mathcal{S} \xrightarrow{(\mathrm{res}^{\mathbb{C}}_{\mathbb{Q}})^*} \mathcal{T} \xrightarrow{q} \mathrm{Gal}_{\overline{\mathbb{Q}}/\mathbb{Q}} \longrightarrow 1$$

is exact.

According to [M], I, Proposition 4.5, $CM/\mathbb{C}$ is equivalent to the category of CM–motives over $\mathbb{C}$. Hence there is a natural action of $\mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$ on $CM/\mathbb{C}$. For each automorphism τ, there is a fibre functor $H_\tau : M \longmapsto H_B(\tau M)$. The right–$\underline{\mathrm{Aut}}^{\otimes}(H_B)$–torsor $\underline{\mathrm{Isom}}^{\otimes}(H_B, H_\tau)$ is represented by the right–$\mathcal{S}$–torsor
${}^{\tau}\mathcal{S} := q^{-1}(\tau)$ ([M], I , Remark 6.3.b)).

Recall the notion of twisting by ${}^{\tau}\mathcal{S}$: assume $\mathcal{S}$ acts on an algebraic variety $Y/\mathbb{Q}$ from the left. Then ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y$, the twist of Y by ${}^{\tau}\mathcal{S}$ is the variety over $\mathbb{Q}$, whose associated sheaf is the sheafification of

$$R \longmapsto {}^{\tau}\mathcal{S}(R) \times Y(R)/\sim,$$

where $(ba, y) \sim (b, ay)$ for $a \in \mathcal{S}(R), b \in {}^{\tau}\mathcal{S}(R), y \in Y(R)$.
Twisting by ${}^{\tau}\mathcal{S}$ is a functor on the category of varieties over $\mathbb{Q}$ with an $\mathcal{S}$–action. If Y is an algebraic group and $\mathcal{S}$ acts by group automorphisms, then ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y$ is again an algebraic group: the unit, which is defined over $\mathbb{Q}$, is given by the class of $(b, 1)$ for any $b \in {}^{\tau}\mathcal{S}(R)$, and the group law is given by

$$\begin{aligned}
({}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y) \times_{\mathbb{Q}} ({}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y) &\xrightarrow{\quad\sim\quad} ({}^{\tau}\mathcal{S} \times_{\mathbb{Q}} {}^{\tau}\mathcal{S}) \times^{(\mathcal{S} \times_{\mathbb{Q}} \mathcal{S})} (Y \times_{\mathbb{Q}} Y) \\
&\xrightarrow{(\triangle \times \mathrm{id}_{Y \times_{\mathbb{Q}} Y})^{-1}} {}^{\tau}\mathcal{S} \times^{\mathcal{S}} (Y \times_{\mathbb{Q}} Y) \\
&\xrightarrow{\quad\mathrm{id} \times \mathrm{mult}\quad} {}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y
\end{aligned}$$

If R is a $\mathbb{Q}$–algebra and ${}^{\tau}\mathcal{S}(R) \neq \emptyset$, then any $b_0 \in {}^{\tau}\mathcal{S}(R)$ defines an isomorphism

$$({}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y) \otimes_{\mathbb{Q}} R \xrightarrow{\sim} Y \otimes_{\mathbb{Q}} R : [b, y] \longmapsto (b_0^{-1} b)(y)\,.$$

Obviously any two such isomorphisms differ by an automorphism of $Y \otimes_{\mathbb{Q}} R$ coming from $\mathcal{S} \otimes_{\mathbb{Q}} R$.

Example: Let $\mathbf{V}$ be a finite–dimensional vector space over $\mathbb{Q}$, and let $\rho : \mathcal{S} \longrightarrow GL(\mathbf{V})$ be an algebraic representation of $\mathcal{S}$, which we may think of as an $\mathcal{S}$–action by group automorphisms on the algebraic group $\mathbf{V} \cong \mathbb{G}_{a,\mathbb{Q}}^{\dim \mathbf{V}}$. The twist ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V}$ is again a vector space. We may identify $GL({}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V})$ with ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} GL(\mathbf{V})$, where the group action of $\mathcal{S}$ on $GL(\mathbf{V})$ is given by inner conjugation via ρ. We therefore get the twisted representation

$$
{}^{\tau}\mathcal{S} \times^{\mathcal{S}} \rho : \mathcal{S} \xrightarrow{\sim} {}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathcal{S} \longrightarrow GL({}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V})\,.
$$

Here, the first isomorphism is given by sending $a \in \mathcal{S}(R)$ to the class of (b, a) for any $b \in {}^{\tau}\mathcal{S}(R')$; note that $\mathcal{S}$ is commutative, hence its action on itself by inner conjugation is trivial. If ρ factors over the algebraic group Y:

$$
\mathcal{S} \xrightarrow{h} Y \longrightarrow GL(\mathbf{V})\,,
$$

then because of the functoriality of twisting, ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V}$ is a representation of ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y$.

Given a neutral Tannakian category $\mathcal{C}$ with two fibre functors ω_1 and ω_2, the interrelation between $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$ and $\underline{\mathrm{Aut}}^{\otimes}(\omega_2)$ is given by twisting:

$$
\underline{\mathrm{Aut}}^{\otimes}(\omega_2) = \underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2) \times^{\underline{\mathrm{Aut}}^{\otimes}(\omega_1)} \underline{\mathrm{Aut}}^{\otimes}(\omega_1)\,,
$$

where $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$ acts on itself by inner conjugation.

It follows that ${}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y$ is the Tannakian dual of the category $\mathrm{Rep}_{\mathbb{Q}}(Y)$ with respect to the fibre functor

$$
\mathrm{Rep}_{\mathbb{Q}}(Y) \xrightarrow{h^*} \mathrm{Rep}_{\mathbb{Q}}(\mathcal{S}) \xrightarrow{H_{\tau}} \mathrm{Vec}_{\mathbb{Q}}\,.
$$

The equivalence of tensor categories $\mathrm{Rep}_{\mathbb{Q}}(Y) \longrightarrow \mathrm{Rep}_{\mathbb{Q}}({}^{\tau}\mathcal{S} \times^{\mathcal{S}} Y)$ predicted by the Tannakian formalism is given explicitely by

$$
\mathbf{V} \longmapsto {}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V}\,.
$$

Now let $(P, \mathfrak{X})$ be mixed Shimura data, and let $(G, \mathcal{H}) = (P, \mathfrak{X})/W$ and $\pi : (P, \mathfrak{X}) \longrightarrow (G, \mathcal{H})$ as before. Because of our axiom vii)', the pure Shimura data $(G, h(\mathcal{H}))$ satisfy (2.1*) of [M], II.

Definition: A point $x \in \mathfrak{X}$ is called *special* if there is an embedding $(T, \mathcal{Y}) \hookrightarrow (P, \mathfrak{X})$ of Shimura data, with T a torus, whose image contains x.

It follows that h_x factors through the base change to $\mathbb{R}$ of a maximal $\mathbb{Q}$–rational subtorus of P. Hence $\pi(h_x)$ is special if and only if it is special in the sense of [M], II, § 2. Furthermore, $x \in \mathfrak{X}$ is special if and only if $h_x \in h(\mathfrak{X})$ is.

Lemma 5.2: A point $x \in \mathfrak{X}$ is special if and only if there is a Levi section $i_x : (G, \mathcal{H}) \longrightarrow (P, \mathfrak{X})$ of π and a special point $h \in \mathcal{H}$ such that $x = i_x(h)$. If this is the case, then x determines h and i_x uiquely.

Proof: The "if"–part of the claim is obvious.

Conversely, assume given a torus embedding $(T, \mathcal{Y}) \xrightarrow{k} (P, \mathfrak{X})$. $\pi \circ k$ defines a torus embedding $(T, \mathcal{Y}) \longrightarrow (G, \mathcal{H})$. Fix a Levi section $i : (G, \mathcal{H}) \longrightarrow (P, \mathfrak{X})$. For any $x \in k(\mathcal{Y})$, the subgroup $i(G)_{\mathbb{C}} \le P_{\mathbb{C}}$ is the centralizer of $i \circ \pi \circ h_x \circ w$, and the map

$$\begin{aligned} W(\mathbb{Q}) &\longrightarrow \left\{ \begin{matrix} \text{Levi decomposition of } P_{\mathbb{C}}\,, \\ \text{that are defined over } \mathbb{Q} \end{matrix} \right\} , \\ p &\longmapsto \mathrm{Cent}_{P_{\mathbb{C}}}(\mathrm{int}(p) \circ i \circ \pi \circ h_x \circ w) = p \cdot i(G)_{\mathbb{C}} \cdot p^{-1} \end{aligned}$$

is a bijection; see the proof of [P1], Proposition 1.16.b).

It remains to show that the Levi subgroup $\mathrm{Cent}_{P_{\mathbb{C}}}(h_x \circ w)$ of $P_{\mathbb{C}}$ is already defined over $\mathbb{Q}$. But this is the case since $h_x \circ w$ itself is rational. q.e.d.

Let $\mu : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{S}_{\mathbb{C}}$ denote the cocharacter given on $\mathbb{C}$–valued points by sending z to $(z, 1)$. So if ι denotes complex conjugation, we have the formula $(\iota + 1)\mu = w_{\mathbb{C}}$.

The functor sending a $\mathbb{Q}$–Hodge structure of CM–type to the underlying semisimple $\mathbb{R}$–Hodge structure defines a morphism $h_{can} : \mathbb{S} \longrightarrow \mathcal{S}_{\mathbb{R}}$, hence a cocharacter $\mu_{can} := h_{can,\mathbb{C}} \circ \mu$ of $\mathcal{S}_{\mathbb{C}}$.

Corollary 5.3: Let $x \in \mathfrak{X}$ be a special point, and let $\mu_x := h_x \circ \mu$. Then there exists a unique morphism $\rho_x : \mathcal{S} \longrightarrow P$ such that the diagram

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \arrow[rr, "\mu_{\mathrm{can}}"] \arrow[dr, "\mu_x"'] & & \mathcal{S}_{\mathbb{C}} \arrow[dl, "\rho_{x,\mathbb{C}}"] \\ & P_{\mathbb{C}} & \end{tikzcd}$$

commutes.

Proof: The image of $\rho_{x,\mathbb{C}}$ must centralize μ_x, hence h_x. Therefore, let $T \subset P$ be the minimal $\mathbb{Q}$–rational torus such that h_x factors through $T_{\mathbb{R}}$. We have to show that $\mu_x : \mathbb{G}_{m,\mathbb{C}} \longrightarrow T_{\mathbb{C}}$ satisfies the Serre condition:

$$(\tau - 1)(\iota + 1)\mu_x = 0 = (\iota + 1)(\tau - 1)\mu_x \quad \text{for all } \tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}} .$$

By the universal property of $\mathcal{S}$ (see [M], I, Proposition 2.4), the existence and uniqueness of ρ_x will then be guaranteed. The left equation is obvious: $(\iota+1)\mu_x = h_x \circ w$ is defined over $\mathbb{Q}$. For the other equation, observe that T, being the almost direct product of $\mathbb{G}_m$ and a torus of compact type, splits over a CM–field. Therefore, $\tau\iota$ and $\iota\tau$ act in the same manner on the set of cocharacters of $T_{\mathbb{C}}$, for any $\tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$. In particular,

$$(\iota + 1)(\tau - 1)\mu_x = (\tau - 1)(\iota + 1)\mu_x = 0 .$$

q.e.d.

We are now in a position to define the Shimura data conjugate to $(P, \mathfrak{X})$ under an automorphism of $\mathbb{C}$. Fix $\tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$, and choose a special point $x \in \mathfrak{X}$. Let $\rho_x : \mathcal{S} \longrightarrow P$ be as in 5.3.

Write ${}^{\tau,x}P := {}^{\tau}\mathcal{S} \times^{\mathcal{S}} P$, where the action of $\mathcal{S}$ on P is inner conjugation via ρ_x. If we let $T_x := \rho_x(\mathcal{S}) \subset P$, then by functoriality of twisting, we may consider $T_x = {}^{\tau}\mathcal{S} \times^{\mathcal{S}} T_x$ as a subtorus of ${}^{\tau,x}P$. So $\tau\mu_x : \mathbb{G}_{m,\mathbb{C}} \longrightarrow T_{x,\mathbb{C}}$ can be regarded as a homomorphism into ${}^{\tau,x}P_{\mathbb{C}}$. Since it commutes with its complex conjugate, it defines a homomorphism

$${}^{\tau}h_x : \mathbb{S} \longrightarrow {}^{\tau,x}P_{\mathbb{R}} .$$

We define $h({}^{\tau,x}\mathfrak{X})$ to be the set of ${}^{\tau,x}P(\mathbb{R}) \cdot {}^{\tau,x}U(\mathbb{C})$–conjugates of ${}^{\tau}h_x$.

Proposition 5.4: $({}^{\tau,x}P, h({}^{\tau,x}\mathfrak{X}))$ are mixed Shimura data.

Proof: We check the axioms of § 1:

i) is obviously fulfilled, since ${}^{\tau}h_x$ is already defined over $\mathbb{R}$. We have $Z({}^{\tau,x}G) = {}^{\tau,x}Z(G)$, and hence the twist ${}^{\tau,x}pr$ of the projection $pr : G \longrightarrow G^{\mathrm{ad}}$ is the projection of ${}^{\tau,x}G$ onto its adjoint group. We know that

$$pr \circ \pi \circ h_x \circ w : \mathbb{G}_m \longrightarrow G^{\mathrm{ad}}$$

is the trivial cocharacter. Now ${}^{\tau}h_x \circ w = (\iota + 1)\tau\mu_x = h_x \circ w$ when considered as a cocharacter of T_x. Since

$$\ker(T_x \longrightarrow P \twoheadrightarrow G^{\mathrm{ad}}) = \ker(T_x \longrightarrow {}^{\tau,x}P \twoheadrightarrow {}^{\tau,x}G^{\mathrm{ad}})$$

under the natural identification of T_x and ${}^{\tau,x}T_x$, the twist ${}^{\tau,x}pr \circ {}^{\tau,x}\pi \circ {}^{\tau}h_x \circ w$ is trivial as well. This shows ii).
Similarly, iv) is proven by observing that $\mathrm{Lie}^{\tau,x}P$ is the representation of ${}^{\tau,x}P$ obtained by twisting the representation $\mathrm{Lie}\, P$ of P.
This also shows that μ_x, considered as a homomorphism

$$\mathbb{G}_{m,\mathbb{C}} \longrightarrow T_{x,\mathbb{C}} \longrightarrow {}^{\tau,x}P_{\mathbb{C}}$$

has the correct eigenvalues on $\mathrm{Gr}^W_{\cdot}(\mathrm{Lie}^{\tau,x}P)_{\mathbb{C}}$. But this is then also true for any conjugate of μ_x, proving iii).
vii)' holds because $Z(G)$ remains unchanged under twists.
For axioms v) and vi), we refer to [L], page 231. q.e.d.

In order to get ${}^{\tau,x}\mathfrak{X}$, we need to replace $h({}^{\tau,x}\mathfrak{X})$ by a finite covering. We discuss the case of a torus first.
So let $(T, \mathcal{Y})$ be Shimura data, with T a torus. Equivalently ([P1], Example 2.6), $\mathcal{Y}$ is a finite set with a transitive action of $\pi_0(T(\mathbb{R}))$, and $h(\mathcal{Y})$ is a single morphism $\mathbb{S} \longrightarrow T_{\mathbb{R}}$.
Choose $y \in Y$. We have ${}^{\tau,y}T = T$, and so ${}^{\tau,y}h(\mathcal{Y}) \circ \mu = \tau\mu_y$.
We define ${}^{\tau,y}(T, \mathcal{Y})$ to be the Shimura data corresponding to the same finite set $\mathcal{Y}$ with the transitive action of $\pi_0(T(\mathbb{R}))$, but with $h(\mathcal{Y})$ replaced by ${}^{\tau,y}h(\mathcal{Y})$.

The following result is due to Pink:

Lemma 5.5: ([P1], Proposition 2.11.)
For any mixed Shimura data $(P, \mathfrak{X})$, the canonical morphism

$$(P, \mathfrak{X}) \longrightarrow (P, \mathfrak{X})/P^{der} \times (P, h(\mathfrak{X}))$$

is an embedding.

Observe that P/P^{der} is a torus, which satisfies vii)' since P does. If we apply 5.5 to the Shimura data ${}^{\tau,x}(P, \mathfrak{X}) = ({}^{\tau,x}P, {}^{\tau,x}\mathfrak{X})$, that we want to construct, we see that there is exactly one way of defining $({}^{\tau,x}P, {}^{\tau,x}\mathfrak{X})$ is a manner compatible with morphisms of Shimura data, such that $({}^{\tau,x}P, h({}^{\tau,x}\mathfrak{X}))$ are the Shimura data already defined:
write $(P, \mathfrak{X})/P^{der} = (T, \mathcal{Y})$, and let $[x] \in \mathcal{Y}$ be the image of x under $\mathfrak{X} \longrightarrow \mathcal{Y}$.
We define ${}^{\tau}x \in {}^{\tau,[x]}\mathcal{Y} \times {}^{\tau,h_x}h(\mathfrak{X})$ to be the element $([x], {}^{\tau}h_x)$.
Finally, ${}^{\tau,x}\mathfrak{X}$ is defined as the set of ${}^{\tau,x}P(\mathbb{R}) \cdot {}^{\tau,x}U(\mathbb{C})$–conjugates of ${}^{\tau}x$.

Corollary 5.6: ${}^{\tau,x}(P,\mathfrak{X}) := ({}^{\tau,x}P, {}^{\tau,x}\mathfrak{X})$ are mixed Shimura data.
Proof: straightforward. q.e.d.

Remark: Our definition of conjugates of Shimura data does not obviously respect the property $\mathfrak{X}{=}h(\mathfrak{X})$. However, as we shall see, if $\mathfrak{X}{=}h(\mathfrak{X})$, then the data $({}^{\tau,x}P, {}^{\tau,x}\mathfrak{X})$ and $({}^{\tau,x}P, h({}^{\tau,x}\mathfrak{X}))$ define the same Shimura *varieties*. So we may conclude a posteriori that indeed ${}^{\tau,x}\mathfrak{X}{=}h({}^{\tau,x}\mathfrak{X})$ and hence that we get back Milne's definition ([M], II, § 4) if $(P,\mathfrak{X}) = (G, h(\mathfrak{X}))$.

Lemma 5.7: Let $\varphi : (P_1,\mathfrak{X}_1) \longrightarrow (P_2,\mathfrak{X}_2)$ be a morphism of Shimura data, and assume $x_1 \in \mathfrak{X}_1$ is special. Then there is a unique morphism

$$ {}^{\tau,x_1}\varphi : {}^{\tau,x_1}(P_1,\mathfrak{X}_1) \longrightarrow {}^{\tau,\varphi(x_1)}(P_2,\mathfrak{X}_2) $$

of Shimura data, whose underlying morphism of groups coincides with

$$ {}^{\tau,x_1}\varphi : {}^{\tau,x_1}P_1 \longrightarrow {}^{\tau,\varphi(x_1)}P_2 \, , $$

the morphism obtained by twisting φ, and which sends ${}^{\tau}x_1$ to ${}^{\tau}\varphi(x_1)$.
Proof: straightforward. q.e.d.

The point $sp(\tau) \in {}^{\tau}\mathcal{S}(\mathbb{A}_f)$ defines an isomorphism

$$ P(\mathbb{A}_f) \xrightarrow{\sim} {}^{\tau,x}P(\mathbb{A}_f) : p_f \longmapsto {}^{\tau,x}p_f := [sp(\tau), p_f] \, . $$

The following is the main result of this paragraph. Since this will reduce the amount of variables that we will have to introduce, we follow Milne and define $M(P,\mathfrak{X})$ as the projective limit of the $M^K(P,\mathfrak{X})$, where K runs through the open compact subgroups of $P(\mathbb{A}_f)$. It is a scheme over $E(P,\mathfrak{X})$, whose complex points are $P(\mathbb{Q})\backslash(\mathfrak{X} \times P(\mathbb{A}_f))$ ([P1], Lemma 3.7).
It is equipped with a right action of $P(\mathbb{A}_f)$ by algebraic morphisms. Denote by ${}^{\tau}M(P,\mathfrak{X})$ the scheme obtained from $M(P,\mathfrak{X})$ via base change by τ.

Theorem 5.8: There is a unique isomorphism

$$ \varphi_{\tau,x} : {}^{\tau}M(P,\mathfrak{X}) \xrightarrow{\sim} M({}^{\tau,x}(P,\mathfrak{X})) $$

of schemes over ${}^{\tau}E(P,\mathfrak{X})$, such that
a) $\varphi_{\tau,x} : {}^{\tau}[(x,1)] \longmapsto [({}^{\tau}x,1)]$,
b) for all $p_f \in P(\mathbb{A}_f)$, we have the equality

$$ \varphi_{\tau,x} \circ {}^{\tau}[\cdot p_f] = [\cdot {}^{\tau,x}p_f] \circ \varphi_{\tau,x} \, . $$

Proof: As in [P1], Lemma 1.7, one shows that the set of $P(\mathbb{A}_f)$–conjugates of $[(x,1)]$ is Zariski–dense in $M(P,\mathfrak{X})_{\mathbb{C}}$, hence there is at most one map satisfying a) and b).
$(P,\mathfrak{X})$ can be "covered" by irreducible Shimura data ([P1], 2.13), and hence we may assume that $(P,\mathfrak{X})$ itself is irreducible.
Certainly, if the theorem holds for some unipotent extension of $(P,\mathfrak{X})$, then it holds for $(P,\mathfrak{X})$ itself.
By 5.7 and [P1], 2.26, we may replace $(P,\mathfrak{X})$ by Shimura data of the form

$$(T,\mathcal{Y}) \times (G',\mathcal{H}') \times (P,\mathfrak{X})\,,$$

where T is a torus, G' is reductive and $\mathcal{H}' = h(\mathcal{H}')$ (see the proof of [P1], 2.26), and $E(P,\mathfrak{X}) = \mathbb{Q}$.
We treat each of these cases separately:
i) Let $(T,\mathcal{Y})$ be Shimura data, with T a torus, and let $E := E(T,\mathcal{Y})$. So $M(T,\mathcal{Y})(\overline{E}) = (\mathcal{Y} \times T(\mathbb{A}_f))/T(\mathbb{Q})$, and $\mathrm{Gal}_{\overline{E}/E}$ acts by multiplication from the left via a homomorphism

$$\psi : \mathrm{Gal}_{\overline{E}/E} \longrightarrow \pi_0((T(\mathbb{R}) \times T(\mathbb{A}_f))/T(\mathbb{Q}))\,,$$

the so–called reciprocity law for $(T,\mathcal{Y})$ (compare [P1], 11.3, 11.4).
So we may identify ${}^{\tau}M(T,\mathcal{Y})(\overline{{}^{\tau}E})$ with $(\mathcal{Y} \times T(\mathbb{A}_f))/T(\mathbb{Q})$, an automorphism $\sigma \in \mathrm{Gal}_{\overline{{}^{\tau}E}/{}^{\tau}E}$ acting via multiplication by $\psi(\tau^{-1}\sigma\tau)$.
Now observe that the reciprocity law for ${}^{\tau,x}(T,\mathcal{Y})$,

$${}^{\tau}\psi : \mathrm{Gal}_{\overline{{}^{\tau}E}/{}^{\tau}E} \longrightarrow \pi_0((T(\mathbb{R}) \times T(\mathbb{A}_f))/T(\mathbb{Q}))$$

is given by $\sigma \longmapsto \psi(\tau^{-1}\sigma\tau)$.
So the above is exactly the description of the $\mathrm{Gal}_{\overline{{}^{\tau}E}/{}^{\tau}E}$–action on the set of $\overline{{}^{\tau}E}$–valued points of $M({}^{\tau,x}(T,\mathcal{Y}))$.
ii) According to [M], II, Theorems 4.2 and 5.5.b), if $\mathcal{H} = h(\mathcal{H})$ and $(G,\mathcal{H})$ is pure, there is a unique isomorphism

$$\widetilde{\varphi_{\tau,x}} :\ {}^{\tau}M(G,\mathcal{H}) \xrightarrow{\sim} M({}^{\tau,x}G, h({}^{\tau,x}\mathcal{H}))$$

with the required properties. The diagram

$$\begin{tikzcd}
{}^{\tau}M(G,\mathcal{H}) \arrow[r] \arrow[d, dashed] & {}^{\tau}M((G,\mathcal{H})/G^{der}) \arrow[d, "\text{i)}", "\wr"'] & \times & {}^{\tau}M(G,\mathcal{H}) \arrow[d, "\widetilde{\varphi_{\tau,x}}", "\wr"'] \\
M({}^{\tau,x}(G,\mathcal{H})) \arrow[r] & M({}^{\tau,x}((G,\mathcal{H})/G^{der})) & \times & M({}^{\tau,x}G, h({}^{\tau,x}\mathcal{H}))
\end{tikzcd}$$

together with the fact that ${}^{\tau}[(x,1)]$ is mapped to $[({}^{\tau}x,1)]$ by the isomorphism on the right shows that the dotted arrow exists. Since its composition with the finite projection to $M({}^{\tau,x}G, h({}^{\tau,x}\mathcal{H}))$ is an isomorphism, it is a closed immersion. But since it is $G(\mathbb{A}_f)$–equivariant, its image is dense, hence it is itself an isomorphism.

iii) Now assume $E(P,\mathfrak{X}) = \mathbb{Q}$. By i), ii) and 5.5, our claim is true for the Shimura data $(G,\mathcal{H}) = (P,\mathfrak{X})/W$.

We first show that the class of $(\rho_x)_*({}^{\tau}\mathcal{S})$ in $H^1(\mathbb{Q},P)$ is trivial. Since $H^1(\mathbb{Q},W)$ is zero, this is equivalent to showing that $(\pi \circ \rho_x)_*({}^{\tau}\mathcal{S}) \in H^1(\mathbb{Q},G)$ is zero. (Alternatively, use 5.2 for this reduction.) But this is the content of [MS], Lemma 7.2.

Choose $b \in (\rho_x)_*({}^{\tau}\mathcal{S})(\mathbb{Q})$. It defines an isomorphism

$$\varphi_b : P \xrightarrow{\sim} {}^{\tau,x}P : p \longmapsto [b,p]\,,$$

which is easily seen to underly an isomorphism $\varphi_b : (P,\mathfrak{X}) \xrightarrow{\sim} {}^{\tau,x}(P,\mathfrak{X})$ of Shimura data. Let $p_b \in \rho_x(\mathcal{S})(\mathbb{A}_f) \leq P(\mathbb{A}_f)$ be defined by the relation

$$b = \rho_x(sp(\tau)) \cdot p_b\,.$$

Then the isomorphism

$$\varphi_{\tau,x} : {}^{\tau}M(P,\mathfrak{X}) = M(P,\mathfrak{X}) \xrightarrow{\sim} M({}^{\tau,x}(P,\mathfrak{X}))$$

defined by $\varphi_{\tau,x} := [\varphi_b] \circ [\cdot p_b]$ is independent of the choice of b. For $p_f \in P(\mathbb{A}_f)$, we have the formula

$$\varphi_{\tau,x} \circ {}^{\tau}[\cdot p_f] = \varphi_{\tau,x} \circ [\cdot p_f] = [\cdot \varphi_b(p_b^{-1} p_f p_b)] \circ \varphi_{\tau,x} = [\cdot {}^{\tau,x}p_f] \circ \varphi_{\tau,x}\,,$$

so $\varphi_{\tau,x}$ has property b).

In order to prove property a), we use 5.2 and the fact that the restriction of $\varphi_{\tau,x}$ to $i_x({}^{\tau}M(G,\mathcal{H}))$ is the isomorphism $\varphi_{\tau,h}$ for the pure Shimura data $(G,\mathcal{H})$; see the proof of [M], II, Theorem 5.5. q.e.d.

If $x' \in \mathfrak{X}$ is a second special point, then $(\rho_x)_*({}^\tau\mathcal{S})$ and $(\rho_{x'})_*({}^\tau\mathcal{S})$ define the same class in $H^1(\mathbb{Q}, P)$ ([MS], page 283).
Choose an isomorphism $\beta : (\rho_x)_*({}^\tau\mathcal{S}) \xrightarrow{\sim} (\rho_{x'})_*({}^\tau\mathcal{S})$. It defines an isomorphism $\varphi_\beta : {}^{\tau,x}(P,\mathfrak{X}) \xrightarrow{\sim} {}^{\tau,x'}(P,\mathfrak{X})$ of Shimura data. The isomorphisms $\varphi_{\beta,\mathbb{A}_f}$ and ${}^{\tau,x}p_f \longmapsto {}^{\tau,x'}p_f$ differ by an inner conjugation. Let ${}^{\tau,x}p_\beta \in {}^{\tau,x}P(\mathbb{A}_f)$ be such that

$$\varphi_\beta \circ \mathrm{ad}({}^{\tau,x}p_\beta^{-1})({}^{\tau,x}p_f) = {}^{\tau,x'}p_f \ , \text{ for all } {}^{\tau,x}p_f \in P(\mathbb{A}_f)\,.$$

Then the isomorphism

$$\varphi(\tau, x', x) : M({}^{\tau,x}(P,\mathfrak{X})) \xrightarrow{\sim} M({}^{\tau,x'}(P,\mathfrak{X}))$$

defined by $\varphi(\tau, x', x) := [\varphi_\beta] \circ [\cdot\, {}^{\tau,x}p_\beta]$ is independent of the choice of β. For $p_f \in P(\mathbb{A}_f)$, we have the formula

$$\varphi(\tau, x', x) \circ [\cdot\, {}^{\tau,x}p_f] = [\cdot\, {}^{\tau,x'}p_f] \circ \varphi(\tau, x', x)\,.$$

Corollary 5.9: For any two special points x and x' of $\mathfrak{X}$, we have

$$\varphi(\tau, x', x) \circ \varphi_{\tau,x} = \varphi_{\tau,x'}\,.$$

Proof: Property b) of 5.8 for the left hand side is easy to check. For a), we again use 5.2 to reduce ourselves to the pure case, which follows from [M], II, Theorem 4.4. q.e.d.

For an open compact subgroup K of $P(\mathbb{A}_f)$, the isomorphism $\varphi_{\tau,x}$ induces an isomorphism $\varphi_{\tau,x,K} : {}^\tau M^K(P,\mathfrak{X}) \xrightarrow{\sim} M^{{}^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X}))$, where ${}^{\tau,x}K \le {}^{\tau,x}P(\mathbb{A}_f)$ denotes the image of K under the isomorphism $P(\mathbb{A}_f) \xrightarrow{\sim} {}^{\tau,x}P(\mathbb{A}_f)$.

Corollary 5.10: The number of elements of the fibres of $h : \mathfrak{X} \longrightarrow h(\mathfrak{X})$ is invariant under twisting.
Proof: left to the reader. q.e.d.

We conclude this paragraph with a description of the complex conjugation ι on a mixed Shimura variety, whose reflex field is real:
let $(P,\mathfrak{X})$ be mixed Shimura data such that $E = E(P,\mathfrak{X})$ is contained in $\mathbb{R}$. Let $x \in \mathfrak{X}$ be special. We define the point $x^\iota \in \mathcal{Y} \times h(\mathfrak{X})$ as follows: $\mu_{x^\iota} = \iota\mu_x$, and $[x^\iota] \in \mathcal{Y}$ is the conjugate under ι of $[x]$. Here, we use the local reciprocity law

$$\psi_\infty : \mathrm{Gal}_{\mathbb{C}/\mathbb{R}} \xrightarrow{\sigma_0} \mathrm{Gal}_{\overline{E\otimes_{\mathbb{Q}}\mathbb{R}}/E\otimes_{\mathbb{Q}}\mathbb{R}} = \pi_0(\mathbb{G}_m(E \otimes_{\mathbb{Q}} \mathbb{R})) \longrightarrow \pi_0(T(\mathbb{R}))$$

for $(T, \mathfrak{Y})$ to define an action of ι on $\mathfrak{Y}$. We shall see in the proof of 5.12 that x^ι in fact lies in $\mathfrak{X}$.

Lemma 5.11: With the above notations, choose a maximal torus $T \leq P$ such that h_x maps into $T_{\mathbb{R}}$. Denote by N the normalizer of $\pi(T)$.

i) There is an $n \in i_x(N)(\mathbb{R})$ such that $nh_x = h_{x^\iota}$.

ii) The map $\eta_x : px \longmapsto p\,x^\iota$ is the unique $P(\mathbb{R})\cdot U(\mathbb{C})$–equivariant antiholomorphic map $\mathfrak{X} \longrightarrow \mathfrak{Y} \times h(\mathfrak{X})$, which sends x to x^ι.

Proof: For i), use 5.2 and [MS], Corollary 4.3.
In order to prove ii), write $P(\mathbb{R})\cdot U(\mathbb{C})/\mathrm{Stab}(x) \cong \mathfrak{X}$. By [P1], Lemma 1.17, the projection $\pi : P \longrightarrow G$ induces an isomorphism of $\mathrm{Stab}(x)$ onto $\mathrm{Stab}(\pi(x))$. It follows from [MS], page 309 that n normalizes $\mathrm{Stab}(x)$ if $\mathfrak{X} = h(\mathfrak{X})$. So η is well defined in this case. But it is obviously well defined if P is a torus, and so it is always well defined because of 5.5.
Given $\mathbf{V} \in \mathrm{Rep}_{\mathbb{R}}(P)$, we obviously have $F^p_{x^\iota}(\mathbf{V}_{\mathbb{C}}) = \overline{F}^p_x(\mathbf{V}_{\mathbb{C}})$ for all p. It follows that the Hodge filtration of $\eta_x^*\mathbf{V}$ varies antiholomorphically on $\mathfrak{X}$, and so η_x is antiholomorphic ([P1], Proposition 1.7.a)). q.e.d.

Corollary 5.12: Let $(P, \mathfrak{X})$ be mixed Shimura data such that $E(P, \mathfrak{X})$ is real. Then η_x maps $\mathfrak{X}$ to $\mathfrak{X}$, and the involution of $M(P, \mathfrak{X})(\mathbb{C})$ defined by complex conjugation is induced by $\eta_x \times \mathrm{id} : \mathfrak{X} \times P(\mathbb{A}_f) \longrightarrow \mathfrak{X} \times P(\mathbb{A}_f)$.
Proof: The case of $(P, \mathfrak{X}) = (T, \mathfrak{Y})$ is easy to check and left to the reader.
Now assume that P is arbitrary but that $\mathfrak{X} = h(\mathfrak{X})$. We first show the second statement, with η_x replaced by the map

$$pr_2 \circ \eta_x : \mathfrak{X} \longrightarrow \mathfrak{X}.$$

Both complex conjugation and the map induced by $(pr_2 \circ \eta_x) \times \mathrm{id}$ are antiholomorphic and equivariant, so it suffices to show that they agree at $[(x, 1)]$. Using Lemma 5.2, we reduce ourselves to the pure case, where the statement follows from [M], II, Theorem 7.2. A diagram similar to the one of the proof of 5.8 shows that $\eta_x(\mathfrak{X}) \subset \mathfrak{X}$, and hence that $\eta_x = pr_2 \circ \eta_x$.
The same diagram shows the claim in the general case. q.e.d.

§ 6 The canonical construction of mixed sheaves: "mixed systems" version

The results of the preceding paragraph allow us to put the constructions of §§ 2–4 together and show that they define mixed systems of smooth sheaves on $M^K(P,\mathfrak{X})$.

Recall ([W2], § 2) the following

Definition: Let k be a number field, X/k smooth, separated and of finite type. $MS^s_{\mathbb{Q}}(X)$, the category of mixed systems of smooth sheaves on X consists of families

$$(\mathbf{V}_l, \mathbf{V}_{DR}, \mathbf{V}_{\infty,\sigma}, I_{l,\overline{\sigma}}, I_{DR,\sigma}, I_{\infty,\sigma} \,|\, l \in \mathbb{N} \text{ prime}\,, \sigma : k \hookrightarrow \mathbb{C}, \overline{\sigma} : \overline{k} \hookrightarrow \mathbb{C})\,,$$

where

a) $\mathbf{V}_l \in \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X)$,

b) $V_{DR} \in \mathrm{BiF}(X)$.

c) $V_{\infty,\sigma} \in \mathrm{Var}_{\mathbb{Q}}(X_\sigma)$. The underlying local system of $\mathbf{V}_{\infty,\sigma}$, together with its weight filtration is supposed to come from a local system over $\mathbb{Z}$. This implies that the local system, tensored with $\mathbb{Q}_l$, can be interpreted as a lisse l–adic sheaf on X_σ.

d) $I_{l,\overline{\sigma}}$ is an isomorphism

$$F_{\mathcal{O}}(\mathbf{V}_{\infty,\overline{\sigma}|_k}) \otimes_{\mathbb{Q}} \mathbb{Q}_l \longrightarrow \overline{\sigma}^* F_l(\mathbf{V}_l)$$

of weight–filtered l–adic sheaves on X_σ. Here, $F_{\mathcal{O}}$ and F_l are suitably defined forgetful functors.

e) $I_{DR,\sigma}$ is a horizontal isomorphism

$$F'_{\mathcal{O}}(\mathbf{V}_{\infty,\sigma}) \longrightarrow \mathbf{V}_{DR} \otimes_{k,\sigma} \mathbb{C}$$

of bifiltered vector bundles on $X_\sigma(\mathbb{C})$. Again, $F'_{\mathcal{O}}$ is a suitable forgetful functor.

It follows that the filtrations in b) are finite and that the Hodge filtration in b) satisfies Griffiths–transversality:

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes_{\mathcal{O}_X} \Omega^1_{X/k} \quad \text{for all } p \in \mathbb{Z}\,.$$

f) Let $c : \mathbb{C} \longrightarrow \mathbb{C}$ denote complex conjugation.
For any $\sigma : k \hookrightarrow \mathbb{C}$, conjugation defines a diffeomorphism

$$c_\sigma : X_\sigma(\mathbb{C}) \longrightarrow X_{c\circ\sigma}(\mathbb{C})\,.$$

For a variation of $\mathbb{Q}$–MHS $\mathbf{W}$ on $X_{c\circ\sigma}(\mathbb{C})$, we define a variation $c_\sigma^*(\mathbf{W})$ on $X_\sigma(\mathbb{C})$ as follows: the local system and the weight filtration are the pull backs via c_σ of the local system and the weight filtration on $\mathbf{W}$, and the Hodge filtration is the pull back of the conjugate of the Hodge filtration on $\mathbf{W}$.
c_σ^* preserves admissibility.
$I_{\infty,\sigma}$ is an isomorphism of variations of $\mathbb{Q}$–MHS

$$\mathbf{V}_{\infty,\sigma} \longrightarrow c_\sigma^*(\mathbf{V}_{\infty,c\circ\sigma})$$

such that $c_{c\circ\sigma}^*(I_{\infty,\sigma}) = I_{\infty,c\circ\sigma}^{-1}$.

For $\rho \in G_k$ we suppose that $I_{l,\overline{\sigma}\rho} = \overline{\sigma}^*(can_\rho)\circ I_{l,\overline{\sigma}}$. Here, can_ρ denotes the canonical isomorphism

$$F_l(\mathbf{V}_l) \longrightarrow \rho^* F_l(\mathbf{V}_l)$$

given by the fact that $F_l(\mathbf{V})$ comes from X.
Furthermore, we require the following:
For each σ, let $c_{\infty,\sigma}$ be the antilinear involution of $F_{\text{diff.}}(\mathbf{V}_{\infty,\sigma})$, the C^∞–bundle underlying $\mathbf{V}_{\infty,\sigma}$, given by complex conjugation of coefficients. Likewise, let $c_{DR,\sigma}$ be the antilinear isomorphism

$$F_{\text{diff.}}(\mathbf{V}_{\infty,\sigma}) \longrightarrow c_\sigma^{-1}(F_{\text{diff.}}(\mathbf{V}_{\infty,c\circ\sigma}))$$

given by complex conjugation of coefficients on the right hand side of the isomorphism in e). Our requirement is the validity of the formula

$$F_{\text{diff.}}(I_{\infty,\sigma}) = c_{DR,\sigma}\circ c_{\infty,\sigma} = c_\sigma^{-1}(c_{\infty,c\circ\sigma})\circ c_{DR,\sigma}\,.$$

In the category of data defined so far, it is possible to define Tate twists $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$.
The last condition we impose is the existence of a system of polarizations: there are morphisms

$$\mathrm{Gr}_n^W \mathbf{V}_l \otimes_{\mathbb{Q}_l} \mathrm{Gr}_n^W \mathbf{V}_l \longrightarrow \mathbb{Q}_l(-n)\,,\ l \in \mathbb{N} \text{ prime}\,, n \in \mathbb{Z},$$
$$\mathrm{Gr}_n^W \mathbf{V}_{DR} \otimes_{\mathcal{O}_X} \mathrm{Gr}_n^W \mathbf{V}_{DR} \longrightarrow \mathbb{Q}_{DR}(-n)\,,\ n \in \mathbb{Z}$$

of l–adic sheaves and flat vector bundles on X, and polarizations

$$\mathrm{Gr}_n^W \mathbf{V}_{\infty,\sigma} \otimes_{\mathbb{Q}} \mathrm{Gr}_n^W \mathbf{V}_{\infty,\sigma} \longrightarrow \mathbb{Q}(-n)\ ,\ \sigma : k \hookrightarrow \mathbb{C}, n \in \mathbb{Z}$$

of variations of $\mathbb{Q}$–MHS such that the $I_{l,\overline{\sigma}}$, $I_{DR,\sigma}$ and $I_{\infty,\sigma}$ and the corresponding morphisms of the mixed system $\mathbb{Q}(-n)$ form commutative diagrams.

Definition: $[\pi]$–$UMS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X}))$ is the full subcategory of $MS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X}))$ of objects admitting a filtration, whose graded objects lie in the subcategory $[\pi]^* MS^s_{\mathbb{Q}}(M^L(G,\mathcal{H}))$.

Now let $(P,\mathfrak{X})$ be mixed Shimura data satisfying Conjecture 4.2, and let K be a neat open compact subgroup of $P(\mathbb{A}_f)$ with image $L \leq G(\mathbb{A}_f)$ under π.

Let $x \in \mathfrak{X}$ be special, and $\tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$. Let $\rho_x : \mathcal{S} \longrightarrow P$ be the morphism of 5.3. For any field F over $\mathbb{Q}$, there is an equivalence of tensor categories given by

$${}^{\tau,x}\_ : \mathrm{Rep}_F(P) \xrightarrow{\sim} \mathrm{Rep}_F({}^{\tau,x}P) : \mathbf{V} \longmapsto {}^{\tau,x}\mathbf{V} := {}^{\tau}\mathcal{S} \times^{\mathcal{S}} \mathbf{V}\,.$$

Theorem 6.1: Let $\varphi_{\tau,x,K} : {}^{\tau}M^K(P,\mathfrak{X}) \xrightarrow{\sim} M^{{}^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X}))$ be the isomorphism of Theorem 5.8.

i) ${}^{\tau,x}(P,\mathfrak{X})$ also satisfies Conjecture 4.2, and there is a canonical isomorphism

$$\eta_{\tau,x,K} : {}^{\tau}\mu_{K,l} \xrightarrow{\sim} \varphi^*_{\tau,x,K} \circ \mu_{{}^{\tau,x}K,l} \circ {}^{\tau,x}\_$$

of functors from $\mathrm{Rep}_{\mathbb{Q}_l}(P)$ to ${}^{\tau}[\pi]$–$U\mathrm{Et}^{l,m}_{\mathbb{Q}_l}({}^{\tau}M^K(P,\mathfrak{X}))$. It is compatible with the action of $P(\mathbb{A}_f)$ and with morphisms of Shimura data.

ii) If τ' is a second automorphism of $\mathbb{C}$, so

$$\varphi_{\tau',{}^{\tau}x,{}^{\tau,x}K} \circ {}^{\tau'}\varphi_{\tau,x,K} = \varphi_{\tau'\tau,x,K} : {}^{\tau'\tau}M^K(P,\mathfrak{X}) \xrightarrow{\sim} M^{{}^{\tau'\tau,x}K}({}^{\tau'\tau,x}(P,\mathfrak{X}))\,,$$

we have

$${}^{\tau'}\varphi^*_{\tau,x,K}(\eta_{\tau',{}^{\tau}x,{}^{\tau,x}K}) \circ {}^{\tau'}\eta_{\tau,x,K} = \eta_{\tau'\tau,x,K} : {}^{\tau'\tau}\mu_{K,l} \xrightarrow{\sim} \varphi^*_{\tau'\tau,x,K} \circ \mu_{{}^{\tau'\tau,x}K,l} \circ {}^{\tau'\tau,x}\_\,.$$

iii) If $x' \in \mathfrak{X}$ is a second special point, and

$$\varphi(\tau,x',x)_K : M^{{}^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X})) \xrightarrow{\sim} M^{{}^{\tau,x'}K}({}^{\tau,x'}(P,\mathfrak{X}))$$

the isomorphism of Corollary 5.9, so

$$\varphi(\tau,x',x)_K \circ \varphi_{\tau,x,K} = \varphi_{\tau,x',K} : {}^{\tau}M^K(P,\mathfrak{X}) \xrightarrow{\sim} M^{{}^{\tau,x'}K}({}^{\tau,x'}(P,\mathfrak{X}))\,,$$

we have a canonical isomorphism

$$\eta(\tau, x', x)_K : \mu_{{}^{\tau,x}K,l} \circ {}^{\tau,x}\_ \xrightarrow{\sim} \varphi(\tau, x', x)^*_K \circ \mu_{{}^{\tau,x'}K,l} \circ {}^{\tau,x'}\_$$

of functors from $\mathrm{Rep}_{\mathbb{Q}_l}(P)$ to $[{}^{\tau,x}\pi]$–$U\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(M^{{}^{\tau,x}K}({}^{\tau,x}(P, \mathfrak{X})))$ such that

$$\varphi^*_{\tau,x,K}(\eta(\tau, x', x)_K) \circ \eta_{\tau,x,K} = \eta_{\tau,x',K} : {}^{\tau}\mu_{K,l} \xrightarrow{\sim} \varphi^*_{\tau,x',K} \circ \mu_{{}^{\tau,x'}K,l} \circ {}^{\tau,x'}\_ \, .$$

Proof: Let $sp_l(\tau) \in {}^{\tau}\mathcal{S}(\mathbb{Q}_l)$ be the l–th component of $sp(\tau)$. It defines an isomorphism

$$\psi_{\tau,x} : P_{\mathbb{Q}_l} \xrightarrow{\sim} {}^{\tau,x}P_{\mathbb{Q}_l} : p \longmapsto [sp_l(\tau), p] \, .$$

We have a natural isomorphism

$$\chi_{\tau,x} : \mathrm{id}_{\mathrm{Rep}_{\mathbb{Q}_l}(P)} \xrightarrow{\sim} \psi^*_{\tau,x} \circ {}^{\tau,x}\_$$

of functors on $\mathrm{Rep}_{\mathbb{Q}_l}(P)$ given by $v \longmapsto [sp_l(\tau), v]$.
Let $\mathbf{V} \in \mathrm{Rep}_{\mathbb{Q}_l}(P)$, and take the constant sheaves $\mathbf{V}$ on ${}^{\tau}M(P, \mathfrak{X})$ and ${}^{\tau,x}\mathbf{V}$ on $M({}^{\tau,x}(P, \mathfrak{X}))$ with their natural $P(\mathbb{A}_f)$– and ${}^{\tau,x}P(\mathbb{A}_f)$–actions, which actually factor through $P(\mathbb{Q}_l)$ and ${}^{\tau,x}P(\mathbb{Q}_l)$. Then by Theorem 5.8, the isomorphism

$$\eta_{\tau,x}(\mathbf{V}) := \varphi_{\tau,x} \times \chi_{\tau,x}(\mathbf{V}) : {}^{\tau}M(P, \mathfrak{X}) \times \mathbf{V} \xrightarrow{\sim} M({}^{\tau,x}(P, \mathfrak{X})) \times {}^{\tau,x}\mathbf{V}$$

is $P(\mathbb{A}_f)$–equivariant and hence induces an isomorphism $\eta_{\tau,x,K}(\mathbf{V})$ of ${}^{\tau}\mu_{K,l}(\mathbf{V})$ and $\varphi^*_{\tau,x,K}(\mu_{{}^{\tau,x}K,l}({}^{\tau,x}\mathbf{V}))$.
We leave it to the reader to check that the claim in ii) is true. The isomorphism $\eta(\tau, x', x)_K$ in iii) is induced by the isomorphism

$$ {}^{\tau,x}\_ \longrightarrow {}^{\tau,x'}\_$$

of functors on $\mathrm{Rep}_{\mathbb{Q}_l}(P)$ given by $[sp_l(\tau), v]_x \longmapsto [sp_l(\tau), v]_{x'}$. q.e.d.

Let $E := E(P, \mathfrak{X})$ be the reflex field of $(P, \mathfrak{X})$.

Theorem 6.2: There is a canonical isomorphism

$$\eta_{\tau,x,K} : {}^{\tau}\mu_{K,DR} \xrightarrow{\sim} \varphi^*_{\tau,x,K} \circ \mu_{{}^{\tau,x}K,DR} \circ {}^{\tau,x}\_$$

of functors from $\mathrm{Rep}_E(P)$ to ${}^{\tau}[\pi]$–$U\mathrm{BiF}({}^{\tau}M^K(P, \mathfrak{X}))$ with properties analogous to the ones stated in 6.1.

Proof: If we write $P = W \rtimes G$ and ${}^{\tau,x}P = {}^{\tau,x}W \rtimes {}^{\tau,x}G$, then the action of ${}^{\tau,x}P$ on $\hat{\mathfrak{U}}(\mathrm{Lie}({}^{\tau,x}W))$ is identical to the one on ${}^{\tau,x}\hat{\mathfrak{U}}(\mathrm{Lie}W)$. Since we have ${}^{\tau}\mu_{K,DR}\,\hat{\mathfrak{U}}(\mathrm{Lie}W) = {}^{\tau}\mathcal{L}og(i,K)_{DR}$ and $\mu_{{}^{\tau,x}K,DR}\,\hat{\mathfrak{U}}(\mathrm{Lie}({}^{\tau,x}W)) = \mathcal{L}og({}^{\tau,x}i,{}^{\tau,x}K)_{DR}$, we apply 3.3, [W2], Theorem 3.5.ii) and 3.5.i) (in this order) to conclude that there is exactly one isomorphism

$${}^{\tau}\mu_{K,DR}\,\hat{\mathfrak{U}}(\mathrm{Lie}W) \xrightarrow{\sim} \varphi^*_{\tau,x,K}\mu_{{}^{\tau,x}K,DR}\,\hat{\mathfrak{U}}(\mathrm{Lie}({}^{\tau,x}W))$$

respecting the unit sections. We take this to be $\eta_{\tau,x,K}$, thus reducing ourselves, as in § 3, to the pure case. We may assume $\mathfrak{X} = h(\mathfrak{X})$. In [M], III, § 4, the theory of conjugates of the standard principal bundles is developed, and our claim follows from [M], III, Theorem 5.1.b) and an isomorphism similar to that at the beginning of the proof of 6.1, which identifies the bundles ${}^{\tau}\mathbf{V}_{\mathbb{C}}$ and $\check{\varphi}^*_{\tau,x}({}^{\tau,x}\mathbf{V}_{\mathbb{C}})$ on ${}^{\tau}\check{\mathcal{H}}$. Here, $\check{\varphi}_{\tau,x}$ is as in [M], III, Proposition 1.3. q.e.d.

Theorem 6.3: If τ fixes E, then there is a canonical isomorphism

$$\eta_{\tau,x,K} : \mu_{K,\infty,\sigma_0} \xrightarrow{\sim} \varphi^*_{\tau,x,K} \circ \mu_{{}^{\tau,x}K,\infty,\sigma_0} \circ {}^{\tau,x}\_$$

of functors from $\mathrm{Rep}_{\mathbb{Q}}(P)$ to $[\pi]_{\mathbb{C}}$–$U\mathrm{Var}_{\mathbb{Q}}(M^K(P,\mathfrak{X})_{\mathbb{C}})$. The underlying transformation of functors to $[\pi]_{\mathbb{C}}$–$U\mathrm{BiF}(M^K(P,\mathfrak{X})_{\mathbb{C}})$ is compatible with the isomorphism of 6.2, and the underlying transformation of functors to local systems is compatible with the isomorphism of 6.1.

Proof: As in the proof of 5.8, the fact that τ fixes E implies that the class of $(\rho_x)_*({}^{\tau}\mathcal{S})$ in $H^1(\mathbb{Q},P)$ is trivial, and the isomorphism $\varphi_{\tau,x}$ is of the following shape: choose $b \in (\rho_x)_*({}^{\tau}\mathcal{S})(\mathbb{Q})$. It defines $\varphi_b : (P,\mathfrak{X}) \xrightarrow{\sim} {}^{\tau,x}(P,\mathfrak{X})$, which is an isomorphism of Shimura data, whose underlying group isomorphism is given by $p \longmapsto [b,p]$. If we define $p_b \in \rho_x(\mathcal{S})(\mathbb{A}_f) \le P(\mathbb{A}_f)$ by the relation

$$b = \rho_x(sp(\tau)) \cdot p_b\,,$$

then $\varphi_{\tau,x}$ is given by $[\varphi_b] \circ [\cdot p_b]$.

Define $\chi_{\tau,x} = \mathrm{id}_{\mathrm{Rep}_{\mathbb{Q}}(P)} \xrightarrow{\sim} \varphi_b^* \circ {}^{\tau,x}\_$ by sending v to $[b,v]$.

For $\mathbf{V} \in \mathrm{Rep}_{\mathbb{Q}}(P)$, the isomorphism

$$\varphi_b \times \chi_{\tau,x}(\mathbf{V}) : \mathfrak{X} \times (P(\mathbb{A}_f)/K) \times \mathbf{V} \xrightarrow{\sim} {}^{\tau,x}\mathfrak{X} \times ({}^{\tau,x}P(\mathbb{A}_f)/\varphi_b(K)) \times {}^{\tau,x}\mathbf{V}$$

is $P(\mathbb{Q})$–equivariant. This is still true if we compose it with right multiplication by p_b, and we arrive at the isomorphism $\eta_{\tau,x,K}(\mathbf{V})$ we were looking for. It is independent of the choice of b. We leave it to the reader to check the various compatibilities. q.e.d.

We are finally in a position to define a tensor functor

$$\mu_{K,MS} : \mathrm{Rep}_{\mathbb{Q}}(P) \longrightarrow [\pi]\text{–}UMS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X}))\,,$$

the "mixed systems" version of the canonical construction, which will include the functors defined in §§ 2–4.

Given $\mathbf{V} \in \mathrm{Rep}_F(P)$, for any special point $x \in \mathfrak{X}$ and any $\tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$, we have an admissible $[{}^{\tau,x}\pi]_{\mathbb{C}}$–unipotent graded–polarizable variation of $\mathbb{Q}$–Hodge structure $\mu_{^{\tau,x}K,\infty,\tau\sigma_0}({}^{\tau,x}\mathbf{V})$ on $M^{^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X}))_{\mathbb{C}}$, a mixed lisse $[{}^{\tau,x}\pi]$–unipotent l–adic sheaf $\mu_{^{\tau,x}K,l}({}^{\tau,x}\mathbf{V}\otimes_{\mathbb{Q}}\mathbb{Q}_l)$ and a bifiltered $[{}^{\tau,x}\pi]$–unipotent flat vector bundle $\mu_{K,DR}({}^{\tau,x}\mathbf{V})$ on $M^{^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X}))$.

Theorems 6.1–6.3 state that all these fit together to form parts a)–e) of the data necessary for a $[\pi]$–unipotent mixed system of smooth sheaves on $M^K(P,\mathfrak{X})$.

Because we already have e), all we have to show for f) is that the diffeomorphism induced by complex conjugation respects the rational structure of the local system. [W2], Theorem 3.6.i) contains this statement for the system $\mathcal{L}og(i,K)$ of logarithmic pro–sheaves. As the finite–dimensional subquotients of $\hat{\mathfrak{U}}(\mathrm{Lie}W)$, together with $\pi^*(\mathrm{Rep}_{\mathbb{Q}}(G))$ generate $\mathrm{Rep}_{\mathbb{Q}}(P)$ as a Tannakian category, we are therefore reduced to the pure case. Also, we may suppose that $\mathcal{H} = h(\mathcal{H})$.

Since the diagram

$$\check{\mathcal{H}} \xleftarrow{\gamma} P(G,\mathcal{H}) \xrightarrow{pr_{13}} M^L(G,\mathcal{H})$$

of § 3 is defined over E, complex conjugation on $M^L(G,\mathcal{H})$ is compatible with complex conjugation on $\check{\mathcal{H}}$.

But the total space of the trivial bundle ${}^{\iota}\mathbf{V}_{\mathbb{C}}$ on ${}^{\iota}\check{\mathcal{H}}$ is given by ${}^{\iota}\check{\mathcal{H}} \times \mathbf{V}_{\mathbb{C}}$, and the isomorphism $\eta_{\iota,x,K} \otimes_E \mathbb{C}$ of ${}^{\iota}\mathbf{V}_{\mathbb{C}}$ and $\check{\varphi}^*_{\iota,x}({}^{\iota,x}\mathbf{V}_{\mathbb{C}})$ in Theorem 6.2 respects the rational structures.

The existence of a system of polarizations is guaranteed by Theorem 1.1.c). So we finally have

Theorem 6.4: If $(P,\mathfrak{X})$ satisfy Conjecture 4.2, then there is a tensor functor

$$\mu_{K,MS} : \mathrm{Rep}_{\mathbb{Q}}(P) \longrightarrow [\pi]\text{–}UMS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X}))\,.$$

Its l–adic component is $\mu_{K,l}$, its de Rham–component is $\mu_{K,DR}$, and its σ_0–Hodge–component is μ_{K,∞,σ_0}. It is compatible with morphisms induced by the action of $P(\mathbb{A}_f)$ and with morphisms of Shimura data. If $x \in \mathfrak{X}$ is a special point, and $\tau \in \mathrm{Aut}_{\mathbb{C}/\mathbb{Q}}$, then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}}(P) \arrow[r, "{{}^{\tau}\mu_{K,MS}}"] \arrow[d, "{{}^{\tau,x}\_}"] & {}^{\tau}[\pi]\text{-}UMS^s_{\mathbb{Q}}({}^{\tau}M^K(P,\mathfrak{X})) \\
\mathrm{Rep}_{\mathbb{Q}}({}^{\tau,x}P) \arrow[r, "{\mu_{{}^{\tau,x}K,MS}}"] & {[{}^{\tau,x}\pi]\text{-}UMS^s_{\mathbb{Q}}(M^{{}^{\tau,x}K}({}^{\tau,x}(P,\mathfrak{X})))} \arrow[u, "{\varphi^*_{\tau,x,K}}"']
\end{tikzcd}$$

of functors, where $\varphi_{\tau,x,K}$ is the isomorphism of Theorem 5.8.

As in § 2, we may reformulate Theorem 6.4: let $M^K(P,\mathfrak{X})^0$ be a connected component of $M^K(P,\mathfrak{X})$, and let $M^L(G,\mathcal{H})^0 := [\pi](M^K(P,\mathfrak{X})^0)$. Fix $x \in M^K(P,\mathfrak{X})^0$, and let $y := [\pi](x)$.

Then if $P_{\overline{x}}$ and $G_{\overline{y}}$ denote the Tannaka duals of $[\pi]$–$UMS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X})^0)$ and $MS^s_{\mathbb{Q}}(M^L(G,\mathcal{H})^0)$, there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \ker[\pi] \arrow[r] \arrow[d, "\wr"'] & P_{\overline{x}} \arrow[r, "{[\pi]}"] \arrow[d, "{\mu^*_{K,MS}}"] & G_{\overline{y}} \arrow[r] \arrow[d, "{\mu^*_{L,MS}}"] & 1 \\
1 \arrow[r] & W \arrow[r] & P \arrow[r, "{[\pi]}"] & G \arrow[r] & 1\,.
\end{tikzcd}$$

In particular, the cohomological derived functor induced by

$$[\pi]^s_* : [\pi]\text{–}UMS^s_{\mathbb{Q}}(M^K(P,\mathfrak{X})^0) \longrightarrow MS^s_{\mathbb{Q}}(M^L(G,\mathcal{H})^0)$$

coincides with cohomology of W. We are confident that once a satisfactory formalism of mixed systems of (not necessarily smooth) sheaves is established, an analogue of Theorems 2.3, 3.5 and 4.7 will hold.

Index of Notations

References

[Bo] A. Borel et al., "Algebraic D–modules", Perspectives in Mathematics 2, Academic Press 1987.

[CKS] E. Cattani, A. Kaplan, W. Schmid, "Degeneration of Hodge structures", Ann. of Math. 123 (1986), pp. 457–535.

[D1] P. Deligne, "Le Groupe Fondamental de la Droite Projective Moins Trois Points", in Y. Ihara, K. Ribet, J.–P. Serre, "Galois Groups over $\mathbb{Q}$", Math. Sci. Res. Inst. Publ. 16, Springer–Verlag 1989, pp. 79–297.

[D2] P. Deligne, "La Conjecture de Weil II", Publ. Math. IHES 52 (1981), pp. 313–428.

[D3] P. Deligne, "Equations Différentielles à Points Singuliers Réguliers", LNM 163, Springer–Verlag 1970.

[D4] P. Deligne, "Motifs et Groupes de Taniyama", in P. Deligne, J.S. Milne, A. Ogus, K.-y. Shih, "Hodge Cycles, Motives, and Shimura varieties", LNM 900, Springer–Verlag 1982, pp. 261–279.

[DM] P. Deligne, J.S. Milne, "Tannakian Categories", in P. Deligne, J.S. Milne, A. Ogus, K.-y. Shih, "Hodge Cycles, Motives, and Shimura varieties", LNM 900, Springer–Verlag 1982, pp. 101–228.

[Ha] R. Hartshorne, "Algebraic Geometry", LNM 52, Springer–Verlag 1977.

[Hu] S.T. Hu, "Homotopy Theory", Academic Press 1959.

[Hub] A. Huber, "Calculation of Derived Functors via Ind–Categories", Jour. of Pure and Appl. Algebra 90 (1993), pp. 39–48.

[Hum] J.E. Humphreys, "Introduction to Lie Algebras and Representation Theory", GTM 9, Springer–Verlag 1980.

[HZ] R.M. Hain, S. Zucker, "Unipotent variations of mixed Hodge structure", Inv. math. 88 (1987), pp. 83–124.

[J] U. Jannsen, “Mixed Motives and Algebraic K–Theory”, LNM 1400, Springer–Verlag 1990.

[Ka] M. Kashiwara, “A Study of Variation of Mixed Hodge Structure”, Publ. RIMS, Kyoto Univ. 22 (1986), pp. 991–1024.

[L] R. Langlands, “Automorphic representations, Shimura varieties, and motives. Ein Märchen”, in A. Borel, W. Casselman, “Automorphic Forms, Representations, and L–functions”, Proc. of Symp. in Pure Math. 33, Part II, AMS 1979, pp. 205–246.

[LR] E. Looijenga, M. Rapoport, “Weights in the Local Cohomology of a Baily–Borel Compactification”, in J.A. Carlson, C.H. Clemens, D.R. Morrison, “Complex Geometry and Lie Theory”, Proc. of Symp. in Pure Math. 53, AMS 1991, pp. 223–260.

[M] J.S. Milne, “Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles”, in L. Clozel, J.S. Milne, “Automorphic Forms, Shimura Varieties, and L–functions”, Volume I, Perspectives in Mathematics 10, Academic Press 1990, pp. 283–414.

[MS] J.S. Milne, K.–y. Shih, “Conjugates of Shimura varieties”, in P. Deligne, J.S. Milne, A. Ogus, K.–y. Shih, “Hodge Cycles, Motives, and Shimura varieties”, LNM 900, Springer–Verlag 1982, pp. 280–356.

[P1] R. Pink, “Arithmetical compactification of Mixed Shimura Varieties”, thesis, Bonner Mathematische Schriften 1989.

[P2] R. Pink, “On l–adic sheaves and their higher direct images in the Baily–Borel compactification”, Math. Ann. 292 (1992), pp. 197–240.

[R] M.S. Raghunathan, “Cohomology of arithmetic subgroups of algebraic groups: I”, Ann. of Math. 86 (1967), pp. 409–424.

[S1] Morihiko Saito, “Modules de Hodge Polarisables”, Publ. RIMS, Kyoto Univ. 24 (1988), pp. 849–995.

[S2] Morihiko Saito, “Mixed Hodge Modules”, Publ. RIMS, Kyoto Univ. 26 (1990), pp. 221–333.

[Sch] W. Schmid, “Variation of Hodge Structure: The Singularities of the Period Mapping”, Inv. math. 22 (1973), pp. 211–319.

[SGA1] A. Grothendieck et al., “Revêtements Etales et Groupe Fondamental”, LNM 224, Springer–Verlag 1971.

[SGA4,III] M. Artin, A. Grothendieck, J.L. Verdier et al., “Théorie des Topos et Cohomologie Etale des Schémas”, Tôme 3, LNM 305, Springer–Verlag 1973.

[W1] J. Wildeshaus, “Polylogarithmic Extensions on Mixed Shimura varieties”, Schriftenreihe des Mathematischen Instituts der Universität Münster, 3. Serie, Heft 12, 1994.

[W2] J. Wildeshaus, “Mixed structures on fundamental groups”, preprint, 1994.

[Z] S. Zucker, “Hodge theory with degenerating coefficients: L_2–cohomology in the Poincaré metric”, Ann. of Math. 109 (1979), pp. 415–476.