The bordism of automorphisms of manifolds from the algebraic L-theory point of view

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Dedicated to William Browder

Introduction

Among other things, Browder [1] initiated the application of surgery theory to the bordism of automorphisms of manifolds and the related study of fibred knots and open book decompositions. In this paper the bordism of automorphisms of high-dimensional manifolds is considered from the point of view of the localization exact sequence in algebraic L-theory.

The mapping torus of an automorphism $f : M \longrightarrow M$ of a closed n-dimensional manifold is a closed $(n+1)$-dimensional manifold

$$T(f) \;=\; M \times [0,1]/\{(x,0) = (f(x),1) \,|\, x \in M\} \; .$$

Given a space X let $\Delta_n(X)$ be the bordism group of pairs (M,f) with M a closed oriented n-dimensional manifold (in one of the standard categories O, PL, TOP) and $f : M \longrightarrow M$ an orientation-preserving automorphism, together with a map $g : M \longrightarrow X$ and a homotopy $gf \simeq g : M \longrightarrow X$. The mapping torus construction of Browder [1, 29.8] defines a morphism of abelian groups

$$T \;:\; \Delta_n(X) \longrightarrow \Omega_{n+1}(X \times S^1) \; ; \; (M,f) \longrightarrow T(f)$$

to the bordism group $\Omega_{n+1}(X \times S^1)$ of closed oriented $(n+1)$-dimensional manifolds N with a map $N \longrightarrow X \times S^1$. The relative group $AB_n(X)$ in the exact sequence

$$\dots \longrightarrow AB_n(X) \longrightarrow \Delta_n(X) \xrightarrow{T} \Omega_{n+1}(X \times S^1) \longrightarrow AB_{n-1}(X) \longrightarrow \dots$$

is the bordism group of oriented $(n+2)$-dimensional manifolds with boundary $(W, \partial W)$ with a map $W \longrightarrow X \times S^1$, such that $\partial W = T(f)$ for some representative (M,f) of an element of $\Delta_n(X)$.

López de Medrano [6] applied the Witt group of automorphisms of symmetric forms to the study of $\Delta_*(\text{pt.})$. Neumann [7] computed the automorphism Witt group over $\mathbb{Z}$, and Kreck [5] used this to compute $\Delta_*(\text{pt.})$ for $* \geq 5$. The automorphism bordism groups $\Delta_*(X)$ are closely related to open book decompositions, as considered by Winkelnkemper [17]. Quinn [8] developed a surgery theory for open book decompositions, in which the obstruction groups are defined for any ring with involution A by

$$W_n(A) \;=\; \begin{cases} \text{Witt group of nonsingular asymmetric forms over } A \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd.} \end{cases}$$

The automorphism bordism groups $AB_*(X)$ are geometrically isomorphic to the open book bordism groups $BB_*(X)$ of [8]

$$AB_*(X) \ = \ BB_*(X) \ .$$

The main result of [8] identifies

$$BB_n(X) \ = \ W_{n+2}(\mathbb{Z}[\pi_1(X)]) \ \ (n \geq 5)$$

for any space X with finitely presented fundamental group $\pi_1(X)$. The main result of this paper (Theorem 3.1) obtains a different algebraic expression for the high-dimensional groups $AB_*(X) = BB_*(X)$, as the surgery obstruction groups of Wall [16]

$$AB_n(X) \ = \ L_{n+2}(\Omega^{-1}\mathbb{Z}[\pi_1(X \times S^1)]) \ \ (n \geq 5)$$

of the following (noncommutative) localization $\Omega^{-1}\mathbb{Z}[\pi_1(X \times S^1)]$ of the Laurent polynomial extension of the group ring $\mathbb{Z}[\pi_1(X)]$

$$\mathbb{Z}[\pi_1(X \times S^1)] \ = \ \mathbb{Z}[\pi_1(X)][z, z^{-1}] \ \ (\overline{z} = z^{-1}) \ .$$

A $k \times k$ matrix ω in the Laurent polynomial extension $A[z, z^{-1}]$ of a ring A is *Fredholm* if the $A[z, z^{-1}]$-module morphism $\omega : A[z, z^{-1}]^k \longrightarrow A[z, z^{-1}]^k$ is injective and the cokernel is f.g. projective as an A-module. In Proposition 1.7 it will be proved that the localization $\Omega^{-1}A[z, z^{-1}]$ inverting the set Ω of Fredholm matrices in $A[z, z^{-1}]$ has the property that a finite f.g. free $A[z, z^{-1}]$-module chain complex C is A-finitely dominated (i.e. A-module chain equivalent to finite f.g. projective A-module chain complex) if and only if

$$H_*(\Omega^{-1}A[z, z^{-1}] \otimes_{A[z,z^{-1}]} C) \ = \ 0 \ .$$

Now suppose that $A = \mathbb{Z}[\pi]$ is a group ring, so that

$$A[z, z^{-1}] \ = \ \mathbb{Z}[\pi \times \mathbb{Z}] \ .$$

If N is a connected finite CW complex with universal cover $\widetilde{N}$ and fundamental group $\pi_1(N) = \pi \times \mathbb{Z}$ the infinite cyclic cover $\overline{N} = \widetilde{N}/\pi$ of N is a connected CW complex with $\pi_1(\overline{N}) = \pi$. The infinite CW complex $\overline{N}$ is finitely dominated if and only if the $\mathbb{Z}[\pi][z, z^{-1}]$-module chain complex $C(\widetilde{N})$ is $\mathbb{Z}[\pi]$-finitely dominated, if and only if

$$H_*(N; \Omega^{-1}\mathbb{Z}[\pi][z, z^{-1}]) \ = \ 0 \ .$$

If N is a closed $(n+1)$-dimensional manifold with $\pi_1(N) = \pi \times \mathbb{Z}$, and $\overline{N} = \widetilde{N}/\pi$ is finitely dominated, then the fibering obstruction $\Phi(N) \in Wh(\pi \times \mathbb{Z})$ of Farrell [4] and Siebenmann [14] is defined, such that $\Phi(N) = 0$ if (and for $n \geq 5$) only if $N = T(f)$ for an automorphism $f : M \longrightarrow M$ of a codimension 1 submanifold $M \subset N$ such that

$f_* = 1 : \pi_1(M) = \pi \longrightarrow \pi$. The mapping torus function

$$\begin{aligned} T\ :\ &\{\text{closed } n\text{-dimensional manifolds } M \text{ with an automorphism } f : M \longrightarrow M \\ &\qquad \text{such that } f_* = 1 : \pi_1(M) = \pi \longrightarrow \pi\} \\ &\longrightarrow \{\text{closed } (n+1)\text{-dimensional manifolds } N \text{ such that } \pi_1(N) = \pi \times \mathbb{Z}, \\ &\qquad H_*(N;\Omega^{-1}\mathbb{Z}[\pi][z,z^{-1}]) = 0 \text{ and } \Phi(N) = 0 \in Wh(\pi \times \mathbb{Z})\}\ ; \\ &\qquad (M,f) \longrightarrow T(f) \end{aligned}$$

is thus a bijection for $n \geq 5$. The relative bordism group $AB_n(X)$ can thus be viewed as the bordism group of oriented $(n+2)$-dimensional manifolds with boundary $(W, \partial W)$ with a map $W \longrightarrow X \times S^1$, such that

$$\begin{aligned} &\pi_1(W) \ = \ \pi_1(\partial W) \ = \ \pi_1(X) \times \mathbb{Z}\ \ , \\ &H_*(\partial W; \Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]) \ = \ 0 \ \ , \ \ \Phi(\partial W) \ = \ 0 \in Wh(\pi_1(X) \times \mathbb{Z})\ . \end{aligned}$$

In Theorem 3.1 the bijection will be used to identify

$$AB_n(X) \ = \ L_{n+2}(\Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]) \ \ (n \geq 5)\ .$$

The open book surgery of Quinn [8] is replaced here by the homology surgery of Cappell and Shaneson [2] and Vogel [15]. Combining the identification of 3.1 with the result of [8] gives geometric identifications

$$L_*(\Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]) \ = \ W_*(\mathbb{Z}[\pi_1(X)])\ .$$

In Ranicki [13] we shall give direct algebraic identifications

$$L_*(\Omega^{-1}A[z,z^{-1}]) \ = \ W_*(A)$$

for any ring with involution A.

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§1. Finite domination

Given a ring A let $A[z,z^{-1}]$ be the Laurent polynomial extension, the ring of polynomials $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ with coefficients $a_j \in A$ such that $\{j \in \mathbb{Z} \,|\, a_j \neq 0\}$ is finite.

Definition 1.1 An $A[z,z^{-1}]$-module chain complex C is A-*finitely dominated* if it is A-module chain equivalent to a finite f.g. projective A-module chain complex.

□

The Novikov completions $A((z))$, $A((z^{-1}))$ of $A[z,z^{-1}]$ are the rings of formal power series defined by

$$A((z)) \ = \ \{\sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \leq 0 \,|\, a_j \neq 0 \in A\} \text{ finite}\} \ ,$$

$$A((z^{-1})) \ = \ \{\sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \geq 0 \,|\, a_j \neq 0 \in A\} \text{ finite}\}$$

with $A((z)) \cap A((z^{-1})) = A[z, z^{-1}]$.

Proposition 1.2 (Ranicki [12]) *A finite f.g. free $A[z, z^{-1}]$-module chain complex C is A-finitely dominated if and only if*

$$H_*(A((z)) \otimes_{A[z,z^{-1}]} C) \ = \ H_*(A((z^{-1})) \otimes_{A[z,z^{-1}]} C) \ = \ 0 \ .$$

□

The two conditions of 1.2 can be united, using the diagonal ring morphism

$$A[z, z^{-1}] \longrightarrow A((z)) \times A((z^{-1})) \ ; \ x \longrightarrow (x, x) \ .$$

A finite f.g. free $A[z, z^{-1}]$-module chain complex C is A-finitely dominated if and only if

$$H_*((A((z)) \times A((z^{-1}))) \otimes_{A[z,z^{-1}]} C) \ = \ 0 \ .$$

Proposition 1.3 *The following conditions on a $k \times k$ matrix ω in $A[z, z^{-1}]$ are equivalent:*

(i) *the $A[z, z^{-1}]$-module morphism $\omega : A[z, z^{-1}]^k \longrightarrow A[z, z^{-1}]^k$ is injective and the cokernel is a f.g. projective A-module,*

(ii) *ω becomes invertible in $A((z)) \times A((z^{-1}))$,*

(iii) *the 1-dimensional f.g. free $A[z, z^{-1}]$-module chain complex*

$$C \ : \ C_1 \ = \ A[z, z^{-1}]^k \ \xrightarrow{\omega} \ C_0 \ = \ A[z, z^{-1}]^k$$

is A-finitely dominated.

Proof. (i) $\Longrightarrow$ (iii) C is A-module chain equivalent to the 0-dimensional f.g. projective A-module chain complex P defined by $P_0 = \text{coker}(\omega)$.
(ii) $\Longleftrightarrow$ (iii) Immediate from 1.2.
(iii) $\Longrightarrow$ (i) The $A[z, z^{-1}]$-module morphism $\omega : A[z, z^{-1}]^k \longrightarrow A[z, z^{-1}]^k$ is injective (i.e. $H_1(C) = 0$) since $A[z, z^{-1}] \longrightarrow A((z)) \times A((z^{-1}))$ is injective. We have to prove that $H_0(C) = \text{coker}(\omega)$ is a f.g. projective A-module. Let

$$\omega \ = \ \sum_{j=-N^+}^{N^-} \omega_j z^j$$

with ω_j a $k\times k$ matrix in A, and $-N^+\leq 0\leq N^-$. Let C^+ be the $A[z]$-module subcomplex of C defined by

$$d^+ \;=\; \omega| \;:\; C_1^+ \;=\; \sum_{i=0}^{\infty} z^iA^k \longrightarrow C_0^+ \;=\; \sum_{j=-N^+}^{\infty} z^jA^k \;,$$

and let C^- be the $A[z^{-1}]$-module subcomplex of C defined by

$$d^- \;=\; \omega| \;:\; C_1^- \;=\; \sum_{i=-\infty}^{-1} z^iA^k \longrightarrow C_0^- \;=\; \sum_{j=-\infty}^{N^--1} z^jA^k \;.$$

The intersection $C^+\cap C^-$ is the 0-dimensional f.g. free A-module chain complex with

$$(C^+\cap C^-)_0 \;=\; \sum_{j=-N^+}^{N^--1} z^jA \;,$$

and there is defined an exact sequence

$$0 \longrightarrow C^+\cap C^- \longrightarrow C^+\oplus C^- \longrightarrow C \longrightarrow 0$$

with

$$A[z,z^{-1}]\otimes_{A[z]} C^+ \;=\; A[z,z^{-1}]\otimes_{A[z^{-1}]} C^- \;=\; C \;.$$

As in the proof of 1.2 there are defined short exact sequences

$$0 \longrightarrow C^+ \longrightarrow (A[[z]]\otimes_{A[z]} C^+)\oplus C \longrightarrow A((z))\otimes_{A[z,z^{-1}]} C \longrightarrow 0 \;,$$

$$0 \longrightarrow C^+\cap C^- \longrightarrow C^+\oplus (A[[z^{-1}]]\otimes_{A[z^{-1}]} C^-) \longrightarrow A((z^{-1}))\otimes_{A[z,z^{-1}]} C \longrightarrow 0 \;.$$

By hypothesis

$$H_*(A((z))\otimes_{A[z,z^{-1}]} C) \;=\; H_*(A((z^{-1}))\otimes_{A[z,z^{-1}]} C) \;=\; 0 \;,$$

so that there are defined A-module isomorphisms

$$\begin{aligned} H_0(C^+) \;&\cong\; H_0(A[[z]]\otimes_{A[z]} C^+)\oplus H_0(C) \;, \\ H_0(C^+\cap C^-) \;&\cong\; H_0(C^+)\oplus H_0(A[[z^{-1}]]\otimes_{A[z^{-1}]} C^-) \\ &\cong\; H_0(A[[z]]\otimes_{A[z]} C^+)\oplus H_0(C)\oplus H_0(A[[z^{-1}]]\otimes_{A[z^{-1}]} C^-) \;. \end{aligned}$$

Thus $H_0(C)$ is (isomorphic to) a direct summand of the f.g. free A-module $H_0(C^+\cap C^-)$, verifying that $H_0(C)$ is a f.g. projective A-module.

□

Definition 1.4 A square matrix ω in $A[z,z^{-1}]$ is *Fredholm* if it satisfies any one of the equivalent conditions of 1.3.

□

Example 1.5 (i) If $\omega = z - h$ for an invertible $k \times k$ matrix h in A then ω is Fredholm: the $A[z,z^{-1}]$-module morphism $\omega : A[z,z^{-1}]^k \longrightarrow A[z,z^{-1}]^k$ is injective with cokernel the f.g. free A-module A^k (z acting by h).
(ii) If $\omega = 1 - zp$ for a projection $k \times k$ matrix $p = p^2$ in A then ω is Fredholm: the $A[z,z^{-1}]$-module morphism $\omega : A[z,z^{-1}]^k \longrightarrow A[z,z^{-1}]^k$ is injective with cokernel the f.g. projective A-module $\mathrm{im}(p)$ (z acting by 1).

□

Cohn [3, pp. 254-255] defines the localization $\Sigma^{-1}R$ for any ring R and any set Σ of square matrices with entries in R to be the ring with generators all the elements of R and all the entries m'_{ij} in formal inverses $M' = (m'_{ij})$ of the matrices $M \in \Sigma$, subject to all the relations holding in R as well as

$$MM' \ = \ M'M \ = \ I \quad (M \in \Sigma) \ .$$

The canonical ring morphism $i : R \longrightarrow \Sigma^{-1}R$ has the universal property that any ring morphism $f : R \longrightarrow S$ such that $f(M)$ is invertible for each $M \in \Sigma$ has a unique factorization

$$f \ : \ R \xrightarrow{i} \Sigma^{-1}R \longrightarrow S \ .$$

In general, $i : R \longrightarrow \Sigma^{-1}R$ may not be injective – for example, if $0 \in \Sigma$ then $\Sigma^{-1}R = 0$ is the zero ring.

Definition 1.6 Let Ω be the set of Fredholm matrices in $A[z,z^{-1}]$, and let $\Omega^{-1}A[z,z^{-1}]$ be the localization of $A[z,z^{-1}]$ inverting Ω.

□

The diagonal ring morphism

$$A[z,z^{-1}] \longrightarrow A((z)) \times A((z^{-1})) \ ; \ x \longrightarrow (x,x)$$

is injective, and has a factorization

$$A[z,z^{-1}] \xrightarrow{i} \Omega^{-1}A[z,z^{-1}] \longrightarrow A((z)) \times A((z^{-1}))$$

so that the canonical ring morphism $i : A[z,z^{-1}] \longrightarrow \Omega^{-1}A[z,z^{-1}]$ is injective.

Given an $A[z,z^{-1}]$-module chain complex C let

$$\Omega^{-1}C \ = \ \Omega^{-1}A[z,z^{-1}] \otimes_{A[z,z^{-1}]} C$$

be the induced $\Omega^{-1}A[z,z^{-1}]$-module chain complex.

Proposition 1.7 *The following conditions on a finite f.g. free $A[z,z^{-1}]$-module chain complex C are equivalent:*
(i) $H_*(\Omega^{-1}C) = 0$,

(ii) $H_*((A((z)) \times A((z^{-1}))) \otimes_{A[z,z^{-1}]} C) = 0$,
(iii) *C is A-finitely dominated,*
(iv) *C is $A[z,z^{-1}]$-module chain equivalent to the algebraic mapping cone $\mathcal{C}(z-h : P[z,z^{-1}] \longrightarrow P[z,z^{-1}])$ for an automorphism $h : P \longrightarrow P$ of a finite f.g. projective A-module chain complex P.*

Proof. (i) $\Longrightarrow$ (ii) Immediate from the existence of a ring morphism

$$A[z,z^{-1}] \longrightarrow A((z)) \times A((z^{-1})) \ .$$

(ii) $\Longrightarrow$ (i) Choose a basis for each C_r, writing

$$C_r \ = \ A[z,z^{-1}]^{k_r} \quad (r \geq 0) \ .$$

There exist $A[z,z^{-1}]$-module morphisms $\Gamma : C_r \longrightarrow C_{r+1}$ $(r \geq 0)$ such that the $A[z,z^{-1}]$-module endomorphisms

$$\omega \ = \ d\Gamma + \Gamma d \ : \ C_r \ = \ A[z,z^{-1}]^{k_r} \longrightarrow C_r \ = \ A[z,z^{-1}]^{k_r} \quad (r \geq 0)$$

induce automorphisms over $A((z)) \times A((z^{-1}))$. Each ω is a Fredholm matrix in $A[z,z^{-1}]$ by 1.3, so that $H_*(\Omega^{-1}C) = 0$.
(ii) $\Longleftrightarrow$ (iii) This is 1.2.
(i) $\Longrightarrow$ (iv) Assume that C is n-dimensional, and let

$$C_r \ = \ A[z,z^{-1}]^{k_r} \quad (0 \leq r \leq n) \ .$$

There exist $A[z,z^{-1}]$-module morphisms $\Gamma : C_r \longrightarrow C_{r+1}$ $(0 \leq r \leq n-1)$ such that the $A[z,z^{-1}]$-module endomorphisms

$$\omega \ = \ d\Gamma + \Gamma d \ : \ C_r \ = \ A[z,z^{-1}]^{k_r} \longrightarrow C_r \ = \ A[z,z^{-1}]^{k_r} \quad (0 \leq r \leq n)$$

are defined by Fredholm matrices ω in $A[z,z^{-1}]$, with $\Gamma^2 = 0$. The $A[z,z^{-1}]$-module morphism

$$d + \Gamma \ = \ \begin{pmatrix} d & 0 & 0 & \cdots \\ \Gamma & d & 0 & \cdots \\ 0 & \Gamma & d & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$C_{odd} \ = \ C_1 \oplus C_3 \oplus C_5 \oplus \dots \longrightarrow C_{even} \ = \ C_0 \oplus C_2 \oplus C_4 \oplus \dots$$

is injective with f.g. projective A-module cokernel. The $(n-1)$-dimensional f.g. projective A-module chain complex P defined by

$$P_r \ = \ \begin{cases} \text{coker}(d+\Gamma : C_{odd} \longrightarrow C_{even}) \text{ if } r = 0 \\ \text{coker}(\omega \oplus \omega \oplus \dots : C_{r+1} \oplus C_{r+3} \oplus \dots \longrightarrow C_{r+1} \oplus C_{r+3} \oplus \dots) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } r = 1, 2, \dots, n-1 \ , \end{cases}$$

$$d \ : \ P_r \longrightarrow P_{r-1} \ ; \ [x] \longrightarrow \begin{cases} [0, x] & \text{if } r = 1 \\ [(d+\Gamma)(x)] & \text{if } r = 2, 3, \dots, n-1 \end{cases}$$

is equipped with an automorphism

$$h \;:\; P \longrightarrow P \;;\; [x] \longrightarrow [zx]$$

such that C is $A[z,z^{-1}]$-module chain equivalent to the algebraic mapping cone $\mathcal{C}(z-h : P[z,z^{-1}]\longrightarrow P[z,z^{-1}])$.
(iv) $\Longrightarrow$ (iii) The algebraic mapping cone $\mathcal{C}(z-h)$ is A-module chain equivalent to P.
(iv) $\Longrightarrow$ (i) Each of the $A[z,z^{-1}]$-module morphisms

$$z-h \;:\; P_r[z,z^{-1}] \longrightarrow P_r[z,z^{-1}] \quad (r \geq 0)$$

induces an $\Omega^{-1}A[z,z^{-1}]$-module isomorphism

$$z-h \;:\; \Omega^{-1}P_r[z,z^{-1}] \longrightarrow \Omega^{-1}P_r[z,z^{-1}] \;,$$

so that

$$H_*(\Omega^{-1}C) \;=\; H_*(z-h : \Omega^{-1}P[z,z^{-1}]\longrightarrow\Omega^{-1}P[z,z^{-1}]) \;=\; 0 \;.$$

□

Example 1.8 If $A = K$ is a field then Ω consists of all the square matrices in $K[z,z^{-1}]$ with non-zero determinant, and $\Omega^{-1}K[z,z^{-1}] = K(z)$ is the function field of K. A finite f.g. free $K[z,z^{-1}]$-module chain complex C is K-finitely dominated if and only if $H_*(K(z) \otimes_{K[z,z^{-1}]} C) = 0$, if and only if the homology K-vector spaces $H_*(C)$ are finite-dimensional.

□

Definition 1.9 (i) The *automorphism category of A* is the exact category $\text{Aut}(A)$ in which an object (P,h) is a f.g. projective A-module P together with an automorphism $h : P\longrightarrow P$, a morphism $f : (P,h)\longrightarrow(P',h')$ is an A-module morphism $f : P\longrightarrow P'$ such that $h'f = fh$, and a sequence $(P,h)\longrightarrow(P',h')\longrightarrow(P'',h'')$ is exact if the A-module sequence $P\longrightarrow P'\longrightarrow P''$ is exact.
(ii) The *automorphism class group* $\text{Aut}_0(A)$ is the class group of the automorphism category

$$\text{Aut}_0(A) \;=\; K_0(\text{Aut}(A)) \;.$$

□

Proposition 1.10 *The torsion group of $\Omega^{-1}A[z,z^{-1}]$ is a direct sum*

$$K_1(\Omega^{-1}A[z,z^{-1}]) \;=\; K_1(A[z,z^{-1}]) \oplus \text{Aut}_0(A) \;.$$

Proof. The relative term $K_1(i)$ in the localization exact sequence of algebraic K-theory

$$\dots \longrightarrow K_1(A[z,z^{-1}]) \xrightarrow{\;i\;} K_1(\Omega^{-1}A[z,z^{-1}]) \xrightarrow{\;\partial\;} K_1(i) \longrightarrow K_0(A[z,z^{-1}]) \longrightarrow \dots$$

is the class group of the exact category of f.g. $\Omega^{-1}A[z,z^{-1}]$-torsion $A[z,z^{-1}]$-modules of homological dimension 1, which is isomorphic to the automorphism category $\mathrm{Aut}(A)$, with

$$\partial \ : \ K_1(\Omega^{-1}A[z,z^{-1}]) \longrightarrow K_1(i) \ = \ \mathrm{Aut}_0(A) \ ; \ \tau(\Omega^{-1}C) \longrightarrow [C,\zeta]$$

sending the torsion $\tau(\Omega^{-1}C)$ for an A-finitely dominated finite based f.g. free $A[z,z^{-1}]$-module chain complex C to the class of the A-module automorphism

$$\zeta \ : \ C \longrightarrow C \ ; \ x \longrightarrow zx \ .$$

The morphism ∂ is a surjection which is split by

$$\begin{aligned}\Delta \ &: \mathrm{Aut}_0(A) \longrightarrow K_1(\Omega^{-1}A[z,z^{-1}]) \ ; \\ &[P,h] \longrightarrow \tau(z-h:\Omega^{-1}P[z,z^{-1}]\longrightarrow\Omega^{-1}P[z,z^{-1}]) \ .\end{aligned}$$

The morphism

$$i \ : \ K_1(A[z,z^{-1}]) \longrightarrow K_1(\Omega^{-1}A[z,z^{-1}]) \ ; \ \tau(C) \longrightarrow \tau(\Omega^{-1}C)$$

is an injection which is split by

$$\Phi \ : \ K_1(\Omega^{-1}A[z,z^{-1}]) \longrightarrow K_1(A[z,z^{-1}]) \ ; \ \tau(\Omega^{-1}C) \longrightarrow \Phi(C)$$

with C an A-finitely dominated ($=\Omega^{-1}A[z,z^{-1}]$-contractible) based f.g. free $A[z,z^{-1}]$-module chain complex and $\Phi(C)\in K_1(A[z,z^{-1}])$ the algebraic fibering obstruction (Ranicki [11, §20]).

□

Example 1.11 Let N be a connected finite CW complex with universal cover $\widetilde{N}$, such that $\pi_1(N)=\pi\times\mathbb{Z}$ and the infinite cyclic cover $\overline{N}=\widetilde{N}/\pi$ is finitely dominated. Let

$$\Lambda \ = \ \mathbb{Z}[\pi_1(N)] \ = \ \mathbb{Z}[\pi][z,z^{-1}]$$

and let $\zeta:\overline{N}\longrightarrow\overline{N}$ be a generating covering translation, inducing $z:C(\widetilde{N})\longrightarrow C(\widetilde{N})$ on the Λ-module chain level. The cellular Λ-module chain complex $C(\widetilde{N})$ is $\Omega^{-1}\Lambda$-contractible, and the $\Omega^{-1}\Lambda$-coefficient Whitehead (or rather Reidemeister) torsion of N is given by

$$\begin{aligned}\tau(N;\Omega^{-1}\Lambda) \ &= \ \tau(\Omega^{-1}C(\widetilde{N})) \\ &= \ (\Phi(N),[\overline{N},\zeta]) \ = \ (\Phi(C(\widetilde{N})),[C(\widetilde{N}),\zeta]) \\ &\in K_1(\Omega^{-1}\Lambda) \ = \ K_1(\Lambda)\oplus\mathrm{Aut}_0(\mathbb{Z}[\pi]) \ .\end{aligned}$$

□

§2. Localization in L-theory

We refer to Ranicki [9] for an exposition of the quadratic and symmetric L-groups $L_*(A)$, $L^*(A)$ of a ring with involution A, which we take to be defined using (unbased) f.g. free A-modules. The quadratic L-group $L_n(A)$ of Wall [16] was identified in [9] with the cobordism group of n-dimensional quadratic Poincaré complexes (C,ψ) over A, with C a f.g. free A-module chain complex.

Extend the involution on A to $A[z,z^{-1}]$ by $\overline{z}=z^{-1}$. The conjugate transpose of a Fredholm matrix $\omega=(a_{ij})$ in $A[z,z^{-1}]$ is a Fredholm matrix $\omega^*=(\overline{a}_{ji})$ in $A[z,z^{-1}]$, with

$$\mathrm{coker}(\omega^*) \;=\; \mathrm{Hom}_A(\mathrm{coker}(\omega),A)\ ,$$

so that $\Omega^{-1}A[z,z^{-1}]$ is also a ring with involution.

We refer to Ranicki [10, §3] for the localization exact sequence in algebraic L-theory, which applies also to $\Omega^{-1}A[z,z^{-1}]$:

Proposition 2.1 *For any ring with involution A there is defined an exact sequence*

$$\dots \longrightarrow L^{n+2}(A[z,z^{-1}]) \xrightarrow{i} L^{n+2}(\Omega^{-1}A[z,z^{-1}]) \xrightarrow{\partial} L\mathrm{Aut}^n(A) \xrightarrow{T} L^{n+1}(A[z,z^{-1}]) \longrightarrow \dots$$

with $L\mathrm{Aut}^n(A)$ the cobordism group of automorphisms of f.g. projective n-dimensional symmetric Poincaré complexes over A, T given by the algebraic mapping torus, and

$$\partial\ :\ L^{n+2}(\Omega^{-1}A[z,z^{-1}]) \longrightarrow L\mathrm{Aut}^n(A)\ ;\ \Omega^{-1}(C,\phi) \longrightarrow (\partial C,\partial\phi,\zeta)$$

for any $\Omega^{-1}A[z,z^{-1}]$-Poincaré f.g. free $(n+2)$-dimensional symmetric complex (C,ϕ) over A, with $\partial C=\mathcal{C}(\phi_0:C^{n+2-*}\longrightarrow C)_{*+1}$, $\zeta:x\longrightarrow zx$.

□

§3. Bordism of automorphisms of manifolds

Let X be a connected space with universal cover $\widetilde{X}$.

The *symmetric signature* map

$$\sigma^*\ :\ \Omega_n(X) \longrightarrow L^n(\mathbb{Z}[\pi_1(X)])\ ;\ (M\longrightarrow X) \longrightarrow \sigma^*(M)\ =\ (C(\widetilde{M}),\phi)$$

is defined as in Ranicki [9], with ϕ the symmetric Poincaré duality structure on the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{M})$ of the pullback cover $\widetilde{M}$ of the oriented n-dimensional manifold M, so that

$$\phi_0\ =\ [M]\cap -\ :\ C(\widetilde{M})^{n-*} \longrightarrow C(\widetilde{M})\ .$$

There are corresponding symmetric signature maps on the automorphism bordism groups $\Delta_*(X), AB_*(X)$ defined in the Introduction.

The *symmetric signature* map on $\Delta_*(X)$ is defined by

$$\begin{aligned}\sigma^* \ : \Delta_n(X) &\longrightarrow L\mathrm{Aut}^n(\mathbb{Z}[\pi_1(X)]) \ ; \\ (M \longrightarrow X, f: M \longrightarrow M) &\longrightarrow (C(\widetilde{M}), \phi, \widetilde{f}: C(\widetilde{M}) \longrightarrow C(\widetilde{M})) \ .\end{aligned}$$

(For $\pi_1(X) = \{1\}$, $n = 2k$ this is the automorphism Witt invariant of López de Medrano [6]).

The *symmetric signature* map on $AB_*(X)$ is defined by

$$\begin{aligned}\sigma^* \ : AB_n(X) &\longrightarrow L^{n+2}(\Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]) \ ; \\ ((W,\partial W) \longrightarrow X \times S^1, \partial W = T(f: M \longrightarrow M)) &\longrightarrow \Omega^{-1}(C(\widetilde{W}, \partial\widetilde{W}), \delta\phi/\phi) \ .\end{aligned}$$

Here, $(C(\widetilde{W}, \partial\widetilde{W}), \delta\phi/\phi)$ is the $(n+2)$-dimensional symmetric complex over $\mathbb{Z}[\pi_1(X)][z,z^{-1}]$ obtained by collapsing the boundary in the $(n+2)$-dimensional symmetric Poincaré pair $(C(\partial\widetilde{W}) \longrightarrow C(\widetilde{W}), (\delta\phi, \phi))$ over $\mathbb{Z}[\pi_1(X)][z,z^{-1}]$ associated to the $(n+2)$-dimensional manifold with boundary $(W, \partial W = T(f: M \longrightarrow M))$. The induced $(n+2)$-dimensional symmetric complex over $\Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]$ is Poincaré since $H_*(\partial W; \Omega^{-1}\mathbb{Z}[\pi_1(X)][z,z^{-1}]) = 0$.

Theorem 3.1 *The symmetric signature maps define a natural transformation of exact sequences*

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \Omega_{n+2}(X\times S^1) & \longrightarrow & AB_n(X) & \longrightarrow & \Delta_n(X) & \xrightarrow{T} & \Omega_{n+1}(X\times S^1) & \longrightarrow & \dots \\ & & \Big\downarrow \sigma^* & & \Big\downarrow \sigma^* & & \Big\downarrow \sigma^* & & \Big\downarrow \sigma^* & & \\ \dots & \longrightarrow & L^{n+2}(\Lambda) & \xrightarrow{i} & L^{n+2}(\Omega^{-1}\Lambda) & \xrightarrow{\partial} & L\mathrm{Aut}^n(A) & \xrightarrow{T} & L^{n+1}(\Lambda) & \longrightarrow & \dots \end{array}$$

with

$$A \ = \ \mathbb{Z}[\pi_1(X)] \quad , \quad \Lambda \ = \ \mathbb{Z}[\pi_1(X\times S^1)] \ = \ A[z,z^{-1}] \ .$$

If $\pi_1(X)$ is finitely presented the symmetric signature maps

$$\sigma^* \ : \ AB_n(X) \longrightarrow L^{n+2}(\Omega^{-1}\Lambda) \quad (n \geq 5)$$

are isomorphisms, and the automorphism bordism groups $\Delta_*(X)$ fit into an exact sequence

$$\dots \longrightarrow L^{n+2}(\Lambda) \longrightarrow \Delta_n(X) \longrightarrow L\mathrm{Aut}^n(A) \oplus \Omega_{n+1}(X\times S^1) \longrightarrow L^{n+1}(\Lambda) \longrightarrow \dots \ .$$

Proof. The unit

$$u \ = \ (1-z)^{-1} \in \Omega^{-1}\Lambda$$

is such that $u+\overline{u}=1$, so there is no difference between the quadratic and symmetric L-groups of $\Omega^{-1}\Lambda$

$$L_*(\Omega^{-1}\Lambda) \;=\; L^*(\Omega^{-1}\Lambda)\ .$$

Let $(V,\partial V)$ be an $(n+1)$-dimensional manifold with boundary with a π_1-isomorphism reference map $(V,\partial V)\longrightarrow X$ such that

$$\pi_1(V) \;=\; \pi_1(\partial V) \;=\; \pi_1(X)\ .$$

By the realization theorems of Wall [16], Cappell and Shaneson [2] and Vogel [15] every element

$$x\in L^{n+2}(\Omega^{-1}\Lambda) \;=\; \Gamma_{n+2}(\Lambda\longrightarrow\Omega^{-1}\Lambda)$$

is the $\Omega^{-1}\Lambda$-homology surgery obstruction $x=\sigma_*(F,B)$ of a normal map of $(n+2)$-dimensional manifolds with boundary

$$(F,B)\ :\ (W,\partial W)\longrightarrow (V,\partial V)\times S^1$$

with

$$\pi_1(W) \;=\; \pi_1(\partial W) \;=\; \pi_1(X)\times\mathbb{Z}$$

and such that $\partial F:\partial W\longrightarrow\partial V\times S^1$ is a $\Omega^{-1}\Lambda$-homology equivalence, i.e. such that the pullback infinite cyclic cover of ∂W

$$\overline{\partial W} \;=\; (\partial F)^*(\partial V\times\mathbb{R})$$

is finitely dominated. Use the direct sum decomposition given by 1.10

$$K_1(\Omega^{-1}\Lambda) \;=\; K_1(\Lambda)\oplus \mathrm{Aut}_0(A)$$

to express the $\Omega^{-1}\Lambda$-coefficient Whitehead torsion of ∂F as

$$\begin{aligned}\tau(\partial F;\Omega^{-1}\Lambda) \;&=\; (-)^n\tau(\partial F;\Omega^{-1}\Lambda)^*\\ &=\; (\Phi(\partial W),[\overline{\partial W},\zeta])-(\Phi(\partial V\times S^1),[\partial V\times\mathbb{R},1\times\zeta_{\mathbb{R}}])\\ &=\; (\Phi(\partial W),[\overline{\partial W},\zeta])-(0,[\partial V,1])\\ &\quad\in K_1(\Omega^{-1}\Lambda) \;=\; K_1(\Lambda)\oplus\mathrm{Aut}_0(A)\ ,\end{aligned}$$

with $\zeta:\overline{\partial W}\longrightarrow\overline{\partial W}$ a generating covering translation. Moreover, for every $\mu\in K_1(\Omega^{-1}\Lambda)$ it is possible to vary (F,B) by an $\Omega^{-1}\Lambda$-coefficient homology cobordism with torsion μ, changing $\tau(\partial F;\Omega^{-1}\Lambda)$ by $\mu+(-)^n\mu^*$. The duality involution on $K_1(\Omega^{-1}\Lambda)$ defined by the conjugate transposition of matrices $(a_{ij})\longrightarrow(\overline{a}_{ji})$ is of the form

$$* \;=\; \begin{pmatrix} * & \beta * \\ 0 & * \end{pmatrix}\ :\ K_1(\Omega^{-1}\Lambda) \;=\; K_1(\Lambda)\oplus\mathrm{Aut}_0(A)\longrightarrow K_1(\Omega^{-1}\Lambda) \;=\; K_1(\Lambda)\oplus\mathrm{Aut}_0(A)$$

with

$$* \ : \ \mathrm{Aut}_0(A) \longrightarrow \mathrm{Aut}_0(A) \ ; \ [P,h] \longrightarrow [P^*,(h^*)^{-1}] \ ,$$

$$\beta \ : \ \mathrm{Aut}_0(A) \longrightarrow K_1(\Lambda) \ ; \ [P,h] \longrightarrow \tau(-zh : P[z,z^{-1}] \longrightarrow P[z,z^{-1}]) \ ,$$

$$\beta * \ = \ - * \beta \ : \ \mathrm{Aut}_0(A) \longrightarrow K_1(A) \ .$$

Now β maps the automorphism class group $\mathrm{Aut}_0(A)$ onto the direct summand $K_1(A) \oplus K_0(A)$ in the Bass decomposition

$$K_1(\Lambda) \ = \ K_1(A) \oplus K_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \ .$$

The duality involution on $K_1(\Lambda)$ interchanges the two $\widetilde{\mathrm{Nil}}_0(A)$-summands, so that every self-dual element $\tau = \pm\tau^* \in K_1(\Omega^{-1}\Lambda)$ can be expressed as

$$\tau \ = \ (\mu_1,\mu_2) \pm (\mu_1,\mu_2)^* + (0,\mu_3) \in K_1(\Omega^{-1}\Lambda) \ = \ K_1(\Lambda) \oplus \mathrm{Aut}_0(A)$$

for some $\mu_1 \in K_1(\Lambda)$, $\mu_2 \in \mathrm{Aut}_0(A)$, $\mu_3 = \pm\mu_3^* \in \ker(\beta)$. Applying this to

$$\tau \ = \ \tau(\partial F;\Omega^{-1}\Lambda) \in K_1(\Omega^{-1}\Lambda)$$

shows that every element $x \in L_{n+2}(\Omega^{-1}\Lambda)$ is realized as the $\Omega^{-1}\Lambda$-coefficient surgery obstruction $\sigma_*(F,B)$ of a normal map $(F,B) : (W,\partial W) \longrightarrow (V,\partial V) \times S^1$ with fibering obstruction

$$\Phi(\partial W) \ = \ 0 \in Wh(\pi_1(X) \times \mathbb{Z}) \ ,$$

so that $\partial W = T(f)$ is the mapping torus of an automorphism $f : M \longrightarrow M$ of a closed n-dimensional manifold M. The surgery obstruction is the difference of the symmetric signatures

$$\sigma_*(F,B) \ = \ \sigma^*(W,\partial W) - \sigma^*(V \times S^1, \partial V \times S^1) \in L_{n+2}(\Omega^{-1}\Lambda) \ = \ L^{n+2}(\Omega^{-1}\Lambda) \ .$$

The construction defines an isomorphism

$$L^{n+2}(\Omega^{-1}\Lambda) \longrightarrow AB_n(X) \ ;$$

$$x \ = \ \sigma_*(F,B) \longrightarrow (W, \partial W = T(f)) - (V \times S^1, \partial V \times S^1 = T(1 : \partial V \longrightarrow \partial V))$$

inverse to the symmetric signature map $\sigma^* : AB_n(X) \longrightarrow L^{n+2}(\Omega^{-1}\Lambda)$.

□

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