ON THE K-THEORY OF FINITE ALGEBRAS OVER WITT VECTORS OF PERFECT FIELDS

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Contents

Introduction

The purpose of this paper is two fold. Firstly, it gives a thorough introduction to the topological cyclic homology theory, which to a ring R associates a spectrum $\mathrm{TC}(R)$. We determine $\mathrm{TC}(k)$ and $\mathrm{TC}(k[\epsilon])$ where k is a perfect field of positive characteristic and $k[\epsilon]$ its dual numbers, and sets the stage for further calculations. Secondly, we show, as conjectured in [M], that the cyclotomic trace from Quillen's $K(R)$ to $\mathrm{TC}(R)$ becomes a homotopy equivalence after p-adic completion when R is a finite algebra over the Witt vectors $W(k)$ of a perfect field of characteristic $p > 0$. This involves a recent relative result of R. McCarthy, stated in theorem A below, the calculation of $\mathrm{TC}(k)$ and Quillen's theorem about $K(k)$, and continuity results for $\mathrm{TC}(R)$ and $K(R)$, the latter basically due to Suslin and coworkers. In particular, we obtain a calculation of the *tangent space* of $K(k)$, *i.e.* the homotopy fiber of the map from $K(k[\epsilon])$ to $K(k)$ given by $\epsilon \mapsto 0$.

The functor $\mathrm{TC}(R)$, and more generally $\mathrm{TC}(L)$ where L is a 'functor with smash product', for short *FSP*, was initially introduced in [BHM], but its more formal properties were maybe not so well elucidated in that paper. The present account focuses upon the concept of *cyclotomic spectra*. These are a special class of equivariant S^1-spectra for which the associated fixed point spectrum (suitably defined) with respect to finite subgroups of the circle are equivalent to the original spectrum. The defining extra property is analogous to the property shared by free loop spaces $\mathcal{L}X$, namely that the fixed set $(\mathcal{L}X)^C$ is homeomorphic to $\mathcal{L}X$, for C finite. Indeed, the S^1-equivariant suspension spectrum of the free loop space is a cyclotomic spectrum. More generally, Bökstedts topological Hochschild homology spectrum $\mathrm{THH}(L)$ is always a cyclotomic spectrum, so they are in rich supply. The construction $\mathrm{TC}(-)$, given in paragraph 3, can be applied to any cyclotomic spectrum, and applied to $\mathrm{THH}(R)$, or more generally to $\mathrm{THH}(L)$, gives $\mathrm{TC}(R)$ or $\mathrm{TC}(L)$. If R (or L) is commutative then $\mathrm{TC}(R)$ (or $\mathrm{TC}(L)$) is a homotopy commutative ring spectrum. It is (-2)-connected in the sense that $\pi_i \mathrm{TC}(R) = 0$ for $i \leq -2$; in general $\pi_{-1} \mathrm{TC}(R) \neq 0$.

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Theorem A. (McCarthy) *Let $R \to \bar{R}$ be a surjection of rings whose kernel is nilpotent. Then the square*

$$\begin{array}{ccc} K(R) & \xrightarrow{\text{trc}} & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K(\bar{R}) & \xrightarrow{\text{trc}} & \mathrm{TC}(\bar{R}) \end{array}$$

becomes homotopy cartesian after profinite completion.

The proof of this result is unfortunately indirect. It is based upon Goodwillie's *calculus of functors* and a reduction of his to the case where R is a split extension of $\bar{R}$ by a square zero ideal.

Let k be a perfect field of characteristic $p > 0$ and let $F\colon W(k) \to W(k)$ be the Frobenius homomorphism of its (p-typical) Witt vectors. The kernel of $F - 1$ is the Witt vectors of $\mathbb{F}_p = k^{\langle F \rangle}$, *i.e.* $\ker(F-1) = \mathbb{Z}_p$. If k is finite then $\mathrm{coker}(F-1) = \mathbb{Z}_p$; it vanishes if k is algebraicly closed, but can be a large group in general. In §4.5 below we calculate $\mathrm{TC}(k)$ to be

Theorem B. *Topological cyclic homology of a perfect field k of positive characteristic is the generalized Eilenberg-MacLane spectrum*

$$\mathrm{TC}(k) = H\mathbb{Z}_p \wedge \Sigma^{-1} H(\mathrm{coker}(F-1)).$$

It follows that the connective cover $\mathrm{TC}(k)[0,\infty)$ is $H(\mathbb{Z}_p, 0)$; this is also the value of $K(k)^\wedge_p$ by [K], [Q], and the cyclotomic trace $\mathrm{trc}\colon K(k)^\wedge_p \to \mathrm{TC}(k)[0,\infty)$ is an equivalence. For a $\mathbb{Z}_p$-algebra we define continuous versions of $K(R)$ and $\mathrm{TC}(R)$ to be

$$K^{\mathrm{top}}(R) = \underset{\longleftarrow}{\mathrm{holim}}\, K(R/p^i), \qquad \mathrm{TC}^{\mathrm{top}}(R) = \underset{\longleftarrow}{\mathrm{holim}}\, \mathrm{TC}(R/p^i),$$

cf. [W].

Theorem C. *Suppose that A is a $W(k)$-algebra which is finitely generated as a $W(k)$-module. Then*

(i) $$K^{\mathrm{top}}(A)^\wedge_p \simeq \mathrm{TC}^{\mathrm{top}}(A)^\wedge_p[0,\infty)$$

(ii) $$\mathrm{TC}^{\mathrm{top}}(A)^\wedge_p \simeq \mathrm{TC}(A)^\wedge_p,$$

(iii) $$K^{\mathrm{top}}(A)^\wedge_p \simeq K(A)^\wedge_p.$$

The first part of this result follows from the two previous theorems. The second part is proved in §5 below. The final third part is a recast of results from [SuY]. This uses quite different methods from the rest of the paper, and is proved in Appendix B. In conclusion we have

Theorem D. *For the rings of theorem C, $K(A)^\wedge_p \simeq \mathrm{TC}(A)^\wedge_p[0,\infty)$.*

It is fair to remark that $\mathrm{TC}(R)^\wedge_p$ is of course not very easy to evaluate. It does however lend itself to analysis by the well-tried methods of algebraic topology more readily than $K(R)$ does. This is demonstrated here for $R = k[\epsilon]$ and in [BM], [BM1] when R is the Witt vectors of a finite field. One might hope in the future to get a thorough grasp of $\mathrm{TC}(A)$ for the rings of theorem C, and maybe even a closed formula when A is a k-algebra.

We next describe the *tangent space* of algebraic K-theory,

$$\tilde{K}(k[\epsilon]) = \mathrm{hofiber}(K(k[\epsilon]) \to K(k)), \qquad \epsilon \mapsto 0,$$

when k is a perfect field of characteristic $p > 0$. We have $\tilde{K}_*(k[\epsilon]) \otimes \mathbb{Q} \cong \tilde{\mathrm{HC}}_{*-1}(k[\epsilon]) \otimes \mathbb{Q} = 0$ by a theorem of Goodwillie, [G2], and on the other hand, by theorem A, $\tilde{K}(k[\epsilon])^\wedge \simeq \tilde{\mathrm{TC}}(k[\epsilon])^\wedge$. Since the latter turns out to be rationally trivial we get in turn

$$\tilde{K}(k[\epsilon]) \simeq \tilde{\mathrm{TC}}(k[\epsilon]).$$

We evaluate the right hand side in paragraph 7 below. The result is best stated in terms of the big Witt vectors. Let $\mathbf{W}(R)$ denote the multiplicative group of the power series with constant term 1, and let $\mathbf{W}_n(R)$ be the quotient of big Witt vectors of length n, *i.e.*

$$\mathbf{W}_n(R) = (1 + XR[\![X]\!])^\times / (1 + X^{n+1}R[\![X]\!])^\times.$$

Sign change $X \mapsto -X$ induces an involution on $\mathbf{W}_n(R)$ whose (-1)-eigenspace will be denoted $\mathbf{W}_n(R)^{\langle -1 \rangle}$. If we write $\tilde{\mathrm{TC}}_n(R) = \pi_n \tilde{\mathrm{TC}}(R)$ then we have from §8.2::

Theorem E. *For the dual numbers* $k[\epsilon]$, $\tilde{\mathrm{TC}}(k[\epsilon])$ *is a generalized Eilenberg-MacLane spectrum whose non-zero homotopy groups are*
(i) $p > 2$: $\tilde{\mathrm{TC}}_{2n-1}(k[\epsilon]) = \mathbf{W}_{2n-1}(k)^{\langle -1 \rangle}$.
(ii) $p = 2$: $\tilde{\mathrm{TC}}_{2n-1}(k[\epsilon]) = k^{\oplus n}$.

We remark that for $p > 2$ there is higher torsion in $\mathrm{TC}_{2n-1}(k[\epsilon])$ in general, and that our results are in agreement with the Evens-Friedlander calculation of $K_i(\mathbb{F}_p[\epsilon])$ for $i \leq 3$ and $p \geq 5$, [EF]. Indeed the above theorem gives $\mathrm{TC}_3(\mathbb{F}_p[\epsilon]) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ for $p \neq 3$ and $\mathrm{TC}_3(\mathbb{F}_3[\epsilon]) = \mathbb{Z}/9$. The dichotomy between $p = 2$ and $p > 2$ in theorem D disappears in the Poincaré series when $k = \mathbb{F}$ is a finite field. One has in all cases

$$\sum_{m=1}^{\infty} (-1)^m |\,\mathrm{TC}_m(\mathbb{F}[\epsilon])| t^m = \frac{qt}{qt^2 - 1}, \qquad q = |\mathbb{F}|.$$

Let us finally mention the following general result is proved in §2.3:

Theorem F. *For any commutative ring* A,

$$\pi_0 \,\mathrm{THH}(A)^{C_{p^n}} \cong W_{n+1}(A),$$

the p*-typical Witt vectors of length* $n+1$.

The cyclotomic structure of $\mathrm{THH}(A)$ induces two maps

$$R, F \colon \mathrm{THH}(A)^{C_{p^n}} \to \mathrm{THH}(A)^{C_{p^{n-1}}}.$$

In earlier writings on topological cyclic homology, and in particular in [BHM], R was called Φ and F was called D. The reason for the change of notation is that $\pi_0(R)$ and $\pi_0(F)$ under the identifications of theorem F become the restriction map and Frobenius homomorphism, respectively, from $W_{n+1}(A)$ to $W_n(A)$. Thus the new notation is in agreement with the notation used for Witt vectors.

We say that a spectrum T is connective if $\pi_i(T) = 0$ when $i < 0$. A space will mean a compactly generated topological space which is weakly Hausdorff, *i.e.* the diagonal $X \subset X \times X$ is closed when the product is given the compactly generated topology. We shall use equivalence to mean a map which induces isomorphisms on homotopy groups, and a G-equivalence to be a G-equivariant map which induces an equivalence on H-fixed sets for all closed subgroups $H \subset G$. Unless otherwise stated, G will denote the circle group S^1.

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1. The Topological Hochschild Spectrum

1.1. Throughout this paper G will denote the circle group, C_r or just C the cyclic group of order r and J the quotient G/C. We recall briefly some notions from equivariant stable homotopy theory. The standard reference is [LMS].

A G-prespectrum indexed on a 'complete G-universe' $\mathcal{U}$ is a collection of G-spaces $t(V)$, one for each finite dimensional sub inner product G-space $V \subset \mathcal{U}$, together with a transitive system of G-maps

$$\sigma: t(V) \to \Omega^{W-V} t(W).$$

Here $W - V$ denotes the orthogonal complement of V in W. It is a G-spectrum if the structure maps σ are all homeomorphisms. A map $f: t \to t'$ of G-prespectra consists of G-maps $f(V): t(V) \to t'(V)$ which commute strictly with the structure maps. The category of G-prespectra indexed on $\mathcal{U}$ is denoted $G\mathcal{PU}$ and $G\mathcal{SU}$ denotes the full subcategory of G-spectra. The forgetful functor $l: G\mathcal{SU} \to G\mathcal{PU}$ has a left adjoint L. It is given by the colimit over the structure maps

$$Lt(V) \cong \varinjlim_{W \subset \mathcal{U}} \Omega^{W-V} t(W),$$

provided that each σ is an inclusion, *i.e.* induces a homeomorphism onto its image. This, for instance, is the case when t is *good* as defined in the appendix. We show in lemma A.1 that any G-prespectrum can be replaced by an equivalent good one, and we shall tacitly assume that G-(pre)spectra be good.

Suppose C is a closed subgroup in G with quotient J and $T \in G\mathcal{SU}$. There are two possible notions of an associated fixed point spectrum in $J\mathcal{SU}^C$, in [LMS] denoted T^C and $\Phi^C T$ respectively. Their V'th spaces are

$$T^C(V) = T(V)^C, \qquad \Phi^C T(V) = \varinjlim_{W \subset \mathcal{U}} \Omega^{W^C - V} t(W)^C,$$

and the structure maps are the evident ones. There is a natural map $s_C: T^C \to \Phi^C T$.

Lemma 1.1. *If t is a bounded below G-prespectrum, then $\Phi^C t \simeq_J \Phi^C Lt$.*

Proof. We assume for convenience that $t(W)^C$ is $\dim(W^C) - 1$-connected; the general case is similar. The composition

$$(\Omega^{W^C} t(W))^C \xrightarrow{\sigma^C} (\Omega^{W^C + Z} t(W+Z))^C \xrightarrow{i^*} (\Omega^{W^C + Z^C} t(W+Z))^C$$

is a map in the limit system which defines $\Phi^C t(0)$. Therefore it induces a homeomorphism on colimits over W. The lemma states that σ^C induces a J-equivalence on colimits over W, so it suffices to show that i^* does.

Let H be a subgroup of G containing C. The map induced by i^* on H-fixed points is a fibration with fiber

$$\mathrm{Map}(S^{W^C + Z^C} \wedge (S^{Z - Z^C}/S^0), t(W+Z))^H.$$

By elementary equivariant obstruction theory the connectivity of an equivariant mapping space $A^H = \mathrm{Map}(X, Y)^H$ satisfies

$$\mathrm{conn}(A^H) \geq \min_{K \leq H} (\mathrm{conn}(Y^K) - \dim(X^K)),$$

when X is a G-CW-complex. In the case at hand we have

$$\begin{aligned} (S^{W^C + Z^C} \wedge (S^{Z - Z^C}/S^0))^K &= S^{W^{CK} + Z^{CK}} \wedge (S^{Z^K - Z^{CK}}/S^0), \text{ if } K \not\supset C \\ &= * \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad, \text{ if } K \supset C. \end{aligned}$$

Thus $\dim(X^K) \leq \dim(W^{CK} + Z^K)$ while $\mathrm{conn}(Y^K) = \dim(W^K + Z^K)$. It follows that the difference $\mathrm{conn}(Y^K) - \dim(X^K) \geq \dim(W^K - W^{CK})$ tends to infinity as W runs through the f.d. sub inner product spaces of $\mathcal{U}$. Under our standard assumption, that t is good, this implies that the map induced by i^* on colimits over W is an equivalence on H-fixed points. □

Next we evaluate the homotopy fiber of $s_C: T^C \to \Phi^C T$ in the case where $C = C_p$ has prime order. We recall that the smash product of a G-space X and a G-prespectrum t is the G-prespectrum with V'th space $X \wedge t(V)$ and the obvious structure maps. If T is a G-spectrum we let $X \wedge T$ denote the G-spectrum $L(X \wedge T)$.

Let $j: \mathcal{U}^G \to \mathcal{U}^C$ be the inclusion of the G-trivial universe and let D be a J-spectrum. We call $j^* D$ with its J-action forgotten for the underlying non-equivariant spectrum D.

Proposition 1.1. *Suppose C is a cyclic p-group. Then for any bounded below G-spectrum T there is a cofibration sequence of non-equivariant spectra*

$$T_{hC} \xrightarrow{N} T^C \xrightarrow{s_{C_p}^{C/C_p}} (\Phi^{C_p} T)^{C/C_p}.$$

Here $T_{hC} = EC_+ \wedge_C j^* T$ is the homotopy orbit spectrum.

Proof. Consider the cofibration sequence of G-spaces

$$EG_+ \xrightarrow{\pi} S^0 \to \tilde{E}G,$$

where π maps EG to the non-basepoint in S^0. We can smash with T and obtain a cofibration sequence of G-spectra which in turn induces a cofibration sequence of non-equivariant spectra

$$[EG_+ \wedge T]^C \to T^C \to [\tilde{E}G \wedge T]^C.$$

The identification of the first term goes in two steps. Let $i: \mathcal{U}^C \to \mathcal{U}$ be the inclusion. The forgetful functor $i^*: G\mathcal{SU} \to G\mathcal{SU}^C$ has a left adjoint i_* given by

$$i_* D(V) = \varinjlim_{W \subset U} \Omega^{W-V}(S^{W-W^C} \wedge D(W^C)).$$

Since the functors i^* and $F(X, -)$, the pointed mapping space functor, commute the same hold for their left adjoints i_* and $X \wedge -$. Thus the counit of the adjunction $i_* \dashv i^*$ induces a map

$$e: i_*(EG_+ \wedge i^* T) \to EG_+ \wedge T.$$

On V'th spaces e spells out to the map

$$e(V): \varinjlim_{W \subset U} \Omega^{W-V}(S^{W-W^C} \wedge EG_+ \wedge T(W^C)) \to \varinjlim_{W \subset U} \Omega^{W-V}(EG_+ \wedge T(W))$$

induced by the spectrum structure maps

$$\tilde{\sigma}: S^{W-W^C} \wedge T(W^C) \to T(W).$$

Since T is bounded below, $e(V)$ is a G-equivalence if $\tilde{\sigma}$ is $\dim W + k(W)$-connected and $k(W) \to \infty$ as W runs through the f.d. sub inner product spaces of $\mathcal{U}$, *cf.* [A]. The composition

$$T(W^C) \xrightarrow{\Sigma} \Omega^{W-W^C}(S^{W-W^C} \wedge T(W^C)) \xrightarrow{\tilde{\sigma}_*} \Omega^{W-W^C} T(W)$$

is a homeomorphism and the suspension map is $2\dim(W^C) - 1$-connected. Hence $e(V)$ is a G-equivalence. Finally we have the transfer equivalence

$$\tau: EG_+ \wedge_C i^* T \simeq [i_*(EG_+ \wedge i^* T)]^C$$

of [LMS, p.97]. Combined with e this identifies the first term.

The natural map $s_{C_p} : T^{C_p} \to \Phi^{C_p} T$ may be written as a composition

$$T^{C_p} \to [\tilde{E}G \wedge T]^{C_p} \xrightarrow{\bar{s}_{C_p}} \Phi^{C_p} T,$$

where $\bar{s}_{C_p}(V)$ is the map on colimits induced by the restriction map to C_p-fixed sets

$$(\Omega^{W-V}(\tilde{E}G \wedge T(W)))^{C_p} \to \Omega^{W^{C_p}-V} T(W)^{C_p}.$$

This is a fibration whose fiber is the equivariant mapping space

$$\mathrm{Map}(S^{W-V}/S^{W_{C_p}-V}, \tilde{E}G \wedge T(W))^{C_p}.$$

If we regard W as a C-space, then W^{C_p} is the singular set. Thus $S^{W-V}/S^{W^{C_p}-V}$ is a based free C-CW-complex, and since $\tilde{E}G$ is non-equivariantly contractible it follows that $\bar{s}_{C_p}$ is a C/C_p-equivalence. □

Example. It is illuminating to consider the case of a suspension G-spectrum $\Sigma_G^\infty X$. We let $E_G H$ denote a universal H-free G-space, that is $E_G H^K \simeq *$ when $H \cap K = 1$ and $E_G H^K = \varnothing$ when $H \cap K \neq 1$. Then on the one hand we have the tom Dieck-Segal splitting

$$(\Sigma_G^\infty X)^C \simeq_J \bigvee_{H \leq C} \Sigma_J^\infty (E_{G/H}(C/H)_+ \wedge_{C/H} X^H),$$

[tD1], and on the other hand lemma 1.1 shows that $\Phi^C(\Sigma_G^\infty X) \simeq_J \Sigma_J^\infty X^C$. Moreover the natural map $(\Sigma_G^\infty X)^C \to \Phi^C(\Sigma_G^\infty X)$ is simply the projection onto the summand $H = C$.

1.2. Suppose C is finite of order r. Then the r'th root $\rho_C\colon G \to J$ is an isomorphism of groups, and a J-space X may be viewed as a G-space $\rho_C^* X$ through ρ_C. We also use ρ_C to view J-spectra as G-spectra.

When D is a J-spectrum indexed on $\mathcal{U}^C$, then the G-spaces

$$\rho_C^* D((\rho_C^{-1})^*(V)),$$

for $V \subset \rho_C^* \mathcal{U}^C$, form a G-spectrum indexed on $\rho_C^* \mathcal{U}^C$. From now on we fix our universe. Let $\mathbb{C}(n) = \mathbb{C}$ with G acting through the n'th power map, $g \cdot z = g^n z$. Then we set

$$\mathcal{U} = \bigoplus_{n \in \mathbb{Z}, \alpha \in \mathbb{N}} \mathbb{C}(n)_\alpha,$$

and note that

$$\rho_C^* \mathcal{U}^C = \bigoplus_{n \in r\mathbb{Z}, \alpha \in \mathbb{N}} \mathbb{C}(n/r)_\alpha.$$

Identifying $\mathbb{Z}$ and $r\mathbb{Z}$ in the usual way we get $U = \rho_C^* \mathcal{U}^C$. Thus a J-spectrum D indexed on $\mathcal{U}^C$ determines a G-spectrum indexed on $\mathcal{U}$ and we denote this $\rho_C^\# D$.

Definition 1.2. A *cyclotomic spectrum* is a G-spectrum indexed on $\mathcal{U}$ together with a G-equivalence

$$r_C\colon \rho_C^\# \Phi^C T \to T$$

for every finite $C \subset G$, such that for any pair of finite subgroups the diagram

$$\begin{array}{ccc} \rho_{C_r}^\# \Phi^{C_r} \rho_{C_s}^\# \Phi^{C_s} T & = & \rho_{C_{rs}}^\# \Phi^{C_{rs}} T \\ \downarrow{\scriptstyle \rho_{C_r}^\# \Phi^{C_r} r_{C_s}} & & \downarrow{\scriptstyle r_{C_{rs}}} \\ \rho_{C_r}^\# \Phi^{C_r} T & \xrightarrow{r_{C_r}} & T \end{array}$$

commutes.

We spell out the definition in the following

Lemma 1.2. *A G-spectrum T, indexed on $\mathcal{U}$, is a cyclotomic spectrum if and only if for each index space $V \subset \mathcal{U}$ and each finite subgroup $C \subset G$ there is a G-map*

$$r_C(V)\colon \rho_C^* T(V)^C \to T(\rho_C^* V^C)$$

subject to the following conditions
i) For each pair $V \subset W \subset \mathcal{U}$ the diagram

$$\begin{array}{ccc} S^{\rho_C^*(W-V)^C} \wedge \rho_C^* T(V)^C & \xrightarrow{1 \wedge r_C(V)} & S^{\rho_C^*(W-V)^C} \wedge T(\rho_C^* V^C) \\ \downarrow{\scriptstyle \rho_C^*(\tilde\sigma)^C} & & \downarrow{\scriptstyle \tilde\sigma} \\ \rho_C^* T(W)^C & \xrightarrow{r_C(W)} & T(\rho_C^* W^C) \end{array}$$

commutes.
ii) For each pair of finite subgroups the diagram

$$\begin{array}{ccc} \rho_{C_{rs}}^* T(V)^{C_{rs}} & \xrightarrow{\rho_{C_s}^*(r_{C_r}(V))^{C_s}} & \rho_{C_s}^* T(\rho_{C_r}^* V^{C_r})^{C_s} \\ \downarrow{\scriptstyle r_{C_{rs}}(V)} & & \downarrow{\scriptstyle r_{C_s}(\rho_{C_r}^* V^{C_r})} \\ T(\rho_{C_{rs}}^* V^{C_{rs}}) & = & T(\rho_{C_s}^*(\rho_{C_r}^* V^{C_r})^{C_s}) \end{array}$$

commutes

iii) For any $V \subset \mathcal{U}$ the induced map on colimits

$$\varinjlim_{W\subset\mathcal{U}} \Omega^{\rho_C^* W^C - V}\rho_C^* T(W)^C \to \varinjlim_{W\subset\mathcal{U}} \Omega^{\rho_C^* W^C - V} T(\rho_C^* W^C)$$

is a G-equivalence. □

In this formulation we could as well take T to be a G-prespectrum t. This gives us the notion of a *cyclotomic prespectrum*. We define a map of cyclotomic (pre)spectra to be a map of G-prespectra which strictly commutes with the φ-maps.

Example. The free loopspace $\mathcal{L}(X)$ is the space of unbased maps from S^1 to X. Rotation of the loops defines a G-action on $\mathcal{L}(X)$. Suppose C is a subgroup of G of order r. Then there is an equivariant homeomorphism $\Delta_C\colon \mathcal{L}(X) \to \rho_C^*\mathcal{L}(X)^C$ given by the r'th power map of the circle S^1. We can use this to give the suspension prespectrum of $\mathcal{L}(X)$ the structure of a cyclotomic prespectrum. Indeed we define

$$r_C(V)\colon \rho_C^*(S^V \wedge \mathcal{L}(X)_+)^C = S^{\rho_C^* V^C} \wedge \rho_C^*\mathcal{L}(X)_+^C \xrightarrow{1\wedge\Delta_C^{-1}} S^{\rho_C^* V^C} \wedge \mathcal{L}(X)_+$$

and i), ii) and iii) in the lemma/definition are readily verified.

The maps s_{C_r} and r_{C_r} give rise a map of G-spectra

$$\rho_{C_{rs}}^{\#} T^{C_{rs}} = \rho_{C_s}^{\#}(\rho_{C_r}^{\#} T^{C_r})^{C_s} \to \rho_{C_s}^{\#}(\rho_{C_r}^{\#}\Phi^{C_r}T)^{C_s} \to \rho_{C_s}^{\#} T^{C_s}$$

and hence a map

(1.2.1) $$R_r\colon T^{C_{rs}} \to T^{C_s}$$

of the underlying non-equivariant spectra, which will play a fundamental role in the following. We call it the *r'th restriction* map.

Let $Z \subset \mathcal{U}$ be a representation. Then, slightly more general, we let T_Z denote the smash product G-spectrum $T \wedge S^Z$. The cyclotomic structure maps give a G-equivalence

(1.2.2) $$r_{C,Z}\colon \rho_C^{\#}\Phi^C T_Z \to T_{\rho_C^* Z^C}.$$

Indeed, for each $V \subset \mathcal{U}$ we have a G-map

$$r_{C,Z}(V)\colon \rho_C^*(S^Z \wedge T(V))^C \xrightarrow{1\wedge r_C(V)} S^{\rho_C^* Z^C} \wedge T(\rho_C^* Z^C),$$

and iii) in the lemma shows that the induced map on colimits over V is a G-equivalence. We note that $T_Z(V-Z) \simeq_G T(V)$. As above we get a map of non-equivariant spectra

(1.2.3) $$R_{r,Z}\colon T_Z^{C_{rs}} \to T^{C_s}_{\rho_{C_r}^* Z^{C_r}}.$$

We can restate proposition 1.1 for cyclotomic spectra as

Theorem 1.2. *For any bounded below cyclotomic spectrum T and any $Z \subset \mathcal{U}$ there is a cofibration sequence of non-equivariant spectra*

$$(T_Z)_{hC_{p^n}} \xrightarrow{N} T_Z^{C_{p^n}} \xrightarrow{R_{p,Z}} T^{C_{p^{n-1}}}_{\rho_{C_p}^* Z^{C_p}}.$$ □

Remark. The spectrification T of a cyclotomic prespectrum t is a cyclotomic spectrum. To see this we define a map $r_C(V)\colon \rho_C^* T(V)^C \to T(\rho_C^* V^C)$ as the map on colimits over $W \subset \mathcal{U}$ induced by the composite

$$\rho_C^*(\Omega^{W-V} t(W))^C \xrightarrow{i^*} \rho_C^*(\Omega^{W^C - V^C} t(W)^C) = \Omega^{\rho_C^*(W^C - V^C)}\rho_C^* t(W)^C$$
$$\xrightarrow{r_C(W)_*} \Omega^{\rho_C^*(W^C - V^C)} t(\rho_C^* W^C).$$

Then i) and ii) are easily checked, and iii) follows from lemma 1.1.

1.3. Suppose T is a cyclotomic spectrum, then so is $\rho_C^{\#}\Phi^C T$ but in general $\rho_C^{\#}T^C$ is not. We proceed to explain the situation. First we recall the notion of a family of subgroups.

A collection $\mathcal{F}$ of subgroups of G is called a family if it is closed under passage to subgroups. A map $f\colon X \to Y$ of G-spaces (G-spectra) is called an $\mathcal{F}$-equivalence if the induced map f^H on H-fixed points is an equivalence for all $H \in \mathcal{F}$, or equivalently, if $f \wedge E\mathcal{F}_+$ is a G-equivalence. Here $E\mathcal{F}$ is the join of the free contractible G/H-spaces $E(G/H)$ for $H \in \mathcal{F}$. It is the terminal object among G-spaces with orbit types G/H, $H \in \mathcal{F}$, and G-homotopy classes of maps; *cf.* [tD]. We let $\mathcal{F}_p$ denote the family of finite p-subgroups in G.

Definition 1.3. ([BM]) A *p-cyclotomic spectrum* is a G-spectrum indexed on $\mathcal{U}$ together with an $\mathcal{F}_p$-equivalence $r_p\colon \rho_{C_p}^{\#}\Phi^{C_p}T \to T$.

Of course a cyclotomic spectrum is p-cyclotomic for every prime p. Also note that for a p-cyclotomic spectrum, theorem 1.2 holds for the prime p.

Proposition 1.3. *Let T be a cyclotomic spectrum. Then $\rho_C^{\#}T^C$ is a p-cyclotomic spectrum if p does not divide the order of C.*

Proof. Let $S = \rho_C^{\#}T^C$, then we want to define a G-map

$$r_p(V)\colon \rho_{C_p}^{*}S(V)^{C_p} \to S(\rho_{C_p}^{*}V^{C_p}),$$

that is, a G-map

$$\rho_{C_p}^{*}(\rho_C^{*}T((\rho_C^{-1})^{*}V)^{C})^{C_p} \to \rho_C^{*}T((\rho_C^{-1})^{*}(\rho_{C_p}^{*}V^{C_p}))^{C}.$$

We have a G-map

$$\begin{aligned}\rho_{C_p}^{*}(\rho_C^{*}T((\rho_C^{-1})^{*}V)^{C})^{C_p} &= \rho_C^{*}(\rho_{C_p}^{*}T((\rho_C^{-1})^{*}V)^{C_p})^{C}\\ &\xrightarrow{\rho_C^{*}(r_{C_p})^{C}} \rho_C^{*}T(\rho_{C_p}^{*}((\rho_C^{-1})^{*}V)^{C_p})^{C}\end{aligned}$$

so we need the following equality to hold

$$\rho_{C_p}^{*}((\rho_C^{-1})^{*}V)^{C_p} = (\rho_C^{-1})^{*}(\rho_{C_p}^{*}V^{C_p}).$$

Now $((\rho_C^{-1})^{*}V)^{C_p}$ and $(\rho_C^{-1})^{*}(V^{C_p})$ are equal when $(p, |C|) = 1$. ☐

1.4. In this section we define the topological Hochschild spectrum. It is a cyclotomic spectrum whose zero'th space is naturally C-equivalent to Bökstedt's topological Hochschild space $\mathrm{THH}(L)$.

We briefly recall the definition of $\mathrm{THH}(L)$ and refer to [B], [BHM] and [PW] for details. Let I be the category whose objects are the finite cardinals $\mathbf{n} = \{1, 2, \ldots, n\}$ and whose morphisms are the injective maps, and let L be a functor with smash product. Then $\mathrm{THH}(L)_\bullet$ is the cyclic space with k-simplices equal to the homotopy colimit

$$\operatorname*{holim}_{\overrightarrow{I^{k+1}}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, L(S^{i_0}) \wedge \ldots \wedge L(S^{i_k}))$$

and with Hochschild-type structure maps. The realization $\mathrm{THH}(L)$ is a G-space. More generally we let $\mathrm{THH}(L; X)_\bullet$ be the cyclic space with k-simplices

$$\operatorname*{holim}_{\overrightarrow{I^{k+1}}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, L(S^{i_0}) \wedge \ldots \wedge L(S^{i_k}) \wedge X),$$

where X acts as a dummy for the cyclic structure maps, *cf.* (6.1.1) below. If X has a G-action then $\mathrm{THH}(L; X)$ becomes a $G \times G$-space, and hence a G-space via the diagonal $\Delta\colon G \to G \times G$.

We define a G-prespectrum $t(L)$ whose 0'th space is $\mathrm{THH}(L)$. Let V be a f.d. sub inner product space of some G-universe $\mathcal{U}$, and let S^V be the one-point compactification. Then

$$t(L)(V) = \mathrm{THH}(L; S^V)$$

and the obvious maps

$$\sigma: t(L)(V) \to \Omega^{W-V} t(L)(W)$$

are G-equivariant and form a transitive system. Finally we let $T(L)$ be the associated G-spectrum of the thickened G-prespectrum $t^\tau(L)$, that is

$$T(L)(V) = \varinjlim_{W \subset U} \Omega^{W-V} t^\tau(L)(W).$$

In order to define the cyclotomic structure maps we need the edgewise subdivision of [BHM] §1.

The realization of a cyclic space becomes a G-space upon identifying G with $\mathbb{R}/\mathbb{Z}$, and hence $C = C_r$ may be identified with $r^{-1}\mathbb{Z}/\mathbb{Z}$. Edgewise subdivision associates to a cyclic space $Z_\bullet$ a simplicial C-space $\operatorname{sd}_C Z_\bullet$. It has k-simplices $\operatorname{sd}_C Z_k = Z_{r(k+1)-1}$ and the generator $r^{-1} + \mathbb{Z}$ of C acts as τ^{k+1}. Moreover, there is a natural homeomorphism

$$D: |\operatorname{sd}_C Z_\bullet| \to |Z_\bullet|,$$

and an $\mathbb{R}/r\mathbb{Z}$-action on $|\operatorname{sd}_C Z_\bullet|$ which extends the simplicial C-action. The map D is G-equivariant when $\mathbb{R}/r\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z}$ through division by r.

We consider the case of $\mathrm{THH}(L;X)_\bullet$. Let us write $G_k^X(i_0, \ldots, i_k)$ for the pointed mapping space

$$F(S^{i_0} \wedge \ldots \wedge S^{i_k}, L(S^{i_0}) \wedge \ldots \wedge L(S^{i_k}) \wedge X).$$

Then the k-simplices of the edgewise subdivision is the homotopy colimit

$$\operatorname{sd}_C \mathrm{THH}(L;X)_k = \operatorname*{\underrightarrow{holim}}_{I^{r(k+1)}} G^X_{r(k+1)-1}.$$

Suppose that X_α is a diagram of C-spaces. Then the homotopy colimit is again a C-space and its C-fixed set is the homotopy colimit of the C-fixed sets X_α^C. However the C-action on $\operatorname{sd}_C \mathrm{THH}(L;X)_k$ does not arise in this way. We consider instead the composite functor $G^X_{r(k+1)-1} \circ \Delta_r$ where $\Delta_r: I^{k+1} \to (I^{k+1})^r$ is the diagonal functor. It has a C-action and the canonical map of homotopy colimits

$$b_k: \operatorname*{\underrightarrow{holim}}_{I^{k+1}} G^X_{r(k+1)-1} \circ \Delta_r \to \operatorname*{\underrightarrow{holim}}_{I^{r(k+1)}} G^X_{r(k+1)-1}$$

is a C-equivariant inclusion and induces a homeomorphism of C-fixed sets. Let R be the regular representation $\mathbb{R}C$ and let iR denote the i-fold direct sum. Then

$$G^X_{r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k) \cong F(S^{i_0 R} \wedge \ldots \wedge S^{i_k R}, L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k})^{\wedge r} \wedge X),$$

where C acts by cyclic permutation on $L(S^i)^{\wedge r}$ and by conjugation on the mapping space. This ends our discussion of edgewise subdivision.

Lemma 1.4. *Let H be a compact Lie group and let $Y_\bullet$ be a simplicial H-space such that Y_k^K is $n(K)$-connected for all k, $n(K) \geq 0$. Suppose X is a based H-CW-complex with finitely many orbit types, and such that $\dim X^K \leq n(K)$ for all $K \leq H$. If $Y_\bullet^K$ is proper in the sense of* [May] *for the occuring orbit types then the natural map*

$$\gamma: |F(X, Y_\bullet)| \to F(X, |Y_\bullet|)$$

is an H-equivalence.

Proof. We prove that γ^H is an equivalence by induction over the H-cells in X. Let X_β be obtained from X_α by adjoining an H-cell $H/K_+ \wedge S^n$. Then we have a simplicial Hurewicz fibration

$$F(S^n, Y_\bullet^K) \to F(X_\beta, Y_\bullet)^H \to F(X_\alpha, Y_\bullet)^H,$$

and the condition that $\dim X^K \leq n(K)$ ensures that its realization is quasi-fibration. We consider the diagram

$$\begin{array}{ccccc} |F(S^n, Y_\bullet^K)| & \longrightarrow & |F(X_\beta, Y_\bullet)^H| & \longrightarrow & |F(X_\alpha, Y_\bullet)^H| \\ \downarrow{\scriptstyle \gamma^n} & & \downarrow{\scriptstyle \gamma^H} & & \downarrow{\scriptstyle \gamma^H} \\ F(S^n, |Y_\bullet^K|) & \longrightarrow & F(X_\beta, |Y_\bullet|)^H & \longrightarrow & F(X_\alpha, |Y_\bullet|)^H. \end{array}$$

The map γ^n is an equivalence by [May, 12.4] and we are done by induction.

Since an H-CW-complex is also a K-CW-complex for $K \leq H$, the same argument shows that γ^K is an equivalence. This concludes the proof. □

Proposition 1.4. *The canonical map $t(L)(V) \to T(L)(V)$ is an $\mathcal{F}$-equivalence, where $\mathcal{F}$ is the family of finite subgroups of G.*

Proof. We must prove that the prespectrum structure map $\sigma\colon t(V) \to \Omega^{W-V} t(W)$ is a C-equivalence for any $C \in \mathcal{F}$. We use edgewise subdivision to get a simplicial C-action and factor σ as

$$|\operatorname{sd}_C \operatorname{THH}(L; S^V)| \to |\Omega^{W-V} \operatorname{sd}_C \operatorname{THH}(L; S^W)| \to \Omega^{W-V} |\operatorname{sd}_C \operatorname{THH}(L; S^W)|,$$

where the latter map is a C-equivalence by the lemma above. It follows from [L] that the simplicial spaces involved are 'good' in the sense of [S] or 'strictly proper' in the sense of [May]. Therefore it is enough to show that the map on homotopy colimits

$$\begin{aligned}\hat{\sigma}_k\colon & \underset{\overrightarrow{I^{k+1}}}{\operatorname{holim}} F(S^{i_0 R} \wedge \ldots \wedge S^{i_k R}, L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k})^{\wedge r} \wedge S^V) \\ & \to \underset{\overrightarrow{I^{k+1}}}{\operatorname{holim}} F(S^{i_0 R} \wedge \ldots \wedge S^{i_k R} \wedge S^{W-V}, L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k})^{\wedge r} \wedge S^W),\end{aligned}$$

induced by the adjoints of evaluation maps, is a C-equivalence. Furthermore we may assume that $W - V = lR$. We consider the map

$$\begin{aligned}\tau_k\colon & \underset{\overrightarrow{I^{k+1}}}{\operatorname{holim}} F(S^{i_0 R} \wedge \ldots \wedge S^{i_k R} \wedge S^{lR}, L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k})^{\wedge r} \wedge S^{lR} \wedge S^V) \\ & \to \underset{\overrightarrow{I^{k+1}}}{\operatorname{holim}} F(S^{i_0 R} \wedge \ldots \wedge S^{(i_k + l)R}, L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k + l})^{\wedge r}),\end{aligned}$$

given by the identification $S^{lR} \cong (S^l)^{\wedge r}$ and the stabilization $L(S^{i_k}) \wedge S^l \to L(S^{i_k + l})$. It is a C-equivalence by [BHM], 3.11 and 3.12, and the approximation theorem [B1], 1.6. The composition $\tau_k \circ \hat{\sigma}_k$ is a map in the limit system and induces therefore a C-equivalence on homotopy colimits. It follows that $\hat{\sigma}_k$ is a C-equivalence. □

1.5. In this section we define the cyclotomic structure on $t(L)$ and $T(L)$. The restriction of a C-map

$$f\colon S^{i_0 R} \wedge \ldots \wedge S^{i_k R} \to L(S^{i_0})^{\wedge r} \wedge \ldots \wedge L(S^{i_k})^{\wedge r} \wedge X$$

to the C-fixed set induces a simplicial map

$$\phi_{C,\bullet}\colon \operatorname{sd}_C \operatorname{THH}(L; X)^C_\bullet \to \operatorname{THH}(L; X^C)_\bullet$$

and we define $r_C\colon \rho_C^* t(L)(V)^C \to t(L)(\rho_C^* V^C)$ as the composite

$$\rho_C^* |\operatorname{THH}(L; S^V)|^C \xrightarrow{D^{-1}} |\operatorname{sd}_C \operatorname{THH}(L; S^V)|^C \xrightarrow{\phi_C} |\operatorname{THH}(L; S^{\rho_C^* V^C})|.$$

The maps $r_C(V)$ induce similar maps in the thickened prespectrum $t^\tau(L)$. In order to show that these makes $t^\tau(L)$ into a cyclotomic prespectrum we need

Lemma 1.5. *Let j be a G-prespectrum and let J be the G-spectrum associated with j^τ. If $J^\Gamma \simeq *$ for any finite subgroup $\Gamma \subset G$ and $j(V)^G \simeq *$ for any $V \subset \mathcal{U}$ then $J \simeq_G *$.*

Proof. Let $\mathcal{F}$ be the family of finite subgroups of the circle, then J is $\mathcal{F}$-contractible. Since $J \wedge E\mathcal{F}_+ \to J$ is an $\mathcal{F}$-equivalence, $J \wedge E\mathcal{F}_+$ is also $\mathcal{F}$-contractible. However $J \wedge E\mathcal{F}_+$ is G-equivalent to an $\mathcal{F}$-CW-spectrum and therefore it is in fact G-contractible by the $\mathcal{F}$-Whitehead theorem, [LMS] p.63. Now

$$(J \wedge E\mathcal{F}_+)(V) \cong \underset{\overrightarrow{W}}{\lim} \Omega^W (j^\tau(V + W) \wedge E\mathcal{F}_+),$$

and $j^\tau(V) \wedge E\mathcal{F}_+ \to j^\tau(V)$ is an G-equivalence since $j(V)^G \simeq *$. Therefore $J \simeq_G J \wedge E\mathcal{F}_+$ and we have already seen that the latter is G-contractible. □

Proposition 1.5. *$t^\tau(L)$ is a cyclotomic prespectrum and $T(L)$ is a cyclotomic spectrum.*

Proof. The remark following theorem 1.2 shows that $T(L)$ is a cyclotomic spectrum if $t^\tau(L)$ is a cyclotomic prespectrum. The map

$$r_C\colon \rho_C^* t(L)(V)^C \to t(L)(\rho_C^* V^C)$$

is G-equivariant by construction so we have left to check the three conditions in lemma 1.2. We leave i) and ii) to the reader and prove iii).

We define an auxiliary functor $a^C\colon G\mathcal{PU} \to G\mathcal{PU}$ as follows. For each $Z \subset \mathcal{U}^C$ choose $V(Z) \subset \mathcal{U}$ such that $V(Z)^C = Z$ and such that the union of all $V(Z)$ is equal to $\mathcal{U}$, then define a^C by

$$a^C t(\rho_C^* Z) = \rho_C^* t(V(Z))^C$$

with the obvious prespectrum structure maps. The maps $r_C(V)$ above defines a map of G-prespectra

$$r_C\colon a^C t \to t.$$

In this formulation lemma 1.2 iii) states that the map of the associated spectra is a G-equivalence; we apply the lemma with j being the homotopy fiber of r_C.

Let Γ be some finite subgroup of G and let H be the inverse image of Γ under the composition

$$G \xrightarrow{\pi_C} G/C \xrightarrow{\rho_C} G.$$

To show that J^Γ is contractible it suffices to show that $(r_C(V))^\Gamma$ is $\dim(V^H) + k(V)$-connected, where $k(V) \to \infty$ as V runs through the f.d. sub inner product spaces of $\mathcal{U}$. Indeed $t(L)$ is connective and one can proceed as in the proof of lemma 1.1. Recall that $r_C(V)$ is defined as the composition of three maps, two of which are homeomophisms. Therefore we need to show that

$$(\phi_C)^\Gamma\colon (\rho_C^* |\operatorname{sd}_C \operatorname{THH}(L; S^V)^C|)^\Gamma \to |\operatorname{THH}(L; S^{\rho_C^* V^C})|^\Gamma$$

is $\dim(V^H) + k(V)$-connected. We recall from [BHM] that $\operatorname{sd}_\Gamma \operatorname{sd}_C = \operatorname{sd}_H$ and that we have a commutative diagram

$$\begin{array}{ccc} \rho_C^* |\operatorname{sd}_H \operatorname{THH}(L; S^V)^C| & \xrightarrow{\hat{\phi}_C} & |\operatorname{sd}_\Gamma \operatorname{THH}(L; S^{\rho_C^* V^C})| \\ {\scriptstyle D}\downarrow\cong & & {\scriptstyle D}\downarrow\cong \\ \rho_C^* |\operatorname{sd}_C \operatorname{THH}(L; , S^V)^C| & \xrightarrow{\phi_C} & |\operatorname{THH}(L; S^{\rho_C^* V^C})| \end{array}$$

In the top row Γ acts simplicially. Furthermore

$$(\hat{\phi}_{C,k})^\Gamma\colon \operatorname{sd}_H \operatorname{THH}(L; S^V)_k^H \to \operatorname{sd}_H \operatorname{THH}(L; S^{V^C})_k^\Gamma$$

is a fibration whose fiber is an equivariant mapping space, and one can proceed as in the proof of lemma 1.1 to show that this is $\dim(V^H) + k(V)$-connected.

It remains to deal with G-fixed points. If $X_\bullet$ is a cyclic space, then $|X_\bullet|^G$ may be idenfied with the subspace in X_0 of those 0-simplices x for which $s_0 x = \tau_1 s_0 x$. In the case of $\operatorname{THH}(L; S^V)$ this is S^{V^G}, and $j(V)^G$ is the homotopy fiber of the identity. □

1.6. In [BHM] C-equivariant deloops of $\operatorname{THH}(L)$ were defined using the Γ-space machine of Segal and Shimakawa. We show in this section that the equivariant deloops obtained in this fashion are C-equivalent to the deloops $t(L)(V)$ defined in 1.4, but first we give a brief discussion of Γ_C-spaces.

Let Γ_C be the category of the finite based C-sets S, whose underlying set is of the form $\mathbf{n} = \{0, 1, \ldots, n\}$, based at 0. A Γ_C-space is a functor A from Γ_C to C-spaces. It is *special* if $A(\mathbf{0}) \simeq_C *$ and if the canonical map is a C-equivalence

$$A(S \vee T) \to A(S) \times A(T),$$

for any $S, T \in \Gamma_C$. A C-spectrum $\mathbf{A}$ defines a special Γ_C-space, $A(S) = S \wedge \mathbf{A}$.

Suppose $X_\bullet\colon \mathbf{\Delta}^{\rm op} \to \Gamma_C$ is a finite simplicial C-set, then $A(X_\bullet)$ is a simplicial C-space, which we want to realize. To get the correct homotopy type, however, we need that $A(S) \to A(T)$ be a closed C-cofibration whenever $S \rightarrowtail T$ is an inclusion. In [S1] Segal obtains this by replacing A by a thickened version τA given by

$$\tau A(S) = \underset{\longrightarrow}{\rm holim}\ ((\Gamma_C \downarrow S) \xrightarrow{{\rm pr}_1} \Gamma_C \xrightarrow{A} {\rm Top}_C),$$

where $(\Gamma_C \downarrow S)$ is the category over S. It has ${\rm id}_S$ as terminal object, so $\tau A(S) \to A(S)$ is a C-homotopy equivalence. Furthermore an injection $f\colon S \rightarrowtail T$ induces an inclusion of over-categories and therefore a closed C-cofibration $\tau A(S) \to \tau A(T)$.

Alternatively one may consider the two sided bar construction $B(A, \Gamma_C, X)$. It is the realization of a simplicial space $B_\bullet(A, \Gamma_C, X)$ with k-simplices

$$\coprod_{S_0,\dots,S_k} A(S_0) \times F(S_0, S_1) \times \cdots \times F(S_{k-1}, S_k) \times F(S_k, X),$$

with the coproduct taken over tuples of finite C-sets in Γ_C. We have

Lemma 1.6. *$B(A, \Gamma_C, |X_\bullet|) \simeq_C |\tau A(X_\bullet)|$, for any $X_\bullet\colon \mathbf{\Delta}^{\rm op} \to \Gamma_C$.*

Proof. A bisimplicial space $Y_{\bullet,\bullet}$ may be realized as $|\mathbf{k} \mapsto |Y_{k,\bullet}||$ or as $|\mathbf{l} \mapsto |Y_{\bullet,l}||$, the two realizations are homeomorphic. Hence $B(A, \Gamma_C, |X_\bullet|) \cong_C |B(A, \Gamma_C, X_\bullet)|$. Now by [Wo] lemma 1.3 the 'evaluation map'

$$B(A, \Gamma_C, X_k) \to A(X_k)$$

is a C-equivalence for all $k \ge 0$. We want the map on realizations to be a C-equivalence. This requires that the simplicial spaces are 'good' in the sense of [S]. The space on the left is good, but the one on the right is not necessarily so. Therefore we must replace it by its thickening $\tau A(X_\bullet)$. □

Following [Sh] we define a C-prespectrum $\mathbf{B}A$ whose V'th space is the quotient

$$B^V A = B(A, \Gamma_C, S^V)/B(A, \Gamma_C, \infty).$$

Finally, recall that $A(\mathbf{1})$ is a C-homotopy commutative, C-homotopy associative H-space, with product $A(\mathbf{1}) \times A(\mathbf{1}) \simeq_C A(\mathbf{1} \vee \mathbf{1}) \to A(\mathbf{1})$.

Proposition 1.6.1. ([Sh]) *If $A(\mathbf{1})$ has a C-homotopy inverse, then $\mathbf{B}A$ is an Ω-C-spectrum, that is the structure maps induce C-equivalences $B^V A \simeq_C \Omega^{W-V} B^W A$.* □

We have two Ω-C-spectra with zero'th space ${\rm THH}(L)$. The first is $\mathbf{B}\,{\rm THH}(L)$, arising from a special Γ_C-structure on ${\rm THH}(L)$, and the other is $t(L)$, defined in 1.4. We know that $t(L)$ is an Ω-C-spectrum by proposition 1.4. To show that they are equivalent we construct a Ω-C-bispectrum, which contains both.

The Γ_C-space on ${\rm THH}(L)$ constructed in [BHM] works equally well for the space $t(L)(V) = {\rm THH}(L, S^V)$. Specifically, in the notation of [BHM] §4:

$$t(L)(V, S) = \bigvee_{\underline{k}: P_0\mathbf{n} \to \mathbb{N}_0} |\,{\rm sd}_C(E_\bullet(\mathbf{n}, \underline{k})_+ \wedge {\rm THH}(L_{\underline{k}}, S^V)_\bullet)|.$$

Here $\mathbf{n}$ is the underlying set of the finite C-set S. In view of [BHM] 4.20 these Γ_C-spaces are special, and we obtain Ω-C-spectra $\mathbf{B}t(L)(V)$ for each V. Hence the equivalence follows from the

Proposition 1.6.2. $B^{W-V} t(L)(V) \simeq_C t(L)(W)$.

Proof. It suffices to treat the case where $W - V$ is the regular representation $R = \mathbb{R}C$. We choose a simplicial model $S^1_\bullet$ for the circle, *e.g.* $S^1_\bullet = \Delta[1]/\partial\Delta[1]$ or $S^1_\bullet = \Lambda[0]$. Then $S^1_\bullet \wedge \dots \wedge S^1_\bullet$ (r times) with C acting by cyclic permutation is a simplicial model $S^R_\bullet$ for S^R. From lemma 1.4 and the lemma above we get

$$\Omega^R B^R t(L)(V \oplus R) \simeq_C |\Omega^R \tau t(L)(V \oplus R, S^R_\bullet)| \simeq_C |\tau t(L)(V, S^R_\bullet)| \simeq_C B^R t(L)(V).$$

Since $\Omega^R B^R$ is C-homotopic to the identity functor the proposition follows. □

1.7. We conclude this paragraph with a list of some additional properties of topological Hochschild homology. We shall need the following extension of Bökstedt's notion of a functor with smash product.

Let L be a functor with smash product. The definition of $\mathrm{THH}(L;X)$ does not require the full functoriality of L. In effect, we only need a collection of spaces $L(S^n)$, $n \geq 0$, with a Σ_n-action together with unit and multiplication maps

$$\mathbf{1}_n\colon S^n \to L(S^n), \qquad \mu_{m,n}\colon L(S^m) \wedge L(S^n) \to L(S^{m+n}), \tag{1.7.1}$$

which are Σ_n-equivariant and $\Sigma_m \times \Sigma_n$-equivariant, respectively, and satisfies the relations

i) $\mu_{n,m} \circ (\mathbf{1}_n \wedge \mathbf{1}_m) = \mathbf{1}_{n+m}$

ii) $\sigma_{m,n} \circ \mu_{m,n} \circ (\mathbf{1}_n \wedge \mathrm{id}) = \mu_{n,m} \circ (\mathrm{id} \wedge \mathbf{1}_n) \circ \mathrm{tw}$

iii) $\mu_{l+m,n} \circ (\mu_{l,m} \wedge \mathrm{id}) = \mu_{l,m+n} \circ (\mathrm{id} \wedge \mu_{m,n})$

In the commutative case we require, in addition, that

iv) $\sigma_{m,n} \circ \mu_{m,n} = \mu_{n,m} \circ \mathrm{tw}$.

where $\sigma_{m,n} \in \Sigma_{n+m}$ permutes the first m and last n elements and tw permutes the two smash factores. We call such a set of data an *FSP defined on spheres*. These are precisely the commutative monoids in the symmetric monoidal category of spectra which has recently been constructed by Jeff Schmidt, [Sch].

We let L be an *FSP* defined on spheres and consider a version of topological Hochschild homology where we replace the index category I by the n-fold product I^n. By the approximation theorem, [B1] theorem 1.6, this will not change the homotopy type:

$$\mathrm{THH}(L^{(n)};X) \simeq \mathrm{THH}(L;X), \qquad \text{for } n > 0. \tag{1.7.2}$$

In more detail, we let $\mathrm{THH}(L^{(n)};X)_\bullet$ be the cyclic space with k-simplices

$$\operatorname*{holim}_{\overrightarrow{(I^n)^{k+1}}} F((S^{i_{10}} \wedge \ldots \wedge S^{i_{n0}}) \wedge \ldots \wedge (S^{i_{1k}} \wedge \ldots \wedge S^{i_{nk}}), L(S^{i_{10}} \wedge \ldots \wedge S^{i_{n0}}) \wedge \ldots \wedge L(S^{i_{1k}} \wedge \ldots \wedge S^{i_{nk}}) \wedge X)$$

and with Hochschild-type cyclic structure maps. Then the realization $\mathrm{THH}(L^{(n)};X)$ is a $G \times \Sigma_n$-space, where the Σ_n-action is induced from the permutation action on I^n. When $X = S^n$ we get another Σ_n-action induced from the Σ_n-action on S^n. Hence $\mathrm{THH}(L^{(n)};S^n)$ becomes a $G \times \Sigma_n \times \Sigma_n$-space which we consider a $G \times \Sigma_n$-space via the diagonal in the second factor.

Proposition 1.7.1. *Let L be a commutative FSP defined on spheres. Then the spaces $\mathrm{THH}(L^{(n)};S^n)$, $n \geq 0$, again form a commutative FSP defined on spheres and the multiplication maps*

$$\mu_{m,n}\colon \mathrm{THH}(L^{(m)};S^m) \wedge \mathrm{THH}(L^{(n)};S^n) \to \mathrm{THH}(L^{(m+n)};S^{m+n})$$

are G-equivariant when the domain is given the diagonal G-action. Moreover, the restriction and Frobenius maps

$$R_r, F_r\colon \mathrm{THH}(L^{(n)};S^n)^{C_{rs}} \to \mathrm{THH}(L^{(n)};S^n)^{C_s}$$

are Σ_n-equivariant, multiplicative and preserve units.

Proof. Let G_k^X be as in 1.4 and let $\mu_n\colon I^n \to I$ be the iterated multiplication functor, *i.e.* concatenation of sets and maps. Then we have

$$\mathrm{THH}(L^{(n)};X)_k = \operatorname*{holim}_{\overrightarrow{(I^n)^{k+1}}} G_k^X \circ \mu_n^{k+1}.$$

We first recall that the canonical map

$$\mathrm{can}\colon \operatorname*{holim}_{\overrightarrow{(I^m)^{k+1}}} G_{k+1}^X \circ \mu_m^{k+1} \wedge \operatorname*{holim}_{\overrightarrow{(I^n)^{k+1}}} G_{k+1}^Y \circ \mu_n^{k+1} \;\to\; \operatorname*{holim}_{\overrightarrow{(I^m)^{k+1} \times (I^n)^{k+1}}} G_{k+1}^X \circ \mu_m^{k+1} \wedge G_{k+1}^Y \circ \mu_n^{k+1}$$

is a homeomorphism, when the spaces are given the compactly generated topology. Next, we note that there are natural transformations

$$G^X_{k+1} \circ \mu^{k+1}_m \wedge G^Y_{k+1} \circ \mu^{k+1}_n \xrightarrow{\lambda} G^{X\wedge Y}_{2k+1} \circ (\mu^{k+1}_m \times \mu^{k+1}_n) \xrightarrow{\sigma} G^{X\wedge Y}_{k+1} \circ \mu^{k+1}_{m+n}.$$

Indeed, λ is concatenation of maps and if $\mathrm{tw}\colon (I^m)^{k+1} \times (I^n)^{k+1} \to (I^{m+n})^{k+1}$ denotes the isomorphism of categories given by

$$\begin{aligned}\mathrm{tw}((i_{10},\dots,i_{m0}),\dots,(i_{1k},\dots,i_{mk}),(j_{10},\dots,j_{n0}),\dots,(j_{1k},\dots,j_{nk})) = \\ (((i_{10},\dots,i_{m0},j_{10},\dots,j_{n0}),\dots,(i_{1k},\dots,i_{mk},j_{1k},\dots,j_{nk})),\end{aligned}$$

then σ is the obvious shuffle permutation covering tw followed by multiplication in L. We may now compose can^{-1} with the map on homotopy colimits induced from λ and σ to get a map

$$\mathrm{THH}(L^{(m)};X)_k \wedge \mathrm{THH}(L^{(n)};Y)_k \to \mathrm{THH}(L^{(m+n)};X\wedge Y)_k.$$

These maps are $\Sigma_m\times\Sigma_n$-equivariant and form, for varying k, a cyclic map. Accordingly, we get a $G\times\Sigma_m\times\Sigma_n$-equivariant map

$$\mu_{m,n}\colon \mathrm{THH}(L^{(m)};X) \wedge \mathrm{THH}(L^{(n)};Y) \to \mathrm{THH}(L^{(m+n)};X\wedge Y)$$

upon realization. If we let $X = S^m$ and $Y = S^n$ we obtain the required product map. The unit map is given as the composition

$$\mathbf{1}_n\colon S^n \to F(S^0\wedge\ldots\wedge S^0, L(S^0\wedge\ldots\wedge S^0)\wedge S^n) \to \mathrm{THH}(L^{(n)};S^n) \to \mathrm{THH}(L^{(n)};S^n)$$

where the first map is given by smashing with the unit map in L, the second is the canonical inclusion in the homotopy colimit and the last is the inclusion of the zero skeleton. We leave it to the reader to verify that the maps $\mathbf{1}_n$ and $\mu_{m,n}$ in fact make the spaces $\mathrm{THH}(L^{(n)};S^n)$ a commutative *FSP* defined on spheres.

The spaces $\mathrm{THH}(L^{(n)};S^n)$ again form a commutative *FSP* defined on spheres and the Frobenius maps F_r are multiplicative. Indeed, the multiplication maps $\mu_{m,n}$ are G-equivariant and the unit maps $\mathbf{1}_n$ factor through the inclusion of the G-fixed set. Finally, we consider the restriction map which, we remember, is defined as the composite

$$\begin{aligned}R_r\colon |\,\mathrm{THH}(L^{(n)};S^n)_\bullet|^{C_{rs}} \xrightarrow{D_{rs}^{-1}} |\,\mathrm{sd}_{C_{rs}}\mathrm{THH}(L^{(n)};S^n)_\bullet^{C_{rs}}| \\ \xrightarrow{\varphi_{C_r}} |\,\mathrm{sd}_{C_s}\mathrm{THH}(L^{(n)};S^n)_\bullet^{C_s}| \xrightarrow{D_s} |\,\mathrm{THH}(L^{(n)};S^n)_\bullet|^{C_s}.\end{aligned}$$

The subdivision $\mathrm{sd}_C\,\mu_{m,n}$ defines a product on the second and third term and the naturality of the homeomorphism D makes it multiplicative. Moreover, $\mathrm{sd}_{C_{rs}}\,\mu_{m,n}$ restricts to $\mathrm{sd}_{C_s}\,\mu_{m,n}$ under $\varphi_{C_r,}.$ and hence φ_{C_s} is multiplicative. □

Next, let L be an *FSP*, then the associated $n\times n$-matrix *FSP* is defined by

$$M_n(L)(X) = F(\mathbf{n}, \mathbf{n}\wedge L(X)),$$

where $\mathbf{n} = \{0,1,\dots,n\}$ with 0 as basepoint. In view of proposition 1.6 above we may restate [BHM] 3.9 and 4.24 as

Proposition 1.7.2. (*Morita invariance*) $T(L) \simeq_G T(M_n(L))$. □

For an *FSP* L and $V \subset \mathcal{U}^G$, we define the underlying spectrum L^S of L by

$$L^S(V) = \underset{\overrightarrow{I}}{\mathrm{holim}}\, F(S^i, L(S^i)\wedge S^V).$$

Lemma 1.7. *Suppose $f\colon L_1 \to L_2$ is a natural transformation such that f^S is an equivalence of spectra. Then $T(f)\colon T(L_1) \to T(L_2)$ is a G-equivalence.* □

2. Witt vectors

2.1. Let A be a commutative ring and let p be a fixed prime. The associated ring $W(A)$ of (p-typical) Witt vectors will play an important role in the sequel, and we briefly recall its definition, refering to [Se], [I] and Bergman's lecture in [Mu] for details. The underlying set $W(A) = A^{\mathbb{N}_0}$; the infinite product. The ring structure is specified by the requirement that the ghost map

$$w\colon W(A) \to A^{\mathbb{N}_0},$$

given by the Witt polynomials,

$$\begin{aligned} w_0 &= a_0 \\ w_1 &= a_0^p + pa_1 \\ w_2 &= a_0^{p^2} + pa_1^p + p^2 a_2 \\ &\vdots \end{aligned} \tag{2.1.1}$$

be a natural transformation of functors from rings to rings. More concretely,

$$\begin{aligned} a + b &= (s_0(a,b), s_1(a,b), \dots) \\ a \cdot b &= (p_0(a,b), p_1(a,b), \dots) \end{aligned}$$

for certain integral polynomials s_i and p_i which depend only on $(a_0, \dots, a_i)$. The integrality follows from the Kummer congruences

$$x^{p^n} \equiv x^{p^{n-1}} (\operatorname{mod} p^n), \qquad x \in \mathbb{Z}.$$

Hence $W(A)$ is well-defined for any ring. The a_i are called the Witt coordinates of the Witt vector $a = (a_0, a_1, \dots)$ and the w_i are called the ghost or phantom coordinates. The element $1 = (1, 0, \dots) \in W(A)$ is the unit.

There are operators

$$\begin{aligned} F\colon W(A) &\to W(A), && \text{(Frobenius homomorphism)} \\ V\colon W(A) &\to W(A), && \text{(Verschiebung map)} \\ \omega\colon A &\to W(A), && \text{(Teichmüller character)} \end{aligned} \tag{2.1.2}$$

characterized by the formulas

$$\begin{aligned} F(w_0, w_1, \dots) &= (w_1, w_2, \dots) \\ V(a_0, a_1, \dots) &= (0, a_0, a_1, \dots) \\ \omega(x) &= (x, 0, 0, \dots). \end{aligned}$$

Any relation which holds true in ghost coordinates also holds in $W(A)$. This is obvious for a $\mathbb{Z}[1/p]$-algebra since the ghost map is a bijection. In general it follows from the functoriality of W: every algebra is the quotient of a p-torsion free algebra which embeds in a $\mathbb{Z}[1/p]$-algebra. It follows that F is a ring homomorphims, that V is additive, that ω is multiplicative, and that we have the relations

$$x \cdot V(y) = V(F(x) \cdot y), \qquad FV = p, \qquad VF = \operatorname{mult}_{V(1)}. \tag{2.1.3}$$

Moreover, when A is an $\mathbb{F}_p$-algebra, $V(1) = p$ and $F = W(\varphi)$ where φ is the Frobenius endomorphism of A, $F(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$. For any $a \in W(A)$,

$$a = \sum_{i=0}^{\infty} V^i(\omega(a_i)), \tag{2.1.4}$$

where the a_i are the Witt coordinates of a.

The additive subgroup $V^n W(A)$ of $W(A)$ is an ideal by (2.1.3) whose quotient

$$W_n(A) = W(A)/V^n W(A)$$

is the ring of Witt vectors of lenght n in A. The elements in $W_n(A)$ are in 1-1 correspondance with tuples $(a_0, \dots a_{n-1})$ with addition and multiplication given by the polynomials s_i and p_i. Hence $W(A)$ is the inverse limit of the $W_n(A)$ over the restriction maps

$$R\colon W_n(A) \to W_{n-1}(A), \qquad R(a_0, \dots, a_{n-1}) = (a_0, \dots, a_{n-2}).$$

It follows that $W(A)$ is complete and separated in the topology defined by the ideals $V^n W(A)$, $n \geq 1$.

Theorem 2.1. (Witt) *If k is a perfect field of positive charactesitic p then $W(k)$ is a complete discrete valuation ring with residue field k and uniformizing element p. In particular, $W(\mathbb{F}_p) = \mathbb{Z}_p$.*

Proof. We have already seen that the ideals $V^n W(k)$ define a complete and separated topology on $W(k)$ and that $W(k)/VW(k) = k$. Therefore, it suffices to show that $V^n W(k)$ is generated by p^n. Now by (2.1.3)

$$p^n \cdot W(k) = V(1)^n \cdot W(k) = V^n(F^n(W(k))),$$

and since $F = W(\varphi)$ is invertible the statement follows. □

We shall also need the ring of big Witt vectors $\mathbf{W}(A)$. Its underlying set is $A^{\mathbb{N}}$ but its n'th ghost coordinate is $\mathbf{w}_n = \sum_{d|n} d\mathbf{a}_d^{n/d}$ and again one requires that $w\colon \mathbf{W}(A) \to A^{\mathbb{N}}$ be a natural transformation of rings. As an abelian group, $\mathbf{W}(A)$ may be identified with the multiplicative group of power series with constant term 1. The isomorphism is given by

$$\psi\colon \mathbf{W}(A) \xrightarrow{\cong} (1 + XA[\![X]\!])^\times, \qquad \psi(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots)(X) = \prod_{i=1}^{\infty}(1 - \mathbf{a}_i X^i). \tag{2.1.5}$$

We call the $\mathbf{a}_i$ (resp. the $\mathbf{w}_i$) the Witt coordinates (resp. the ghost coordinates) of a Witt vector. Again there are Frobenius and Verschiebung operators, one for each $n \geq 0$, defined by

$$\begin{aligned} F_n(\mathbf{w}_m) &= \mathbf{w}_{nm} \\ V_n(\mathbf{a}_m) &= \begin{cases} \mathbf{a}_{m/n}, & \text{if } n|m \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1.6}$$

We note that under the isomorphism (2.1.5),

$$F_n(P(X)) = \prod_{i=1}^{n} P(\xi_i), \qquad V_n(P(X)) = P(X^n),$$

where $\xi_1, \dots, \xi_n$ are the formal n'th roots of X. The formulas

$$\begin{aligned} &F_r(xy) = F_r(x)F_r(y) \\ &V_r(F_r(x)y) = xV_r(y) \\ &F_rV_r = r, \qquad V_rF_r = V_r(1) \\ &F_rV_s = V_sF_r, \qquad \text{if } \ (r,s){=}1. \end{aligned} \tag{2.1.7}$$

are easily verified in ghost coordinates.

We call a subset $S \subset \mathbb{N}$ a *truncation set* if it is stable under division. Since $\mathbf{w}_n$ only involves the $\mathbf{a}_d$ where $d|n$, we may replace $\mathbb{N}$ above by any truncation set S to obtain a ring $\mathbf{W}_S(A)$ with underlying set A^S. If $S \subset S'$ are two truncation sets then the obvious projection $\mathbf{W}_{S'}(A) \to \mathbf{W}_S(A)$ is a ring homomorphism. One can use (2.1.6) to define

$$F_n\colon \mathbf{W}_S(A) \to \mathbf{W}_{S/n}(A), \qquad V_n\colon \mathbf{W}_{S/n}(A) \to \mathbf{W}_S(A).$$

We note that $W(A) = \mathbf{W}_{\{1,p,p^2,\dots\}}(A)$ such that $F = F_p$ and $V = V_p$. Moreover, if $\langle n\rangle = \{d \mid d \text{ divides } n\}$ then $W_{s+1}(A) = \mathbf{W}_{\langle p^s\rangle}(A)$.

2.2. In section 2.3 below we relate Witt vectors to $\pi_0 T(A)^{C_n}$ but first we recall some notions from abstract induction theory, *cf.* [D] and [tD]. We shall only need this when G is the circle group, but in this section G may be any compact Lie group.

We let $\mathrm{Or}(G;\mathcal{F})$ denote the category of canonical orbits G/H with H finite, and all G-maps. Let M be an abelian group valued bifunctor on $\mathrm{Or}(G;\mathcal{F})$, *i.e.* $M = (M^*, M_*)$ is a pair of functors from $\mathrm{Or}(G;\mathcal{F})$ to abelian groups with M^* contravariant and M_* covariant, and $M^*(G/H) = M_*(G/H)$ for all H. M is called a Mackey functor if $i_*i^* = \mathrm{id}$ for any isomorphism $i: G/H \to G/H$ and if the double coset formula holds: if $H, H' \subset K$ and $K = \coprod_i Hx_iH'$ then

$$(2.2.1) \qquad (\pi_H^K)^* \circ (\pi_{H'}^K)_* = \sum_i (\pi^H_{H\cap(x_iH'x_i^{-1})})_* \circ (\pi^{x_iH'x_i^{-1}}_{H\cap x_iH'x_i^{-1}})^* \circ r_{x_i}^*,$$

where $\pi_H^K: G/H \to G/K$ is the projection and $r_x: G/xHx^{-1} \to G/H$ is right multiplication by x.

A Green functor is a Mackey functor M for which $M(G/H)$ is a ring, and such that for all $f: G/H \to G/K$, f^* is a ring homomorphism and f_* is a map of $M(G/K)$-bimodules when $M(G/H)$ is considered an $M(G/K)$-bimodule via f^*, *i.e.*

$$f_*(xf^*(y)) = f_*(x)y, \qquad f_*(f^*(y)x) = yf_*(x),$$

for any $x \in M(G/H)$ and $y \in M(G/K)$.

Now, suppose T is a G-spectrum indexed on a complete G-universe $\mathcal{U}$. For $H \subset G$ the fixed point spectrum is given by

$$T^H \cong F(G/H_+, T)^G.$$

A G-map $f: G/H \to G/K$ induces a map $f^*: T^K \to T^H$. If H and K are finite, one has the equivariant transfer $f^!: \Sigma_G^\infty G/K_+ \to \Sigma_G^\infty G/H_+$. The map $f^!$ depends on choosing a G-embedding of G/H into some $V \subset U$; one gets $f^!: S^V \wedge G/K_+ \to S^V \wedge G/H_+$ and defines f_* to be the composite

$$T^H \cong F(S^V \wedge G/K_+, \Sigma^V T)^G \xrightarrow{(f^!)^*} F(S^V \wedge G/K_+, \Sigma^V T) \cong T^K.$$

The homotopy class of f_* is independant of the choice of embedding. Now the statements of [LMS], IV 6.3, 5.6. and 5.8. easily translate to the following

Proposition 2.2. *Let G be a compact Lie group and let T be a G-spectrum indexed on a complete G-universe. The functor which to G/H assigns $\pi_*(T^H)$ and to $f: G/H \to G/K$ assigns the homomorphisms f^* and f_* is a Mackey functor on $\mathrm{Or}(G;\mathcal{F})$. If T is a G-ring spectrum then this becomes a Green functor.If $H \subset K$ and $\pi_H^K: G/H \to G/K$ is the canonical projection then the composite $(\pi_H^K)_* \circ (\pi_H^K)^*$ is multiplication by $(\pi_H^K)_*(1)$.* □

Let $H \subset G$ be a finite subgroup. In section 1.1 we used the norm map $N: T_{hH} \to T^H$. We can include H in EH as an orbit to get a map

$$\iota_H: T = T \wedge_H H_+ \to T \wedge_H EH_+ = T_{hH}.$$

Later in the paper we need the following

Lemma 2.2. *Let T be as above and let $\pi_H: G \to G/H$ be the projection. Then the diagram*

$$\begin{array}{ccc} T & \xrightarrow{\pi_{H*}} & T^H \\ \downarrow{\scriptstyle \iota_H} & & \| \\ T_{hH} & \xrightarrow{N} & T^H \end{array}$$

is homotopy commutative.

Proof. We consider the diagram

$$\begin{array}{ccccc} T \wedge_H H_+ & \xrightarrow{\simeq} & (T \wedge H_+)^H & \xrightarrow{c} & T^H \\ \downarrow & & \downarrow & & \| \\ T \wedge_H EH_+ & \xrightarrow{\simeq} & (T \wedge EH_+)^H & \xrightarrow{c} & T^H \end{array}$$

where the equivalences on the left are as in the proof of proposition 1.1 and where the maps c collapses H (resp. EH) to a point. The left hand square homotopy commutes and the right hand square is strictly commutative. We claim that the following diagram homotopy commutes

$$\begin{array}{ccc} F(G_+,T)^G & \xrightarrow{\pi_{H*}} & F(G/H_+,T)^G \\ \| & & \| \\ F(H_+,T)^H & \xrightarrow{\pi'_{H*}} & F(H/H_+,T)^H \\ \downarrow\simeq & & \downarrow\simeq \\ (T\wedge H_+)^H & \xrightarrow{c} & (T\wedge H/H_+)^H. \end{array}$$

For the lower square this follows from [LMS] V.9.7. For the upper square note that we have a pullback

$$\begin{array}{ccc} G & \xrightarrow{\pi_H} & G/H \\ \uparrow\text{incl} & & \uparrow\text{incl} \\ H & \xrightarrow{\pi'_H} & H/H. \end{array}$$

Therefore, the corresponding square of transfers, and hence the upper square homotopy commutes. □

2.3. We apply the general theory discussed above to the topological Hochschild spectrum $T(A)$. Let $\pi_r^{rs}: G/C_s \to G/C_{rs}$ be the projection, $s \geq 1$. We have the maps, with V_r only well-defined up to homotopy:

$$\begin{aligned} F_r &= (\pi_r^{rs})^*: T(A)^{C_{rs}} \to T(A)^{C_s}, \\ V_r &= (\pi_r^{rs})_*: T(A)^{C_s} \to T(A)^{C_{rs}}. \end{aligned} \tag{2.3.1}$$

They are called the *r'th Frobenius* and the *r'th Verschiebung*, respectively. We shall write F (resp. V) instead of F_p (resp. V_p) when the subgroups considered are p-groups. We note that F_r is just the obvious inclusion map $T(A)^{C_{rs}} \to T(A)^{C_s}$. Recall from (1.2.1) the restriction maps $R_r: T(A)^{C_{rs}} \to T(A)^{C_s}$. On homotopy groups we have

Lemma 2.3. *For any commutative ring A the following relations hold on $\pi_*(T(A)^{C_\bullet})$:*

$$\begin{array}{ll} (1) & F_r(xy) = F_r(x)F_r(y) \\ (2) & V_r(F_r(x)y) = xV_r(y) \\ (3) & F_rV_r = r, \quad V_rF_r = V_r(1) \\ (4) & F_rV_s = V_sF_r, \quad \textit{if } (r,s)=1 \\ (5) & R_rF_s = F_sR_r, \quad R_rV_s = V_sR_r \end{array}$$

Proof. Relations (1), (2), (3) and (4) follow from proposition 2.2 since $T(A)$ is a G-ring spectrum when A is commutative. For example, the double coset formula (2.2.1) shows that

$$F_rV_r = 1 + t + t^2 + \ldots + t^{r-1},$$

where $t \in C_{rs}$ is any generator. But the C_{rs}-action extends to the circle G and is therefore trivial on homotopy groups, so $F_rV_r = r$. Finally, (5) is an immediate consequence of the fact that $R_r: \rho^{\#}_{C_{rs}} T(A)^{C_{rs}} \to \rho^{\#}_{C_s} T(A)^{C_s}$ is G-equivariant. □

Proposition 2.3. *For any commutative ring A the sequence*

$$0 \to \pi_0 T(A) \xrightarrow{V^n} \pi_0 T(A)^{C_{p^n}} \xrightarrow{R} \pi_0 T(A)^{C_{p^{n-1}}} \to 0$$

is exact.

Proof. The fundamental cofibration sequence of theorem 1.2 gives a long exact sequence of homotopy groups

$$\cdots \to \pi_1 T(A)^{C_{p^{n-1}}} \xrightarrow{\partial} \pi_0 T(A)_{hC_{p^n}} \xrightarrow{N} \pi_0 T(A)^{C_{p^n}} \xrightarrow{R} \pi_0 T(A)^{C_{p^{n-1}}} \to 0.$$

We claim that the map $\iota_{C_{p^n}}: T(A) \to T(A)_{hC_{p^n}}$ induces an isomorphism on $\pi_0(-)$. Indeed, the skeleton filtration of EC_{p^n} gives rise to a first quadrant spectral sequence

$$E^2 = H_*(C_{p^n}; \pi_* T(A)) \Rightarrow \pi_* T(A)_{hC_{p^n}}$$

whose edge homomorphism is induced by $\iota_{C_{p^n}}$. Since $T(A)$ is a connective spectrum the claim follows. Moreover, lemma 2.2 shows that $V^n = N \circ \iota_{C_{p^n}}$.

It remains to show that $V^n: \pi_0 T(A) \to \pi_0 T(A)^{C_{p^n}}$ is injective. Since $F^n V^n = p^n$ by lemma 2.2 (3), we are done if A has no p-torsion. To treat the general case suppose that $A \to \bar{A}$ is a surjection of rings and that A has no p-torsion. We consider the diagram

$$\begin{array}{ccccccc}
\pi_1 T(A)_{hC_{p^n}} & \xrightarrow{N} & \pi_1 T(A)^{C_{p^n}} & \xrightarrow{R} & \pi_1 T(A)^{C_{p^{n-1}}} & \xrightarrow{\partial} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_1 T(\bar{A})_{hC_{p^n}} & \xrightarrow{N} & \pi_1 T(\bar{A})^{C_{p^n}} & \xrightarrow{\bar{R}} & \pi_1 T(\bar{A})^{C_{p^{n-1}}} & &
\end{array}$$

in which the rows are exact. We prove by induction on n that the vertical maps are surjective. Since A is commutative

$$\pi_i T(A) \cong \mathrm{HH}_i(A), \qquad i = 0, 1.$$

Therefore, the spectral sequence of the skeleton filtraion gives a short exact sequence

$$0 \to \mathrm{HH}_1(A) \xrightarrow{\iota} \pi_1 T(A)_{hC_{p^n}} \to A/p^n A \to 0. \tag{2.3.2}$$

But $\mathrm{HH}_1(-)$ preserves surjections so the proof is complete by induction. □

The proposition shows that there is a set bijection $\pi_0 T(A)^{C_{p^n}} \cong A^{n+1}$. We proceed to define a preferred bijection. Consider for any finite subgroup $C_r \subset G$ the diagonal map (notation as in 1.4)

$$\Delta_r: \mathrm{THH}(A)_0 \xrightarrow{\partial} (\mathrm{sd}_{C_r} \mathrm{THH}(A)_0)^{C_r} \to |(\mathrm{sd}_{C_r} \mathrm{THH}(A)_\bullet)^{C_r}| \xrightarrow{D} |\mathrm{THH}(A)_\bullet^{C_r}|.$$

The first map is given by $f \mapsto f \wedge \ldots \wedge f$ (r factors), the second is the inclusion of the zero-skeleton and D is the homeomorphism from 1.4.

Lemma 2.3. *The composition $R_r \circ \Delta_r$ and $F_r \circ \Delta_r$ are equal to the inclusion of the zero-skeleton, $i: \mathrm{THH}(A)_0 \to \mathrm{THH}(A)$ and the r'th power endomorphism of the topological monoid $\mathrm{THH}(A)_0$ followed by i, respectively.*

Proof. The claim for $R_r \circ \Delta_r$ is obvious from the definitions, *cf.* 1.5. To prove the claim for $F_r \circ \Delta_r$ recall that for any simplicial space $Z_\bullet$ the homeomorphism $D: |\mathrm{sd}_{C_r} Z_\bullet| \to |Z_\bullet|$ is homotopic to the realization of the simplicial map which in degree k is

$$d_0^{(k+1)(r-1)}: Z_{(k+1)r-1} \to Z_k.$$

This follows from the proof of [BHM] proposition 2.5. But the composite

$$\mathrm{THH}(A)_0 \xrightarrow{\partial} \mathrm{sd}_{C_r} \mathrm{THH}(A)_0 = \mathrm{THH}(A)_{r-1} \xrightarrow{d_0^{r-1}} \mathrm{THH}(A)_0$$

is precisely the r'th power endomorphism. □

Theorem 2.3. *Let A be a commutative ring. Then there is natural isomorphism of rings*

$$I\colon W_{n+1}(A) \to \pi_0 T(A)^{C_{p^n}}$$

such that $RI = IR$, $FI = IF$ and $VI = IV$.

Proof. The inclusion of the zero-skeleton $\pi_0 \operatorname{THH}(A)_0 \cong \pi_0 \operatorname{THH}(A)$ is an isomorphism because A is commutative and both groups are copies of A. Hence by the lemma,

$$R_r \circ \Delta_r = \mathrm{id}, \qquad F_r \circ \Delta_r = r. \tag{2.3.3}$$

Now an easy induction argument based on proposition 2.3 shows that the sequence

$$0 \to \pi_0 T(A)^{C_{p^{n-1}}} \xrightarrow{V} \pi_0 T(A)^{C_{p^n}} \xrightarrow{R^n} \pi_0 T(A) \to 0$$

is exact, and since Δ_{p^n} gives a natural splitting of R^n (as a set map), we may define a bijection

$$I\colon W_{n+1}(A) \to \pi_0 T(A)^{C_{p^n}}, \qquad I(a_0, \dots, a_n) = \sum_{i=0}^{n} V^i(\Delta_{p^{n-i}}(a_i)). \tag{2.3.4}$$

As an immediate consequence of (2.3.3) we have that $RI = IR$, $FI = IF$ and $VI = IV$. In particular, if we define

$$\bar{w}\colon \pi_0 T(A)^{C_{p^n}} \to \prod_{i=0}^{n} A$$

by $\bar{w}_i = R^i F^{n-i}$, then $\bar{w} \circ I = w$. It remains to be seen that I is a ring homomorphism. If A has no p-torsion this is obvious because w is injective. In the general case, suppose $A \to \bar{A}$ is a surjection of rings where A is without p-torsion and consider the diagram

$$\begin{array}{ccc} W_{n+1}(A) & \xrightarrow{I} & \pi_0 T(A)^{C_{p^n}} \\ \downarrow & & \downarrow \\ W_{n+1}(\bar{A}) & \xrightarrow{I} & \pi_0 T(\bar{A})^{C_{p^n}}. \end{array}$$

The vertical maps are both surjective and the upper horizontal map is a ring homomorphism. Hence so is the lower horizontal map. □

Recall from proposition 1.3 that $\rho^{\#}_{C_r} T(A)^{C_r}$ is p-cyclotomic if p does not divide r. In analogy with proposition 2.3 we have short exact sequences

$$0 \to \pi_0 T(A)^{C_r} \xrightarrow{V_{p^s}} \pi_0 T(A)^{C_{p^s r}} \xrightarrow{R_p} \pi_0 T(A)^{C_{p^{s-1} r}} \to 0$$

and induction on the prime divisors of n gives us a natural bijection

$$I\colon \mathbf{W}_{\langle n \rangle}(A) \to \pi_0 T(A)^{C_n}, \qquad I(a_d | d \text{ divides } n) = \sum_{d|n} V_d(\Delta_{n/d}(a_{n/d})).$$

We can argue as above to get

Addendum 2.3. *Let A be a commutative ring. Then*

$$I\colon W_{\langle n \rangle}(A) \to \pi_0 T(A)^{C_n},$$

is natural isomorphism of rings such that $R_r I = I R_r$, $F_r I = I F_r$ and $V_r I = I V_r$, where $\langle n \rangle$ denotes the truncation set of natural numbers which divides n.

3. Topological Cyclic Homology

3.1. This section is inspired by T. Goodwillie's paper [G1].

Let $\mathbb{I}$ be the category where objects are the natural numbers, $\operatorname{ob}\mathbb{I} = \{1, 2, 3, \dots\}$, and with two morphisms $R_r, F_r\colon n \to m$, whenever $n = rm$, subject to the relations

$$\begin{aligned} &R_1 = F_1 = \mathrm{id}_n \\ &R_r R_s = R_{rs}, \quad F_r F_s = F_{rs} \\ &R_r F_s = F_s R_r. \end{aligned}$$

For a prime p, we let $\mathbb{I}_p$ be the full subcategory with $\operatorname{ob}\mathbb{I}_p = \{1, p, p^2, \dots\}$. A cyclotomic spectrum T defines a functor from $\mathbb{I}$ to the category of non-equivariant spectra. Indeed when $n = rm$ we have two maps of non-equivariant spectra

$$R_r, F_r\colon T^{C_n} \to T^{C_m}.$$

The map R_r was defined in (1.2.1) and F_r is the inclusion of fixed points spectra. The relations above is a consequence of the compatibility condition in definition 1.2.

Topological cyclic homology at p, denoted $\mathrm{TC}(T;p)$, was defined in [BHM]. In the present formulation it is the homotopy limit of the restriction of the functor defined above to $\mathbb{I}_p$.

Definition 3.1. If T is a cyclotomic spectrum, then

$$\mathrm{TC}(T;p) = \underset{\overleftarrow{\mathbb{I}_p}}{\operatorname{holim}}\, T^{C_{p^s}}, \quad \mathrm{TC}(T) = \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, T^{C_n}.$$

For a functor with smash product L, $\mathrm{TC}(L) = \mathrm{TC}(T(L))$ and similarly for $\mathrm{TC}(L;p)$.

Remarks. i) The homotopy limit which defines $\mathrm{TC}(T;p)$ may be formed in two steps. First we can take the homotopy limit over F_p (resp. R_p). Since R_p and F_p commute, R_p (resp. F_p) induces a self-map of this homotopy limit, and we may take the homotopy fixed points. More precisely, let

(3.1.1) $$\mathrm{TR}(T;p) = \underset{\overleftarrow{R_p}}{\operatorname{holim}}\, T^{C_{p^s}}, \qquad \mathrm{TF}(T;p) = \underset{\overleftarrow{F_p}}{\operatorname{holim}}\, T^{C_{p^s}},$$

then F_p induces an endomorphism of $\mathrm{TR}(T;p)$ and R_p an endomorphism of $\mathrm{TF}(T;p)$ and

$$\mathrm{TC}(T;p) \cong \mathrm{TR}(T;p)^{h\langle F_p\rangle} \cong \mathrm{TF}(T;p)^{h\langle R_p\rangle}.$$

Here $\langle F_p\rangle$ is the free monoid on F_p and $X^{h\langle F_p\rangle}$ denotes the $\langle F_p\rangle$-homotopy fixed points of X. It is naturally equivalent to homotopy fiber the of $\mathrm{id} - D$, which was the definition used for $\mathrm{TC}(T;p)$ in [BHM].

There is a similar description of $\mathrm{TC}(L)$. Let

(3.1.2) $$\mathrm{TR}(T) = \underset{\overleftarrow{R}}{\operatorname{holim}}\, T^{C_n}, \qquad \mathrm{TF}(T) = \underset{\overleftarrow{F}}{\operatorname{holim}}\, T^{C_n},$$

then

$$\mathrm{TC}(T) = \mathrm{TR}(T)^{hF} = \mathrm{TF}(T)^{hR},$$

where the decoration hF denotes the homotopy fixed set of the multiplicative monoid of natural numbers acting on $\mathrm{TR}(T)$ through the maps F_s, $s \geq 1$.

ii) The inclusion $\mathbb{I}_p \subset \mathbb{I}$ induces a map $\mathrm{TC}(T) \to \mathrm{TC}(T;p)$ which is a (spacewise) fibration. Similarly the inclusions $\{1\} \subset \mathbb{I}_p$ induce fibrations $\mathrm{TC}(T,p) \to T$. In section 3.5 below we prove:

Theorem 3.1. *The projections $\mathrm{TC}(T) \to \mathrm{TC}(T;p)$ induce an equivalence of $\mathrm{TC}(T)$ with the fiber product of the $\mathrm{TC}(T;p)$'s over T. Moreover, the functors agree after p-completion, $\mathrm{TC}(T)^{\wedge}_{p} \simeq \mathrm{TC}(T;p)^{\wedge}_{p}$.*

3.2. We evaluate the realizations of the index categories $\mathbb{I}_p$ and $\mathbb{I}$:

$$|\,\mathbb{I}_p| \simeq S^1, \quad |\,\mathbb{I}\,| \cong \textstyle\prod'_p |\,\mathbb{I}_p| \simeq \prod'_p S^1, \tag{3.2.1}$$

where $\prod'$ denotes the weak product over the prime numbers. Indeed, the full subcategory $\mathbb{I}_{p,1} \subset \mathbb{I}_p$ whose objects are $\{1, p\}$ has realization $|\,\mathbb{I}_{p,1}| \cong S^1$, and by theorem A of [Q] the inclusion functor $K\colon \mathbb{I}_{p,1} \to \mathbb{I}_p$ is a homotopy equivalence provided that the under-categories $(p^n \downarrow K)$ are contractible for all $p^n \in \operatorname{ob} \mathbb{I}_p$. If we write R^rF^s for the object $(p^\epsilon, R^rF^s\colon p^n \to p^\epsilon)$ in $(p^n \downarrow K)$, then $(p^n \downarrow K)$ is the category

$$R^n \leftarrow R^{n-1} \to R^{n-1}D \leftarrow R^{n-2}D \to \ldots \leftarrow F^{n-1} \to F^n.$$

Its realization is $|(p^n \downarrow K)| \cong [0, 2n]$ which is contractible.

Let $S = \{p_1, \ldots, p_s\}$ be a finite set of primes and let $\mathbb{I}_S$ be the full subcategory of $\mathbb{I}$ whose objects are the numbers $p_1^{n_1} \ldots p_s^{n_s}$, $n_i \geq 0$. Then as categories $\mathbb{I} \cong \varinjlim \mathbb{I}_S$ and $\mathbb{I}_S \cong \mathbb{I}_{p_1} \times \ldots \times \mathbb{I}_{p_s}$. Since realization commutes with colimits and finite products we obtaion (3.2.1).

3.3. Let $\Sigma^{-1}\mathbb{Q}/\mathbb{Z}$ be a Moore spectrum with integral homology $\mathbb{Q}/\mathbb{Z}$ concentrated in degree -1, and let T be any spectrum. Then the profinite completion of T is the function spectrum

$$T^{\wedge} = F(\Sigma^{-1}\mathbb{Q}/\mathbb{Z}, T).$$

We may replace $\mathbb{Q}/\mathbb{Z}$ by its p-part $\mathbb{Q}/\mathbb{Z}_{(p)}$ to obtain the p-completion $T_p^{\wedge}$. Since $\mathbb{Q}/\mathbb{Z}$ is the direct sum over the primes p of its p-parts, the profinite completion $T^{\wedge}$ is the product of the p-completions $T_p^{\wedge}$. One proves immediately that the homotopy groups of $T^{\wedge}$ are given by the exact sequence

$$0 \to \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \pi_s T) \to \pi_s(T^{\wedge}) \to \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \pi_{s-1}T) \to 0.$$

Let T be G-spectrum indexed on a trivial G-universe and consider the homotopy orbit spectrum $T_{hC_{p^n}} = T \wedge_{C_{p^n}} EC_{p^n +}$. There are transfer maps

$$t_n^m\colon T_{hC_{p^n}} \to T_{hC_{p^m}}, \quad n > m$$

associated to the projections $T_{hC_{p^m}} \to T_{hC_{p^n}}$, *cf.* [LMS, p.186], and we have the following key lemma.

Lemma 3.3. *Suppose T is a bounded below G-spectrum. Then the homotopy fiber of t_n^m is a p-complete spectrum; in particular it is profinite complete.*

Proof. We can assume that $m = 0$, and we will write $t_n = t_n^0$. Let $\mathfrak{B}_p$ denote the Serre class of abelian p-groups A which are annihilated by some $N_A > 0$. If we can prove that $t_{n*}\colon \pi_* T_{hC_{p^n}} \to \pi_* T$ is an isomorphism modulo $\mathfrak{B}_p$, then the homotopy fiber of t_n will have homotopy groups in $\mathfrak{B}_p$, and therefore it will be p-complete by [BK], p.166.

The composition $\pi_* T \xrightarrow{\mathrm{pr}_*} \pi_* T_{hC_{p^n}} \xrightarrow{t_{n*}} \pi_* T$ is multiplication p^n and therefore an isomorphism modulo $\mathfrak{B}_p$. Hence we may as well show, that pr_* is an isomorphism modulo $\mathfrak{B}_p$. We have the right halfplane homology type spectral sequence (see §4.2 below)

$$E^2_{s,t} = H_s(C_{p^n}; \pi_t T) \Rightarrow \pi_{s+t} T_{hC_{p^n}}.$$

Since the C_{p^n}-action on T is the restriction of the G-action, $\pi_t T$ is a trivial C_{p^n}-module and therefore

$$E^2_{0,t} = \pi_t T, \quad E^2_{s,t} \in \mathfrak{B}_p, \ s > 0.$$

Furthermore the edge homomorphism $E^2_{0,t} \to E^\infty_{0,t}$ is the surjection by pr_* of $\pi_t T$ onto its image in $\pi_t T_{hC_{p^n}}$. It follows that the edge homomorphism is an isomorphism modulo $\mathfrak{B}_p$ and that $E^\infty_{s,t} \in \mathfrak{B}_p$ when $s > 0$. Since T is bounded below, pr_* is an isomorphism modulo $\mathfrak{B}_p$. □

3.4. Let $K\colon I \to J$ be a functor and $\mathbb{C}$ a category which have all limits; then the forgetfull functor $K^*\colon \mathbb{C}^J \to \mathbb{C}^I$ has a leftadjoint R. If $T\colon I \to \mathbb{C}$ is a functor, then the right Kan extension of T along K is the functor

$$RT(j) = \varprojlim((j \downarrow K) \xrightarrow{\mathrm{pr}_1} I \xrightarrow{T} \mathbb{C}),$$

cf. [ML]. We apply this to the inclusion $K\colon \mathbb{I}_1 \to \mathbb{I}$ of the full subcategory on $\{1\}$. The under-category $(n \downarrow K)$ is the discrete category on the set of morphism $\mathbb{I}(n,1)$ and a functor from $\mathbb{I}_1$ to spectra is just a spectrum T. Thus the right Kan extension is simply a product of copies of T,

$$RT(n) = F(\mathbb{I}(n,1)_+, T),$$

where $\mathbb{I}(n,1) = \{R_{n/d}F_d | d \text{ divides } n\}$.

If $n = p_1^{n_1} \dots p_s^{n_s}$ then $\#\mathbb{I}(n,1) = (n_1+1) + \dots + (n_s+1)$.

Lemma 3.4. $\operatorname{holim}_{\overleftarrow{\mathbb{I}}} RT \cong F(|(\mathbb{I} \downarrow 1)|_+, T) \simeq T$.

Proof. Let $\mathbb{S}$ denote the categoty of spectra. We recall from [BK] that $\operatorname{holim}_{\overleftarrow{\mathbb{I}}}$ is right adjoint to the functor

$$- \wedge |(\mathbb{I} \downarrow \cdot)|_+ \colon \mathbb{S} \to \mathbb{S}^{\mathbb{I}}$$

which takes a spectrum T to the diagram $n \mapsto T \wedge |(\mathbb{I} \downarrow n)|_+$. We have the commutative diagram of functors

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{-\wedge|(\mathbb{I}\downarrow 1)|_+} & \mathbb{S} \\ {\scriptstyle -\wedge|(\mathbb{I}\downarrow\cdot)|_+}\big\downarrow & & \big\| \\ \mathbb{S}^{\mathbb{I}} & \xrightarrow{K^*} & \mathbb{S}^{\mathbb{I}_1}. \end{array}$$

All the functors in the square have right adjoints and accordingly these also commute; this proves the first claim. Finally $(1, \mathrm{id}\colon 1 \to 1)$ is terminal object in $(\mathbb{I} \downarrow 1)$, which therefore has contractible realization $|(\mathbb{I} \downarrow 1)|$. □

3.5. From now on T will be a cyclotomic spectrum, *e.g.* $T = T(L)$. The counit of the adjunction above supplies a map of $\mathbb{I}$-diagrams $\epsilon\colon T^{C_-} \to RT(-)$ such that

$$\epsilon_n\colon T^{C_n} \to F(\mathbb{I}(n,1)_+, T)$$

is the adjoint of the 'evaluation' map $\mathbb{I}(n,1)_+ \wedge T^{C_n} \to T$.

Lemma 3.5. *The homotopy fiber of ϵ_n is a profinitely complete spectrum.*

Proof. Suppose first that $n = p^s$ is a prime power. By induction it is enough to show that the iterated homotopy fiber, *i.e.* the homotopy fiber of the induced map on homotopy fibers, of the square

$$\begin{array}{ccc} T^{C_{p^s}} & \xrightarrow{R_p} & T^{C_{p^{s-1}}} \\ {\scriptstyle F_p}\big\downarrow & & {\scriptstyle F_p}\big\downarrow \\ T^{C_{p^{s-1}}} & \xrightarrow{R_p} & T^{C_{p^{s-2}}} \end{array}$$

is profinitely complete. We recall from the theorem 1.2 the cofibration sequence

$$T_{hC_{p^s}} \to T^{C_{p^s}} \xrightarrow{R_p} T^{C_{p^{s-1}}}.$$

It determines the horizontal homotopy fibers in the square above. Furthermore the map induced by the vertical arrows F_p precisely correspond to the transfer map t_s^{s-1}, and so lemma 3.3 shows that the iterated homotopy fiber is a p-complete spectrum.

Next we consider the general case and write $n = p^s k$ with $(k,p) = 1$. Then T^{C_k} is a p-cyclotomic spectrum by proposition 1.3 and the lemma follows by induction over the prime divisors in n. □

We let $\Phi(T)$ denote the fiber of the fibration $\mathrm{TC}(T) \to T$; similarly for $\Phi(T;p)$.

Corollary 3.5. *$\Phi(T)$ is profinitely complete, and $\Phi(T;p)$ is p-complete.*

Proof. Homotopy limits commute with profinitely completion, so by the lemma the homotopy fiber of

$$\epsilon_*\colon \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, T^{C_n} \to \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, RT$$

is a profinitely complete spectrum. Finally under the equivalence of lemma 3.4 we can identify ϵ_* with the projection $\mathrm{TC}(T) \to T$. □

Proof of theorem 3.1. We first show that $\mathrm{TC}(T)^\wedge_p \simeq \mathrm{TC}(T;p)^\wedge_p$ via the projection. To this end we define a new full subcategory $\mathbb{I}_{p'}$ of $\mathbb{I}$. It has as objects the set $\{k|(k,p)=1\}$ of positive integers prime to p. Then $\mathbb{I} \cong \mathbb{I}_p \times \mathbb{I}_{p'}$ so

$$\mathrm{TC}(T) = \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, T^{C_n} \cong \underset{\overleftarrow{\mathbb{I}_p}}{\operatorname{holim}}(\underset{\overleftarrow{\mathbb{I}_{p'}}}{\operatorname{holim}}\, T^{C_{p^s k}}).$$

We may proceed as in lemma 3.5 and show that the fiber spectrum of the projection

$$\underset{\overleftarrow{\mathbb{I}_{p'}}}{\operatorname{holim}}\, T^{C_{p^s k}} \to T^{C_{p^s}}$$

vanishes after p-completion. This proves the last claim in proposition 3.1. We have left to show that the map from $\mathrm{TC}(T)$ to the fiber product over T of $\mathrm{TC}(T;p)$, indexed by the primes p, is an equivalence. This is the same as to show that $\Phi(T) \to \prod_p \Phi(T;p)$ is a homotopy equivalence. Now a profinitely complete spectrum $\Phi(T)$ is equivalent to the product of its p-completions, with p varying over the primes. Since $\Phi(T)^\wedge_p \simeq \Phi(T;p)^\wedge_p$ by the above and $\Phi(T;p)^\wedge_p \simeq \Phi(T;p)$ by corollary 3.5, we are done. □

3.6. We recall from 1.7 that if L is a commutative *FSP* defined on spheres, then the Σ_n-spaces $\mathrm{THH}(L^{(n)};S^n)$ again form a commutative *FSP* defined on spheres. The same holds for the C-fixed sets $\mathrm{THH}(L^{(n)};S^n)^C$ and the restriction and Frobenius maps are Σ_n-equivariant and multiplicative. In particular, the homotopy limit

$$\mathrm{TC}(L^{(n)};S^n) = \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}\, \mathrm{THH}(L^{(n)};S^n)^{C_r}$$

carries a Σ_n-action.

Proposition 3.6. *Let L be a commutative FSP defined on spheres. Then the spaces $\mathrm{TC}(L^{(n)};S^n)$ again form a commutative FSP defined on spheres. The associated spectrum is equivalent to $\mathrm{TC}(L)$.*

Proof. In view of proposition 1.7 it is enough to prove that a homotopy limit of commutative *FSP*'s defined on spheres is again a commutative *FSP* defined on spheres. So let L_i be a J-diagram of *FSP*'s defined on spheres. We define the product on the homotopy limit by

$$\mu_{m,n}\colon \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_{j_1}(S^m) \wedge \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_{j_2}(S^n) \xrightarrow{\text{can}} \underset{\overleftarrow{J\times J}}{\operatorname{holim}}\, L_{j_1}(S^m) \wedge L_{j_2}(S^n)$$
$$\xrightarrow{\Delta^*} \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_j(S^m) \wedge L_j(S^n) \to \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_j(S^{m+n}),$$

where the second map is induced from the diagonal $\Delta\colon J \to J \times J$ and the last map is induced from the multiplication in L_j. The first map is the canonical map, defined as follows: We have the counits

$$\epsilon_j\colon |(J \downarrow j)|_+ \wedge \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_j(S^n) \to L_j(S^n)$$

and since $(J \times J \downarrow (j_1,j_2)) \cong (J \downarrow j_1) \times (J \downarrow j_2)$ we get

$$\epsilon_{j_1} \wedge \epsilon_{j_2}\colon |(J \times J \downarrow (j_1,j_2))|_+ \wedge \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_{j_1}(S^m) \wedge \underset{\overleftarrow{J}}{\operatorname{holim}}\, L_{j_2}(S^n) \to L_{j_1}(S^m) \wedge L_{j_2}(S^n).$$

The canonical map is the adjoint, *cf.* [BK]. Similarly the unit is the adjoint of the composition

$$|(J\downarrow j)|_+\wedge S^n \xrightarrow{\mathrm{pr}_2} S^n \xrightarrow{\mathbf{1}_n} L_j(S^n).$$

We prove that the product is commutative and leave the remaining verifications to the reader. We have the commutative diagram

$$\begin{array}{ccc}
\underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^m)\wedge \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_2}(S^n) & \xrightarrow{\mathrm{can}} & \underset{J\times J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^m)\wedge L_{j_2}(S^n) \\
 & & \Big\downarrow{\scriptstyle \mathrm{Tw}_*} \\
\Big\downarrow{\scriptstyle \mathrm{Tw}} & & \underset{J\times J}{\underleftarrow{\operatorname{holim}}}\, L_{j_2}(S^n)\wedge L_{j_1}(S^m) \\
 & & \Big\downarrow{\scriptstyle \mathrm{tw}^*} \\
\underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^n)\wedge \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_2}(S^m) & \xrightarrow{\mathrm{can}} & \underset{J\times J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^n)\wedge L_{j_2}(S^m),
\end{array} \tag{3.6.1}$$

where Tw permutes the smash factores and where tw is the functor which permutes the two factores in $J\times J$. Indeed, the adjoints of the two compositions $\mathrm{tw}^*\circ\mathrm{Tw}_*\circ\mathrm{can}$ and $\mathrm{can}\circ\mathrm{Tw}$ are equal. Now consider the diagram

$$\begin{array}{ccccc}
\underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^m)\wedge \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_2}(S^n) & \xrightarrow{\Delta^*\circ\mathrm{can}} & \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_j(S^m)\wedge L_j(S^n) & \xrightarrow{\mu_{m,n*}} & \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_j(S^{m+n}) \\
\Big\downarrow{\scriptstyle \mathrm{Tw}} & & \Big\downarrow{\scriptstyle \mathrm{Tw}_*} & & \Big\downarrow{\scriptstyle \sigma_{m,n*}} \\
\underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_1}(S^n)\wedge \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_{j_2}(S^m) & \xrightarrow{\Delta^*\circ\mathrm{can}} & \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_j(S^n)\wedge L_j(S^m) & \xrightarrow{\mu_{n,m*}} & \underset{J}{\underleftarrow{\operatorname{holim}}}\, L_j(S^{n+m})
\end{array}$$

The commutative of the square on the left follows from (3.6.1) and the fact that $\Delta=\Delta\circ\mathrm{tw}$ as functors from J to $J\times J$. Finally, the commutativity of L_j implies that the right hand square is commutative. This completes the proof. □

Given any commutative *FSP* L we have from proposition 3.6 a sequence of spectra

$$\mathrm{TC}(L),\ \ \mathrm{TC}^2(L),\ \ \mathrm{TC}^3(L),\dots$$

upon iterating the construction. In view of theorem B of the introduction and the calculation of $\mathrm{TC}(\mathbb{Z}_p)$ in [BM] it would seem a very interesting question in homotopy theory to determine the iterates $\mathrm{TC}^n(\mathbb{F}_p)$. In particular one may wonder about the socalled chromatic filtration of $\mathrm{TC}^2(\mathbb{Z}_p)$ or $\mathrm{TC}^3(\mathbb{F}_p)$.

4. Topological cyclic homology of perfect fields

4.1. To each ring R there is associated a functor with smash product, which we denote $\widetilde{R}$. It takes a based space X to the configuration space of particles in X with labels in R, *i.e.* the space of formal linear combinations $\sum r_i x_i$ modulo the relation $r\cdot * = 0\cdot * = *$. It is a generalized Eilenberg-MacLane space with

$$\pi_*\widetilde{R}(X)\cong \widetilde{H}_*(X;R),$$

the reduced singular homology groups of X with R-coefficients.

In this paragraph we evaluate $\mathrm{TC}(\widetilde{R})$ in the case where $R = k$ is a perfect field of characteristic $p > 0$. We note that $\mathrm{TC}(\widetilde{k}) \simeq \mathrm{TC}(\widetilde{k}; p)$ by 3.5. For $T(\widetilde{k})$ and its fixed sets are p-complete by theorem 1.2. In the sequel we write $T(R)$ and $\mathrm{TC}(R)$ instead of $T(\widetilde{R})$ and $\mathrm{TC}(\widetilde{R})$.

We begin with the basic calculation when $k = \mathbb{F}_p$ is the prime field. The general case follows by a descent argument given in section 4.5 below. The strategy for obtaining information about $\mathrm{TC}(\mathbb{F}_p)$ is to compare the fixed sets which defines it with the corresponding homotopy fixed sets.

For any C-spectrum $T \in C\mathcal{SU}$, with C finite, there is a norm cofibration sequence of spectra, which we now recall. Following [GM] one defines

$$\begin{array}{lll} T_{hC} & = j^*T \wedge_C EC_+ & \text{(homotopy orbit)} \\ T^{hC} & = F(EC_+, T)^C & \text{(homotopy fixed points)} \\ \hat{\mathbb{H}}(C;T) & = [\widetilde{E}C \wedge F(EC_+, T)]^C & \text{(Tate spectrum)} \end{array}$$

Here $j: \mathcal{U}^C \to \mathcal{U}$, $\widetilde{E}C$ is the unreduced suspension of EC (as in §1) and the smash product in the definition of $\hat{\mathbb{H}}$ takes place in $C\mathcal{SU}$, *i.e.*

$$\hat{\mathbb{H}}(C;T)(V) = \varinjlim_{W \subset \mathcal{U}} F(S^{W-V}, \widetilde{E}C \wedge F(EC_+, T(W)))^C$$

One has

$$[F(EC_+, T) \wedge EC_+]^C \simeq [T \wedge EC_+]^C \simeq T_{hC},$$

cf. the proof of proposition 1.1. Thus one can smash the cofibration sequence of C-spaces

(4.1.1) $$EC_+ \to S^0 \to \widetilde{E}C$$

with $F(EC_+, T) \in C\mathcal{SU}$ and take C-fixed points to get the 'norm cofibration sequence' of [GM],

$$T_{hC} \xrightarrow{N^h} T^{hC} \xrightarrow{R^h} \hat{\mathbb{H}}(C;T).$$

We now assume that $T \in G\mathcal{SU}$ (where, we remember $G = S^1$), and let C be a cyclic p-subgroup. In proposition 1.1 we identified $[\widetilde{E}C \wedge T]^C$ with $(\Phi^{C_p}T)^{C/C_p}$. Therefore, we may smash the obvious inclusion

$$\gamma: T \to F(EG_+, T)$$

to obtain a C/C_p-equivariant map $\hat{\gamma}: \Phi^{C_p}T \to \hat{\mathbb{H}}(C_p;T)$ and taking fixed sets and using that $\hat{\mathbb{H}}(C_p;T)^{C/C_p} = \hat{\mathbb{H}}(C;T)$ we obtain from (4.1.1) a cofibration diagram

(4.1.2) $$\begin{array}{ccccc} T_{hC} & \xrightarrow{N} & T^C & \longrightarrow & (\Phi^{C_p}T)^{C/C_p} \\ \| & & \downarrow \Gamma_C & & \downarrow \hat{\Gamma}_C \\ T_{hC} & \xrightarrow{N^h} & T^{hC} & \xrightarrow{R^h} & \hat{\mathbb{H}}(C;T). \end{array}$$

For a cyclotomic spectrum $\rho^{\#}_{C_p}\Phi^{C_p}T \simeq_G T$ and (4.1.2) reduces to

Proposition 4.1. ([BM]) *For a (p-)cyclotomic spectrum T there is a commutative diagram*

$$\begin{array}{ccccc} T_{hC_{p^n}} & \xrightarrow{N} & T^{C_{p^n}} & \xrightarrow{R} & T^{C_{p^{n-1}}} \\ \| & & \downarrow \Gamma_n & & \downarrow \hat{\Gamma}_n \\ T_{hC_{p^n}} & \xrightarrow{N^h} & T^{hC_{p^n}} & \xrightarrow{R^h} & \hat{\mathbb{H}}(C_{p^n};T) \end{array}$$

in which the rows are cofibration sequences of non-equivariant spectra. □

The point of this is that there are spectral sequences

$$
(4.1.3) \qquad \begin{array}{rcll} \hat{E}^2_{r,s}(C;T) & = & \hat{H}^{-r}(C;\pi_s T) & \Rightarrow \pi_{r+s}\hat{\mathbb{H}}(C;T) \\ E^2_{r,s}(T^{hC}) & = & H^{-r}(C;\pi_s T) & \Rightarrow \pi_{r+s}T^{hC} \\ E^2_{r,s}(T_{hC}) & = & H_r(C;\pi_s T) & \Rightarrow \pi_{r+s}T_{hC} \end{array}
$$

which in favorable cases can be used to completely calculate the homotopy exact sequence of the norm fibration sequence, *cf.* [BM], §2. The spectral sequences are associated with the skeleton filtration, and for $\hat{E}^r$ a filtration due to Greenlees. One may then attempt a calculation of the actual fixed points, and hence $\mathrm{TC}(T;p)$, starting with a calculation of

$$\hat{\Gamma}_1\colon \pi_* T \to \pi_*\hat{\mathbb{H}}(C_p;T).$$

This was the strategy used in [BM] for $T = T(\widetilde{\mathbb{Z}_p})$ and will below be used for $T = T(\widetilde{\mathbb{F}_p})$.

The spectral sequences in (4.1.3) are strongly interrelated. For any C-spectrum T there is a map of spectral sequences

$$
(4.1.4) \qquad R^{h,r}\colon E^r_{s,t}(T^{hC}) \to \hat{E}^r_{s,t}(C;T)
$$

which is an isomorphism for $r = 2$ and $s < 0$ and an epimorphism for $r > 2$ and $s < 0$. Similarly, there is a map of degree -1

$$
(4.1.5) \qquad \partial^r\colon \hat{E}^r_{s,t}(C;T) \to E^r_{s-1,t}(T_{hC})
$$

which is an isomorphism for $r = 2$ and $s \geq 2$ and a monomorphism for $r > 2$ and $s \geq 2$. The situation for $r = 2$ and $s = 0, 1$ is described by the exact sequence

$$0 \to \hat{E}^2_{1,*}(C;T) \xrightarrow{\partial^2} E^2_{0,*}(T_{hC}) \xrightarrow{N} E^2_{0,*}(T^{hC}) \xrightarrow{R^{h,2}} \hat{E}^2_{0,*}(C;T) \to 0,$$

where N is the norm map $N\colon H_0(C;\pi_* T) \to H^0(C;\pi_* T)$. For $r > 2$ the relationship is explained in [BM], §2.

4.2. We now recall Bökstedt's and Breen's basic result on $\pi_* T(\mathbb{F}_p)$ and sketch briefly the proof, in Bökstedt's formulation.

Since $T(R)$ is the realization of a simplicial space it has a skeleton filtration, and there is a first quadrant spectral sequence

$$E^2(R) = \mathrm{HH}_*(\mathcal{A}_R) \Rightarrow H_*(T(R);\mathbb{F}_p)$$

where $\mathcal{A}_R = H_*(HR;\mathbb{F}_p)$ and $H_*(-)$ is spectrum homology. When $R = \mathbb{F}_p$, $\mathcal{A}_R$ is the dual Steenrod algebra, *i.e.* $\mathcal{A}_{\mathbb{F}_p} = \mathcal{A}$ where

$$\mathcal{A} = \begin{cases} S_{\mathbb{F}_p}\{\xi_1, \xi_2, \dots\} \otimes \Lambda_{\mathbb{F}_p}\{\tau_0, \tau_1, \dots\}, & p \text{ odd} \\ S_{\mathbb{F}_p}\{\xi_1, \xi_2, \dots\}, & p = 2. \end{cases}$$

Here $\deg \xi_i = 2(p^i - 1)$ (or $2^i - 1$ if $p = 2$), $\deg \tau_i = 2p^i - 1$ and $S_{\mathbb{F}_p}$ resp. $\Lambda_{\mathbb{F}_p}$ denotes the symmetric resp. the exterior algebra over $\mathbb{F}_p$. Since $\mathcal{A}$ is a connected Hopf algebra one has with $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}$

$$\mathrm{HH}_*(\mathcal{A}) = \mathrm{Tor}^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) \cong \mathcal{A} \otimes \mathrm{Tor}^{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p),$$

see [CE] p. 194, and

$$\mathrm{Tor}^{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \cong \begin{cases} \Lambda_{\mathbb{F}_p}\{\sigma\xi_1, \sigma\xi_2, \dots\}, & p = 2 \\ \Lambda_{\mathbb{F}_p}\{\sigma\xi_1, \sigma\xi_2, \dots\} \otimes \Gamma_{\mathbb{F}_p}\{\sigma\tau_0, \sigma\tau_1, \dots\}, & p \text{ odd} \end{cases}$$

where $\Gamma_{\mathbb{F}_p}\{-\}$ is the divided power algebra, *e.g.*

$$\Gamma_{\mathbb{F}_p}\{\sigma\tau_i\} \cong \bigoplus_{j\geq 0} S_{\mathbb{F}_p}\{\gamma_{p^j}(\sigma\tau_i)\}/(\gamma_{p^j}(\sigma\tau_i)^p).$$

The (bi-)degrees of the generators are

$$\deg(\sigma\xi_i) = (1, 2(p^i-1)) \quad (\text{resp. } (1, 2^i-1) \text{ for } p=2)$$
$$\deg(\gamma_{p^j}(\sigma\tau_i)) = (p^j, p^j(2p^i-1)).$$

Let $H\mathbb{F}_p \to T(\mathbb{F}_p)$ be the inclusion of the 0-skeleton and consider the composition

$$\sigma\colon S^1_+ \wedge H\mathbb{F}_p \to S^1_+ \wedge T(\mathbb{F}_p) \xrightarrow{\mu} T(\mathbb{F}_p). \tag{4.2.1}$$

Then $\sigma\xi_i$ and $\sigma\tau_i$ are the images under σ_* of $[S^1]\otimes\xi_i$ and $[S^1]\otimes\tau_i$. There are homology operations in $H_*(T(\mathbb{F}_p))$, which commute with σ. The homology operations in $H_*(H\mathbb{F}_p)$ were examined by Steinberger in [BMMS] chap. III, theorem 2.3, and twenty years before by L. Kristensen (unpublished). The result we need is that

$$Q^{p^i}(\tau_i) = \tau_{i+1}, \quad Q^{p^i}(\xi_i) = \xi_{i+1}, \quad \beta\tau_i = \xi_i.$$

Here τ_i and ξ_i are not the usual Milnor generators, but the images of these under the canonical anti automorphism (antipode) of $\mathcal{A}$.

For degree reasons there are no differentials in the spectral sequence when $p = 2$. In the case of odd primes the first possible non-zero differential is d^{p-1}. Bökstedt proves in [B1] that

$$d^{p-1}(\gamma_{p^j}(\sigma\tau_i)) = (\gamma_{p^{j-1}}(\sigma\tau_i)\cdot\ldots\cdot\gamma_p(\sigma\tau_i))^{p-1}\cdot\sigma\xi_{i+1}.$$

This can be viewed as a 'Kudo principle' since $\sigma\xi_{i+1} = \beta Q^{p^i}(\sigma\tau_i)$ by the above. In any case one gets for odd p

$$E^p = \mathcal{A}\otimes S_{\mathbb{F}_p}\{\sigma\tau_i | i\geq 0\}$$

and for degree reasons $E^p = E^\infty$. Finally the homology operations solve the extension problems,

$$(\sigma\tau_i)^p = Q^{p^i}(\sigma\tau_i) = \sigma Q^{p^i}(\tau_i) = \sigma\tau_{i+1}$$

so that

$$H_*(T(\mathbb{F}_p);\mathbb{F}_p) \cong \mathcal{A}\otimes S_{\mathbb{F}_p}\{\sigma\tau_0\}$$

and hence $\pi_*T(\mathbb{F}_p) \cong S_{\mathbb{F}_p}\{\sigma\tau_0\}$.

Let $[S^1]\in\pi_1^S(S^1_+)$ be the image of the generator in $\pi_2^S(S^2)$ under the boundary map $\partial\colon\pi_2^S(S^2)\to\pi_1^S(S^1_+)$ of the cofibration $S^1_+\to S^0\to S^2$. Let $\bar\sigma\in\pi_2(T(\mathbb{F}_p);\mathbb{F}_p)$ be the image of $[S^1]\wedge\tau_0$ under the map in (4.2.1), and let $\sigma\in\pi_2T(\mathbb{F}_p)$ be the preimage of $\bar\sigma$ under the reduction to $\mathbb{F}_p$-coefficients, which is an isomorphism. We have proved

Theorem 4.2. ([Br], [B]) $\pi_*T(\mathbb{F}_p) = S_{\mathbb{F}_p}\{\sigma\}$. □

The above calculation shows that $T(\mathbb{F}_p)$ is a wedge of Eilenberg-MacLane spectra. But this is also clear from the beginning because the composition

$$T(R)\simeq S^0\wedge T(R)\to HR\wedge T(R)\to T(R)\wedge T(R)\xrightarrow{\mu} T(R)$$

is homotopic to the identity, so that $T(R)$ is a retract of $HR\wedge T(R)$ which is always a wedge of Eilenberg-MacLane spectra.

4.3. We return to the spectral sequences of 4.1 for $\pi_*(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p); \mathbb{F}_p)$;

$$\hat{E}^2 = \hat{H}^*(C_{p^n}; \pi_*(T(\mathbb{F}_p); \mathbb{F}_p)) = \Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes \Lambda_{\mathbb{F}_p}\{e_1\} \otimes S_{\mathbb{F}_p}\{\bar{\sigma}\}, \tag{4.3.1}$$

where $\deg u_n = (-1, 0)$, $\deg t = (-2, 0)$, $\deg e_1 = (0, 1)$ and $\deg \bar{\sigma} = (0, 2)$. Indeed, the Bockstein exact sequence which relates integral and modulo p homotopy groups gives $\pi_*(T(\mathbb{F}_p); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}\{e_1\} \otimes S_{\mathbb{F}_p}\{\sigma\}$. The Bockstein on e_1 is 1, so that the odd degree homotopy groups map isomorphically onto the even dimensionally ones. We also consider the spectral sequence for $\pi_*\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p))$, (integral homotopy groups)

$$\hat{\boldsymbol{E}}^2 = \hat{H}^*(C_{p^n}; \pi_* T(\mathbb{F}_p)) = \Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\}. \tag{4.3.2}$$

There is a map of spectral sequences res: $\hat{\boldsymbol{E}}^r \to \hat{E}^r$ which is injective for $r = 2$. Both spectral sequences are homology type and lie in the second quadrant, $\hat{\boldsymbol{E}}^r$ is multiplicative and $\hat{E}^r$ is a module over $\hat{\boldsymbol{E}}^r$.

Lemma 4.3. *The non-zero differentials in $\hat{E}^r$ are generated from $d^2 e_1 = t\bar{\sigma}$ in the module structure over $\hat{\boldsymbol{E}}^r$. In particular,*

$$\pi_*(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)); \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\}.$$

Proof. For degree reasons there are no d^2-differentials in $\hat{\boldsymbol{E}}^r$. Therefore, if $d^2 e_1 = t\bar{\sigma}$ we get

$$\hat{E}^3 = \Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\}$$

and there can be no further differentials. The idea of the proof is to compare with the spectral sequence which calculates $\pi_*(\hat{\mathbb{H}}(G; T(\mathbb{F}_p)); \mathbb{F}_p)$. It has E^2-term

$$\hat{H}^*(BG; \pi_*(T(\mathbb{F}_p); \mathbb{F}_p)) \cong S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes \Lambda_{\mathbb{F}_p}\{e_1\} \otimes S_{\mathbb{F}_p}\{\bar{\sigma}\},$$

and there is a map from this spectral sequence to $\hat{E}^r$ which injects the E^2-term. The differential $d^2\colon E^2_{0,1} \to E^2_{2,2}$ in this spectral sequence is the composite

$$\pi_1(T(\mathbb{F}_p); \mathbb{F}_p) \xrightarrow{[S^1]} \pi_2(S^1_+ \wedge T(\mathbb{F}_p); \mathbb{F}_p) \xrightarrow{\mu} \pi_2(T(\mathbb{F}_p); \mathbb{F}_p),$$

cf. [BM], §5. The first map is exterior multiplication by $[S^1] \in \pi_1^S(S^1_+)$ and the second map is induced by the S^1-action on $T(\mathbb{F}_p)$. Hence, $d^2 e_1 = t\bar{\sigma}$ as claimed. □

Corollary 4.3. *The integral homotopy groups $\pi_*\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p))$ are cyclic $\mathbb{Z}_p$-modules.*

Proof. We may compare the spectral sequence (4.3.2) with the spectral sequence for $\pi_*\hat{\mathbb{H}}(G; T(\mathbb{F}_p))$ to see that t and σ are permanent cycles. Hence there is a differential

$$d^{2r+1} u_n = t^{r+1}\sigma^r, \tag{4.3.3}$$

for some $r \geq 1$, or there are no differentials at all. (We prove in lemma 4.4 that $r = n$.) On the other hand, the mod p spectral sequence shows that the extensions in the passage from $\hat{\boldsymbol{E}}^\infty$ to the actual homotopy groups are maximally non-trivial. Hence the claim. □

We use that $\hat{\Gamma}_1\colon T(\mathbb{F}_p) \to \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))$ is a map of ring spectra to determine the differential (4.3.3). Since $\hat{\Gamma}$ preserves the unit, and as $\pi_0 T(\mathbb{F}_p) = \mathbb{F}_p$ and $\pi_0\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))$ is cyclic,

$$\hat{\Gamma}_*\colon \pi_0 T(\mathbb{F}_p) \to \pi_0\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))$$

is an isomorphism. This can only happen if $r = 1$ in (4.3.3), that is, if

$$d^3 u_1 = t^2\sigma$$

in (4.3.2). It is then easy to solve the spectral sequence to get

$$\pi_*\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)) = S_{\mathbb{F}_p}\{\hat{\sigma}, \hat{\sigma}^{-1}\} \tag{4.3.4}$$

where $\hat{\sigma}$ is a generator of degree 2.

Proposition 4.3. *The map $\hat{\Gamma}_1 \colon T(\mathbb{F}_p) \to \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))$ induces an equivalence of connective covers.*

Proof. Since $\hat{\Gamma}$ is multiplicative theorem 4.2 and (4.3.4) show that it is enough to prove that

$$\hat{\Gamma}_* \colon \pi_2 T(\mathbb{F}_p) \to \pi_2 \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))$$

is an isomorphism. We have $T(\mathbb{F}_p) \simeq_G \rho^{\#}_{C_p} \Phi^{C_p} T(\mathbb{F}_p)$ and the following commutative square of G/C_p-spectra,

$$\begin{array}{ccc} \Phi^{C_p} T(\mathbb{F}_p) & \xrightarrow{\partial} & \Sigma T(\mathbb{F}_p)_{hC_p} \\ \downarrow{\scriptstyle \hat{\gamma}^{C_p}} & & \| \\ \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)) & \xrightarrow{\partial^h} & \Sigma T(\mathbb{F}_p)_{hC_p}, \end{array}$$

cf. (4.1.2). Thus we may instead prove that

$$\partial_* \colon \pi_2 \Phi^{C_p} T(\mathbb{F}_p) \to \pi_1 T(\mathbb{F}_p)_{hC_p}$$

is an isomorphism. One has (by the spectral sequence) $\pi_i T(\mathbb{F}_p)_{hC_p} \cong \mathbb{F}_p$ for $i = 0, 1$, see (2.3.1).

Theorem 4.2 translate under the equivalence $T(\mathbb{F}_p) \simeq_G \rho^{\#}_{C_p} \Phi^{C_p} T(\mathbb{F}_p)$ to the statement that

$$\bar{\mu}_* \colon \pi_1^S(G/C_{p+}) \otimes \pi_1(\Phi^{C_p} T(\mathbb{F}_p); \mathbb{F}_p) \to \pi_2(\Phi^{C_p} T(\mathbb{F}_p); \mathbb{F}_p)$$

is surjective, and the generator of the right hand group is the mod p reduction of the generator of the integral group $\pi_2 \Phi^{C_p} T(\mathbb{F}_p)$. Since ∂ is a G/C_p-equivariant map it is therefore enough if we prove that the two maps

(a) $$\bar{\partial}_* \colon \pi_1(\Phi^{C_p} T(\mathbb{F}_p); \mathbb{F}_p) \to \pi_0(T(\mathbb{F}_p)_{hC_p}; \mathbb{F}_p)$$

(b) $$\bar{\mu}_* \colon \pi_1^S(G/C_{p+}) \otimes \pi_0 T(\mathbb{F}_p)_{hC_p} \to \pi_1 T(\mathbb{F}_p)_{hC_p}$$

are epimorphisms. Claim (a) follows from the diagram

$$\begin{array}{ccccc} \pi_1(\Phi^{C_p} T(\mathbb{F}_p); \mathbb{F}_p) & \xrightarrow{\bar{\partial}_*} & \pi_0(T(\mathbb{F}_p)_{hC_p}; \mathbb{F}_p) & \xrightarrow{\bar{N}_*} & \pi_0(T(\mathbb{F}_p)^{C_p}; \mathbb{F}_p) \\ \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow \\ \pi_1 \Phi^{C_p} T(\mathbb{F}_p) & \xrightarrow{\partial_*} & \pi_0 T(\mathbb{F}_p)_{hC_p} & \xrightarrow{N_*} & \pi_0 T(\mathbb{F}_p)^{C_p}. \end{array}$$

because $\pi_0 T(\mathbb{F}_p)^{C_p} \cong \mathbb{Z}/p^2$ and $\pi_1(\Phi^{C_p} T(\mathbb{F}_p); \mathbb{F}_p) = \mathbb{Z}/p$ by theorems 2.3 and 4.2. To prove (b) we note that the map

$$T(\mathbb{F}_p) \wedge_{C_p} G_+ \to T(\mathbb{F}_p) \wedge_{C_p} EG_+,$$

given by the inclusion $G \subset EG$ induces an isomorphism on $\pi_i(-)$ for $i = 0, 1$, and use the G-homeomorphism

$$T(\mathbb{F}_p) \wedge_{Cp} G_+ \cong |T(\mathbb{F}_p)| \wedge G/C_{p+}$$

where the bars on the right indicate $T(\mathbb{F}_p)$ with trivial G-action. □

Addendum 4.3. *The maps*

$$\Gamma_n \colon T(\mathbb{F}_p)^{C_{p^n}} \to T(\mathbb{F}_p)^{hC_{p^n}}, \qquad \hat{\Gamma}_n \colon T(\mathbb{F}_p)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p))$$

induce equivalences of connective covers.

Proof. Since the spectra are all p-complete it is enough to show that the maps induce isomorphism on $\pi_*(-; \mathbb{F}_p)$ in non-negative degrees. For $n = 1$, this follows from the lemma and from a 5-lemma argument based on proposition 4.1. In the general case we have $\hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p)) = \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{C_{p^{n-1}}}$ and $\hat{\Gamma}_{n-1} = \hat{\gamma}^{n-1}$, where $\hat{\gamma}$ is the G-equivariant map

$$T(\mathbb{F}_p) \to \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p)).$$

We can now compare with the homotopy fixed point situation via the diagram

$$\begin{array}{ccc} T(\mathbb{F}_p)^{C_{p^{n-1}}} & \xrightarrow{\Gamma_{n-1}} & T(\mathbb{F}_p)^{hC_{p^{n-1}}} \\ \downarrow{\scriptstyle \hat{\gamma}^{C_{p^{n-1}}}} & & \downarrow{\scriptstyle \hat{\gamma}^{hC_{p^{n-1}}}} \\ \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{C_{p^{n-1}}} & \xrightarrow{G} & \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{hC_{p^{n-1}}}. \end{array}$$

Since $\hat{\gamma} = \hat{\Gamma}_1$ is a non-equivariant equivalence on connective covers by the lemma, so is $\hat{\gamma}^{hC_{p^{n-1}}}$. Inductively, Γ_{n-1} may be assumed to be an equivalence on connective covers, so it remains to show that G is. There is a commutative diagram

$$\begin{array}{ccc} \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))_{hC_{p^{n-1}}} & \xrightarrow{N} & \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{C_{p^{n-1}}} \\ \| & & \downarrow{\scriptstyle G} \\ \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))_{hC_{p^{n-1}}} & \xrightarrow{N^h} & \rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{hC_{p^{n-1}}} \end{array}$$

and we claim

(i) $$\pi_*(\hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p));\mathbb{F}_p) \cong \pi_*(\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{hC_{p^{n-1}}};\mathbb{F}_p)$$

(ii) $$\hat{\mathbb{H}}(C_{p^{n-1}};\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))) \simeq 0$$

Given these claims, (ii) and the norm cofibration sequence for the $C_{p^{n-1}}$-spectrum $\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))$ show that N^h is an equivalence, and hence that $\pi_*(G;\mathbb{F}_p)$ is a surjection of abstractly isomorphic finite groups, thus an isomophism.

It remains to prove (i) and (ii). This uses the spectral sequences of (4.1.3),

$$\begin{aligned} H^*(C_{p^{n-1}};\pi_*(\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p));\mathbb{F}_p)) &\Rightarrow \pi_*(\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{hC_{p^{n-1}}};\mathbb{F}_p) \\ \hat{H}^*(C_{p^{n-1}};\pi_*(\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p));\mathbb{F}_p)) &\Rightarrow \pi_*(\hat{\mathbb{H}}(C_{p^{n-1}};\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p)));\mathbb{F}_p) \end{aligned}$$

We have already proved that

$$\pi_*(\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p));\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}\{\hat{e}_1\} \otimes S_{\mathbb{F}_p}\{\hat{\sigma},\hat{\sigma}^{-1}\}$$

with $\deg \hat{e}_1 = 1$, $\deg \hat{\sigma} = 2$. The two E^2-terms are consequenctly

$$\begin{aligned} E^2 &= \Lambda_{\mathbb{F}_p}\{u_{n-1}\} \otimes S_{\mathbb{F}_p}\{t\} \otimes \Lambda_{\mathbb{F}_p}\{\hat{e}_1\} \otimes S_{\mathbb{F}_p}\{\hat{\sigma},\hat{\sigma}^{-1}\} \\ \hat{E}^2 &= \Lambda_{\mathbb{F}_p}\{u_{n-1}\} \otimes S_{\mathbb{F}_p}\{t,t^{-1}\} \otimes \Lambda_{\mathbb{F}_p}\{\hat{e}_1\} \otimes S_{\mathbb{F}_p}\{\hat{\sigma},\hat{\sigma}^{-1}\}. \end{aligned}$$

Combining lemma 4.3 and proposition 4.3 one has that $d^2(\hat{e}_1) = t\hat{\sigma}$ in both cases. This differential and its multiplicative consequences are the only ones. Hence

$$E^3 = \Lambda_{\mathbb{F}_p}\{u_{n-1}\} \otimes S_{\mathbb{F}_p}\{\hat{\sigma},\hat{\sigma}^{-1}\}$$

and $E^3 = E^\infty$, so $\pi_*(\hat{\mathbb{H}}(C_p;T(\mathbb{F}_p))^{hC_{p^{n-1}}};\mathbb{F}_p)$ has a copy of $\mathbb{F}_p$ in each degree. Now compare with corollary 4.3 to prove (i). For (ii), note that

$$d^2(\hat{e}_1 t^{-1}\hat{\sigma}^{-1}) = 1$$

so that $\hat{E}^3 = 0$. □

Remark 4.4. S. Tsalides, [T], has given a quite different and more general proof of addendum 4.3, assuming lemma 4.3.

4.4. We can now give a complete description of the fixed point structure of $T(\mathbb{F}_p)$. We begin by solving the spectral sequences in (4.3.2).

Lemma 4.4. *In the spectral sequence* $\hat{\boldsymbol{E}}^r$ *which converges to* $\pi_*\hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p))$ *the differentials are multiplicatively generated from* $d^{2n+1}u_n = t^{n+1}\sigma^n$ *and the fact that* t *and* σ *are permanent cycles. In particular,*

$$\pi_*\hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p)) = S_{\mathbb{Z}/p^n}\{\hat{\sigma},\hat{\sigma}^{-1}\},$$

where $\deg\hat{\sigma} = 2$.

Proof. We may combine addendum 4.3 and theorem 2.3 to get

$$\pi_0\hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p)) \cong \pi_0 T(\mathbb{F}_p)^{C_{p^{n-1}}} \cong \mathbb{Z}/p^n.$$

Now the claim for the differentials follows from corollary 4.3 and its proof. We get

$$\hat{\boldsymbol{E}}^{2n+2} = S_{\mathbb{F}_p}\{t,t^{-1},\sigma\}/(t^{n+1}\sigma^n)$$

and since all elements are in even total degree there are no further differentials. □

Proposition 4.4. *The integral homotopy groups of the fixed point spectra* $T(\mathbb{F}_p)^{C_{p^n}}$ *is a copy of* $\mathbb{Z}/p^{n+1}$ *in each positive even degree,*

$$\pi_*T(\mathbb{F}_p)^{C_{p^n}} = S_{\mathbb{Z}/p^{n+1}}\{\sigma_n\},$$

where $\deg\sigma_n = 2$*. Moreover,* $F(\sigma_n) = \sigma_{n-1}$*,* $V(\sigma_{n-1}) = p\sigma_n$ *and* $R(\sigma_n) = p\lambda_n\sigma_{n-1}$ *where* $\lambda_n \in \mathbb{Z}/p^n$ *is a unit.*

Proof. The claim for the homotopy groups is immediate from addendum 4.3 and the lemma. We have the following commutative square

$$\begin{array}{ccc} T(\mathbb{F}_p)^{C_{p^n}} & \xrightarrow{F} & T(\mathbb{F}_p)^{C_{p^{n-1}}} \\ \downarrow{\scriptstyle \hat{\gamma}^{C_{p^n}}} & & \downarrow{\scriptstyle \hat{\gamma}^{C_{p^{n-1}}}} \\ \hat{\mathbb{H}}(C_{p^n};T(\mathbb{F}_p)) & \xrightarrow{F^h} & \hat{\mathbb{H}}(C_{p^{n-1}};T(\mathbb{F}_p)), \end{array}$$

where the vertical maps are the equivalences of addendum 4.3 and F^h is the obvious inclusion of fixed sets. It induces the restriction map in Tate cohomology,

$$\text{res}\colon \hat{H}^*(C_{p^n};\pi_*T(\mathbb{F}_p)) \to \hat{H}^*(C_{p^{n-1}};\pi_*T(\mathbb{F}_p))$$

on the E^2-term of the spectral sequences $\hat{\boldsymbol{E}}^r$. Since this is an isomorphism in even degrees it follows that we can choose the generators σ_n such that $F\sigma_n = \sigma_{n-1}$. Next, $V(\sigma_{n-1}) = VF(\sigma_n) = p\sigma_n$. Finally, the calculation of R follows from the exact sequence

$$\pi_2T(\mathbb{F}_p)^{C_{p^n}} \xrightarrow{R} \pi_2T(\mathbb{F}_p)^{C_{p^{n-1}}} \xrightarrow{\partial} \pi_1T(\mathbb{F}_p)_{hC_{p^n}} \xrightarrow{N} \pi_1T(\mathbb{F}_p)^{C_{p^n}}$$

since $\pi_1T(\mathbb{F}_p)_{hC_{p^n}} \cong \mathbb{F}_p$ and $\pi_1T(\mathbb{F}_p)^{C_{p^n}} = 0$. □

4.5. In this section we extend the extend proposition 4.4 to any perfect field k of positive characteristic.

Lemma 4.5. *If* k *is a perfect field of positive characteristic then* $\mathrm{HH}_*(k) = k$.

Proof. We choose a transcendence basis $\{X_i | i \in I\}$ of k over $\mathbb{F}_p$. Since k is perfect it contains as a subfield the field

$$l = \varinjlim_r \mathbb{F}_p(X_i^{p^{-r}} \mid i \in I).$$

Moreover, k is a separable algebraic extension of l. For l is perfect by construction, and any algebraic extension of a perfect field is separable. We may write $k = \varinjlim k_\alpha$ where the colimit runs over the finite extensions $l \subset k_\alpha \subset k$. Each k_α/l is a finite separable extensions and hence étale. Therefore, $\mathrm{HH}_*(k_\alpha) \cong k_\alpha \otimes_l \mathrm{HH}_*(l)$, [WG], and since Hochschild homology commutes with filtered colimits,

$$\mathrm{HH}_*(k) \cong k \otimes_l \mathrm{HH}_*(l).$$

Now $\mathrm{HH}_*(l) = l$. Indeed, by [HKR]

$$\mathrm{HH}_*(\mathbb{F}_p[X_i|\ i \in I]) \cong \Omega^*_{\mathbb{F}_p[X_i|\ i\in I]/\mathbb{F}_p}$$

and both sides commute with filtered colimits and locationzation, so $\mathrm{HH}_*(l) \cong \Omega^*_{l/\mathbb{F}_p}$. Now since l is perfect $\Omega_{l/\mathbb{F}_p} = 0$, as $dx = d(y^p) = py^{p-1}dy = 0$. ☐

We thank Chuck Weibel for help with the argument above.

Corollary 4.5. $\pi_* T(k) \cong k \otimes \pi_* T(\mathbb{F}_p)$.

Proof. We consider the spectral sequence $E^r(R)$ of 4.2 with $R = k$. The inclusion $\mathbb{F}_p \to k$ defines $\mathcal{A} \to \mathcal{A}_k$, and since the target is a k-algebra we get a ring homomorphism

$$k \otimes \mathcal{A} \to \mathcal{A}_k.$$

This is in fact an isomorphism. For as an abelian group k is just a direct sum of copies of $\mathbb{F}_p$ and taking homology commutes with direct sums. We get

$$\mathrm{HH}_*(\mathcal{A}_k) \cong \mathrm{HH}_*(k \otimes \mathcal{A}) \cong \mathrm{HH}_*(k) \otimes \mathrm{HH}_*(\mathcal{A}) \cong k \otimes \mathrm{HH}_*(\mathcal{A}),$$

where the last equality is the lemma above. Thus $E^2(k) \cong k \otimes E^2(\mathbb{F}_p)$ and since $E^r(k)$ is a spectral sequence of k-modules

$$E^\infty(k) \cong k \otimes E^\infty(\mathbb{F}_p).$$

The statement follows. ☐

Suppose that T is any C-ring spectrum and that X is any C-space. Then $(T \wedge X)^C$ is a T^C-module spectrum. The action map is the composition

$$T^C \wedge (T \wedge X)^C \to (T \wedge T \wedge X)^C \xrightarrow{\mu \wedge 1} (T \wedge X)^C. \tag{4.5.1}$$

When T is $T(A)$ and X is any of the C-spaces in (4.1.1) this shows that

$$T(A)_{hC_{p^n}} \xrightarrow{N} T(A)^{C_{p^n}} \xrightarrow{R} T(A)^{C_{p^{n-1}}}$$

is a cofibration sequence of $T(A)^{C_{p^n}}$-module spectra. In particular, the associated long exact sequence of homotopy groups

$$\dots \xrightarrow{\partial} \pi_i T(A)_{hC_{p^n}} \xrightarrow{N} \pi_i T(A)^{C_{p^n}} \xrightarrow{R} \pi_i T(A)^{C_{p^{n-1}}} \xrightarrow{\partial} \dots$$

is a sequence of $W_{n+1}(A)$-modules. Moreover, (4.1.3) is a spectral sequence of $W_{n+1}(A)$-modules,

$$E^2 = H_*(C_{p^n}; (F^n)^* \pi_* T(A)) \Rightarrow \pi_* T(A)_{hC_{p^n}}, \tag{4.5.2}$$

where $F^n\colon W_{n+1}(A) \to A$ is the iterated Frobenius. Indeed, the identification of the E^1-term uses the transfer equivalence $(T \wedge \Sigma^r C_+)^C \simeq \Sigma^r T$, and under this equivalence (4.5.1) becomes

$$T^C \wedge \Sigma^r T \xrightarrow{\mathrm{incl} \wedge 1} T \wedge \Sigma^r T \xrightarrow{\mu} \Sigma^r T,$$

which gives (4.5.2).

Theorem 4.5. *For any perfect field k of positive characteristic p,*

$$\pi_* T(k)^{C_{p^n}} \cong S_{W_{n+1}(k)}\{\sigma_n\}, \qquad \deg \sigma_n = 2,$$

and $F(\sigma_n) = \sigma_{n-1}$, $V(\sigma_{n-1}) = p\sigma_n$ and $R(\sigma_n) = p\lambda_n \sigma_{n-1}$ where $\lambda_n \in W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$ is a unit.

Proof. We argue by induction on n starting from the case $n = 1$ was established in corollary 4.5 above. Let $W = W_{n+1}(k)$ and consider the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & W \otimes \pi_i T(\mathbb{F}_p)_{hC_{p^n}} & \xrightarrow{N} & W \otimes \pi_i T(\mathbb{F}_p)^{C_{p^n}} & \xrightarrow{R} & W \otimes \pi_i T(\mathbb{F}_p)^{C_{p^{n-1}}} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_i T(k)_{hC_{p^n}} & \xrightarrow{N} & \pi_i T(k)^{C_{p^n}} & \xrightarrow{R} & \pi_i T(k)^{C_{p^{n-1}}} & \longrightarrow \cdots
\end{array}$$

By induction the right hand vertical map is an isomorphism. Indeed,

$$W_{n+1}(k) \otimes \pi_* T(\mathbb{F}_p)^{C_{p^{n-1}}} \cong W_n(k) \otimes \pi_* T(k)^{C_{p^{n-1}}} \cong \pi_* T(k)^{C_{p^{n-1}}}.$$

Therefore, we are done by induction if we prove that the left hand vertical map is an isomorphism. We let φ_k denote the Frobenius automorphism on k and consider the diagram

$$\begin{array}{ccccc}
W_{n+1}(\mathbb{F}_p) & \xrightarrow{W_{n+1}(\varphi_{\mathbb{F}_p}^n)} & W_{n+1}(\mathbb{F}_p) & \xrightarrow{R^n} & \mathbb{F}_p \\
\downarrow & & \downarrow & & \downarrow \\
W_{n+1}(k) & \xrightarrow{W_{n+1}(\varphi_k^n)} & W_{n+1}(k) & \xrightarrow{R^n} & k.
\end{array}$$

The left hand square is cocartesian because the horizontal maps are isomorphisms and the right square is cocartesian because p generates the maximal ideal of $W_{n+1}(k)$. Moreover, the compositions of the horizontal maps are equal to F^n and therefore we have

$$W_{n+1}(k) \otimes (F^n)^* \pi_* T(\mathbb{F}_p) \cong (F^n)^* k \otimes \pi_* T(\mathbb{F}_p) \cong (F^n)^* T(k).$$

Now the spectral sequence discussed above supplies the conclusion. □

Proof of theorem B. Theorem 4.5 shows that $\mathrm{TR}(k) = HW(k)$, with the notation of (3.1.1). Moreover, $F: \mathrm{TR}(k) \to \mathrm{TR}(k)$ corresponds to the Frobenius on Witt vectors, and hence we obtain an exact sequence

$$0 \to \mathrm{TC}_0(k) \to W(k) \xrightarrow{1-F} W(k) \to \mathrm{TC}_{-1}(k) \to 0.$$

When $k = \mathbb{F}_p$ we have $1 - F = 0$, proving $\mathrm{TC}(\mathbb{F}_p) \cong H\mathbb{Z}_p \vee \Sigma^{-1} H\mathbb{Z}_p$. In particular, $\mathrm{TC}(\mathbb{F}_p)$ is an Eilenberg-MacLane spectrum. For general k, $\mathrm{TC}(k)$ is a module spectrum over $\mathrm{TC}(\mathbb{F}_p)$ and hence an Eilenberg-MacLane spectrum. □

Remark 4.5. We may also extend addendum 4.3 and lemma 4.4 to general perfect fields. The map $\hat{\Gamma}_n$ in proposition 4.1 shows that $\pi_* \hat{\mathbb{H}}(C_{p^n}; T(k))$ is a $W_n(k)$-module, and we claim that

$$\pi_* \hat{\mathbb{H}}(C_{p^n}; T(k)) \cong W_n(k) \otimes \pi_* \hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)). \tag{4.5.2}$$

Indeed, the spectral sequence of (4.1.3) is a spectral sequence of $W_{n+1}(k)$-modules,

$$\hat{E}^2 = \hat{H}^*(C_{p^n}; (F^n)^* \pi_* T(k)) \Rightarrow \pi_* \hat{\mathbb{H}}(C_{p^n}; T(k)),$$

where $F^n: W_{n+1}(k) \to k$ is the iterated Frobenius. This follows from the discussion preseeding theorem 4.5. Therefore, we can repeat the proof of theorem 4.5 and get that

$$W_{n+1}(k) \otimes \pi_* \hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)) \cong \pi_* \hat{\mathbb{H}}(C_{p^n}; T(k)).$$

Since the $W_{n+1}(k)$-module structure on $\pi_* \hat{\mathbb{H}}(C_{p^n}; T(k))$ comes from the $W_n(k)$-module structure via the restriction map $R: W_{n+1}(k) \to W_n(k)$ we get the claimed isomorphism.

Quite similarly, the proof of addendum 4.3 generalizes to show that

$$\Gamma_n: T(k)^{C_{p^n}} \to T(k)^{hC_{p^n}}, \qquad \hat{\Gamma}_{n-1}: T(k)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}; T(k)) \tag{4.5.3}$$

induce equivalences of connective covers.

5. Topological cyclic homology of finite $W(k)$-algebras

5.1. If the ring R is given as an inverse limit of rings R_n, $R = \varprojlim R_n$, then one can define continuous versions of $K(R)$ and $\mathrm{TC}(R)$ by setting

$$K^{\mathrm{top}}(R) = \operatorname{holim}_{\longleftarrow} K(R_n), \qquad \mathrm{TC}^{\mathrm{top}}(R) = \operatorname{holim}_{\longleftarrow} \mathrm{TC}(R_n),$$

cf. [W]. One may then ask when the natural maps from $K(R)$ to $K^{\mathrm{top}}(R)$ and $\mathrm{TC}(R)$ to $\mathrm{TC}^{\mathrm{top}}(R)$ are equivalences.

The cyclotomic trace from $K(R)$ to $\mathrm{TC}(R)$ defines by naturality a corresponding map between the continuous versions, so we have a diagram

$$\begin{array}{ccc} K(R) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K^{\mathrm{top}}(R) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}^{\mathrm{top}}(R). \end{array}$$

Let k be a perfect field of positive characteristic p and let $W(k)$ be its ring of Witt vectors. We have the following result about the above diagram:

Theorem 5.1. *Let A be a $W(k)$-algebra which is finitely generated as a $W(k)$-module.*
(i) *The cyclotomic trace induces an equivalence $K^{\mathrm{top}}(A)^{\wedge}_p \simeq \mathrm{TC}^{\mathrm{top}}(A)^{\wedge}_p[0,\infty)$.*
(ii) *The natural map $\mathrm{TC}(A)^{\wedge}_p \to \mathrm{TC}^{\mathrm{top}}(A)^{\wedge}_p$ is an equivalence.*
In both statements the superscript top *refers to the p-adic topology on A.*

We note that since $W(k)$ is a P.I.D. the structure theorem for finitely generated modules shows that A is p-adically complete: $A = \varprojlim A/A_n$, where $A_n = A/p^nA$.

Proof of 5.1(i). By McCarthy's theorem A of the introduction it suffices to prove that

$$\mathrm{trc}\colon K(A_1)^{\wedge}_p \to \mathrm{TC}(A_1)^{\wedge}_p[0,\infty)$$

is an equivalence. As a finite dimensional k-algebra, A_1 is artinian, and hence its radical $J = \mathrm{rad}(A_1)$ is nilpotent. Therefore, by one more application of theorem A we are reduced to prove that $K(A_1/J)^{\wedge}_p \simeq \mathrm{TC}(A_1/J)^{\wedge}_p[0,\infty)$. Now A_1/J is semi-simple, and since both functors preserve product it suffices to prove that

$$\mathrm{trc}\colon K(\bar{A})^{\wedge}_p \to \mathrm{TC}(\bar{A})^{\wedge}_p[0,\infty)$$

is an equivalence for a central simple k-algebra. If the class of $\bar{A}$ in the Brauer group $\mathrm{Br}(k)$ is trivial, *i.e.* if $\bar{A} \cong M_n(k)$, then we are done by theorem B since both $K(-)$ and $\mathrm{TC}(-)$ are Morita invariant, *cf.* 1.7.

However, $\mathrm{Br}(k)$ might not vanish for perfect fields in general; one knows only that the p-primary part of $\mathrm{Br}(k)$ vanishes, [Se], chap. X, §4. Let K be a Galois splitting field for $\bar{A}$ and G the Galois group of K/k, and let $L = K^{G_p}$ where G_p is a Sylow p-subgroup of G. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(G;K^\times) & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(K) \\ & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow & & \| \\ 0 & \longrightarrow & H^2(G_p;K^\times) & \longrightarrow & \mathrm{Br}(L) & \longrightarrow & \mathrm{Br}(K). \end{array}$$

Since ${}_p\mathrm{Br}(k) = 0$, $[\bar{A}]$ is p'-torsion in $H^2(G;K^\times)$ and since $H^2(G_p;K^\times)$ has vanishing p'-torsion one must have $[\bar{A} \otimes_k L] = 0$ in $\mathrm{Br}(L)$. Thus

$$\bar{B} = \bar{A} \otimes_k L \cong M_r(L).$$

On the other hand, L is perfect (being an algebraic extension of k), so by the previous remarks the middle vertical map in the diagram below is an isomorphims:

$$\begin{array}{ccccc} K(\bar{A})^{\wedge}_{p} & \xrightarrow{i_*} & K(\bar{B})^{\wedge}_{p} & \xrightarrow{i^*} & K(\bar{A})^{\wedge}_{p} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TC}(\bar{A})^{\wedge}_{p} & \xrightarrow{i_*} & \mathrm{TC}(\bar{B})^{\wedge}_{p} & \xrightarrow{i^*} & \mathrm{TC}(\bar{A})^{\wedge}_{p}. \end{array} \tag{5.1.1}$$

Both the horizontal compositions are isomorphisms since $\bar{B}$ is a free $\bar{A}$-algebra of rank $|L:k|$, prime to p. This is well-known for K-theory and for TC we may argue as follows. First, the composition

$$\mathrm{HH}_*(\bar{A}) \xrightarrow{i_*} \mathrm{HH}_*(\bar{B}) \xrightarrow{i^*} \mathrm{HH}_*(\bar{A})$$

is an isomorphism. The spectral sequence of 4.2 then implies that the composition

$$T(\bar{A}) \xrightarrow{i_*} T(\bar{B}) \xrightarrow{i^*} T(\bar{A})$$

is an equivalence. The obvious inductive argument, using the cofibration sequence of theorem 1.2 shows that

$$T(\bar{A})^{C_{p^n}} \xrightarrow{i_*} T(\bar{B})^{C_{p^n}} \xrightarrow{i^*} T(\bar{A})^{C_{p^n}}$$

is an equivalence. The same will then be the case for the lower horizontal composition in (5.1.1). It follows now from (5.1.1) that $K(\bar{A})^{\wedge}_{p} \simeq \mathrm{TC}(\bar{A})^{\wedge}_{p}[0,\infty)$. □

The proof of theorem 5.1 (ii) occupies the rest of this paragraph. It is based on the corresponding statement for Eilenberg-MacLane spectra,

$$HA \simeq \operatorname{holim}_{\longleftarrow} HA_n.$$

Indeed, $\pi_* \operatorname{holim}_{\longleftarrow} HA_n = \lim_{\longleftarrow} \pi_* HA_n$ by [BK], XI.7, and Eilenberg-MacLane spectra are characterized by their homotopy groups. Let us write $HA^{(r)}$ for the r-fold smash product of HA.

Lemma 5.1. *Let A be as in theorem 5.1. Then the natural map*

$$HA^{(r)} \to \operatorname{holim}_{\longleftarrow} HA_n^{(r)},$$

becomes an equivalence upon p-completion.

Proof. We begin with the special case where $A = W(k)$ and $A_n = W_n(k)$. Completion of a spectrum at a prime p is the same as localization with respect to the Moore spectrum S^0/p, and hence the thing to show is that

$$\pi_*(HA^{(r)}; \mathbb{F}_p) \xrightarrow{\cong} \lim_{\longleftarrow} \pi_*((HA_n)^{(r)}; \mathbb{F}_p),$$

see [Bo]. We have

$$HA \wedge S^0/p \simeq HA_1, \qquad HA_n \wedge S^0/p \simeq HA_1 \vee \Sigma HA_1,$$

and moreover the map $HA_1^{n+1} \to HA_n$ induced from the reduction when smashed with S^0/p becomes the self-map of $HA_1 \vee \Sigma HA_1$ which is the identity on the first factor and trivial on the suspension factor. These remarks follows easily from the cofibration diagram

$$\begin{array}{ccccc} HA_n & \xrightarrow{p} & HA_n & \longrightarrow & HA_n \wedge S^0/p \\ \uparrow & & \uparrow & & \uparrow \\ HA & \xrightarrow{p} & HA & \longrightarrow & HA_1 \\ \uparrow{\scriptstyle p^n} & & \uparrow{\scriptstyle p^n} & & \uparrow{\scriptstyle 0} \\ HA & \xrightarrow{p} & HA & \longrightarrow & HA_1. \end{array}$$

Thus we have

$$\pi_*((HA_n)^{(r)};\mathbb{F}_p) \cong H_*((HA_n)^{(r-1)};k) \oplus H_{*-1}((HA_n)^{(r-1)};k)$$

and the maps in the inverse limit system are trivial on the second summand. This gives

$$\varprojlim \pi_*((HA_n)^{(r)};\mathbb{F}_p) \cong \varprojlim H_*((HA_n)^{(r-1)};k).$$

Let $\mathcal{A} = H_*(HA;k)$. Then

$$H_*(HA_n;k) = \mathcal{A} \otimes_k \Lambda_k\{\epsilon_n\}, \qquad \deg \epsilon_n = 1$$

and the map induced from the reduction map $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$ sends ϵ_{n+1} to zero. Indeed, the cofiber of $HA_n \wedge Hk$ of $p^n \wedge \mathrm{id}\colon HA \wedge Hk \to HA \wedge Hk$ is $HA \wedge C_n$, where

$$C_n = \text{cofiber}(p^n\colon Hk \to Hk) = Hk \vee \Sigma Hk,$$

and $C_n \to C_{n-1}$ maps the first wedge summand by the identity and the second trivially. If follows that

$$\varprojlim \pi_*((HA_n)^{(r)};\mathbb{F}_p) \cong \mathcal{A}^{\otimes(r-1)} \cong \pi_*(HA^{(r)};\mathbb{F}_p)$$

where the tensor product if over k.

If A is a free $W(k)$-module of finite rank we can use that

$$HA \simeq HW(k) \vee \ldots \vee HW(k), \qquad HA_n \simeq HW_n(k) \vee \ldots \vee HW_n(k)$$

to get the conclusion. Finally, for general A, let $T(A)$ be the submodule of torsion elements, and let $F(A)$ be the free quotient. Since $W(k)$ is a local P.I.D. and since $T(A)$ is finitely generated $p^eT(A) = 0$ for a suitable exponent e. Hence

$$HA_n = HT(A) \wedge HF(A)_n$$

and the map $HA_n \to HA_{n-1}$ is the identity on $HT(A)$ for $n > e$. Since

$$\underset{\longleftarrow}{\text{holim}}\, HF(A)_n^{(r)} \simeq HF(A)^{(r)}$$

for all r by the above, the same follows for HA_n upon decomposing $HA_n^{(r)}$. □

5.2. We next consider the continuity of $\mathrm{THH}(R)$. This is the realization of the simplicial space with k-simplices

$$\mathrm{THH}_k(R) = \underset{\overrightarrow{I^{k+1}}}{\text{holim}}\, F(S^{i_0} \wedge \ldots \wedge S^{i_k}, \widetilde{R}(S^{i_0}) \wedge \ldots \wedge \widetilde{R}(S^{i_k}) \wedge X).$$

The k-simplices is a spectrum with n'th space $\mathrm{THH}_k(R, S^n)$, *cf.* 1.4, and in fact it is one way to make sense of the smash product $HR^{(k+1)}$. Thus we can restate lemma 5.1 as

$$\mathrm{THH}_k(A)_p^{\wedge} \simeq \underset{\longleftarrow}{\text{holim}}\, \mathrm{THH}_k(A_n).$$

We want to prove the similar statement for the geometric realization $\mathrm{THH}(A)$ of the simplicial spectrum $\mathrm{THH}_\bullet(A)$.

In general it is a sticky point to commute realizations with inverse limits. For example realization does not in general commute with infinite products. A counter example is provided by $\prod S^1_\bullet$, where $S^1_\bullet$ is the simplicial circle with one non-degenerate 1-simplex. However for Kan complexes there are no problems, and we can take advantage of the fact that $\mathrm{THH}_k(R)$ is equivalent to $\Omega\,\mathrm{THH}_k(R;S^1)$.

More precisely, we consider the tri-simplicial set

$$X_{k,l}(R) = G_k\,\mathrm{THH}_l(R;S^1),$$

where $G_\bullet Y$ denotes the Kan loop group of the singular set $\mathrm{Sin}_\bullet Y$, and write $X(R)$ for the realization of the diagonal complex, $X(R) = |\partial X(R)_\bullet| \simeq \mathrm{THH}(R)$.

Lemma 5.2. *Suppose that $HR^{(k)} = \underleftarrow{\mathrm{holim}}\, HR_n^{(k)}$ for all $k \geq 1$. Then*

$$\mathrm{THH}(R)^{\wedge}_p \simeq \underleftarrow{\mathrm{holim}}\, \mathrm{THH}(R_n)^{\wedge}_p.$$

Proof. We may rephrase the assumption to give that

$$|X_{\bullet,l}(R)|^{\wedge}_p \simeq \underleftarrow{\mathrm{holim}}\, |X_{\bullet,l}(R_n)|^{\wedge}_p.$$

Since simplicial groups are Kan we have

$$\underleftarrow{\mathrm{holim}}\, |X_{\bullet,l}(R_n)| \simeq |\underleftarrow{\mathrm{holim}}\, X_{\bullet,l}(R_n)|$$
$$\underleftarrow{\mathrm{holim}}\, |\delta X_{\bullet,\bullet}(R_n)| \simeq |\underleftarrow{\mathrm{holim}}\, \delta X_{\bullet,\bullet}(R_n)|.$$

Indeed, the homotopy groups of the realization of a Kan complex can be combinatorially defined, the homotopy limit of Kan complexes is again Kan, and one has a spectral sequence

$$E^2_{s,r} = \underleftarrow{\lim}^{(-s)} \pi_r \delta X_{\bullet,\bullet}(R_n) \Rightarrow \pi_{r+s}\, \underleftarrow{\mathrm{holim}}\, \delta X_{\bullet,\bullet}(R_n),$$

see [BK], p. 309. There is also a spectral sequence

$$E^2_{s,r} = \underleftarrow{\lim}^{(-s)} \pi_r |\delta X_{\bullet,\bullet}(R_n)| \Rightarrow \pi_{r+s}\, \underleftarrow{\mathrm{holim}}\, |\delta X_{\bullet,\bullet}(R_n)|,$$

and it maps to the former by a map which is an isomorphism on E^2; the claim follows. Thus we have

$$\begin{aligned}\mathrm{THH}(R) &\simeq |[l] \mapsto |\underleftarrow{\mathrm{holim}}\, X_{\bullet,l}(R_n)|| \cong |\delta\, \underleftarrow{\mathrm{holim}}\, \delta X_{\bullet,\bullet}(R_n)| \\ &\cong |\underleftarrow{\mathrm{holim}}\, \delta X_{\bullet,\bullet}(R_n)| \simeq \underleftarrow{\mathrm{holim}}\, |\delta X_{\bullet,\bullet}(R_n)| \\ &\simeq \underleftarrow{\mathrm{holim}}\, \mathrm{THH}(R_n).\end{aligned}$$

□

The above lemma works equally well for $\mathrm{THH}(R; S^n)$, so with the notation of §1, the underlying non-equivariant spectrum of $T(R)^{\wedge}_p$ is equivalent to that of $\underleftarrow{\mathrm{holim}}\, T(R_n)^{\wedge}_p$.

Proof of theorem 5.1. We first note that after p-completion

$$T(A)^{C_{p^m}} \simeq \underleftarrow{\mathrm{holim}}\, T(A_n)^{C_{p^m}}$$

for each m. This follows inductively from theorem 1.2 since for bounded below spectra taking homotopy orbits commutes with homotopy inverse limits,

$$\underleftarrow{\mathrm{holim}}(T(A_n)_{hC_{p^m}}) \simeq (\underleftarrow{\mathrm{holim}}\, T(A_n))_{hC_{p^m}}.$$

Second, we have a cofibration sequence of spectra

$$\mathrm{TC}(A)^{\wedge}_p \to \underset{m}{\underleftarrow{\mathrm{holim}}}[T(A)^{C_{p^m}}]^{\wedge}_p \xrightarrow{R-F} \underset{m}{\underleftarrow{\mathrm{holim}}}[T(A)^{C_{p^{m-1}}}]^{\wedge}_p$$

since $\mathrm{TC}(A)^{\wedge}_p \simeq \mathrm{TC}(A;p)^{\wedge}_p$ by proposition 2.1, and we have a similar cofibration sequence for each A_n. Finally, homotopy inverse limits commute. □

Addendum 5.2. *Suppose R is a ring which is finitely generated as a $\mathbb{Z}$-module and let $R_p = R \otimes \mathbb{Z}_p$. Then the natural map from $\mathrm{TC}(R)^\wedge_p$ to $\mathrm{TC}(R_p)^\wedge_p$ is a homotopy equivalence.*

We leave the argument which is very similar to the proof of theorem 5.1 to the reader, and note that this property clearly distinguishes $K(R)^\wedge$ from $\mathrm{TC}(R)^\wedge$ for non-complete rings. For in the commutative square

$$\begin{array}{ccc} K(R)^\wedge_p & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(R)^\wedge_p \\ \big\downarrow & & \big\downarrow \\ K(R_p)^\wedge_p & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(R_p)^\wedge_p \end{array}$$

the left hand vertical map is not in general an equivalence. For example a result of C. Soulé, [So] shows that for $R = \mathbb{Z}$ and $p = 691$, $K_{22}(\mathbb{Z})$ does not map injectively into $K_{22}(\mathbb{Z}_p)$. In general the Lichtenbaum-Quillen conjecture asserts that the numerators of the Bernoulli numbers enter into the torsion subgroup of $K_i(\mathbb{Z})$ but they do not enter into the structure of $\mathrm{TC}(\mathbb{Z}_p)^\wedge_p \simeq K(\mathbb{Z}_p)^\wedge_p$.

Remark 5.2. Suppose A is a complete discrete valuation rings with perfect residue fields of positive characteristic. One may ask if $\mathrm{TC}(A)^\wedge_p \simeq \mathrm{TC}^{\mathrm{top}}(A)^\wedge_p$ when the topology is given by powers of the maximal ideal, *i.e.* $A_n = A/\mathfrak{m}^n$. In the unequal characteristic case this follows from theorem 5.1 since the $\mathfrak{m}$-adic topology agrees with the p-adic topology. However, in the equal characteristic case, where $A = k[\![X]\!]$ lemma 5.1 fails, and it seems unlikely that the theorem should hold. The problem is that

$$\pi_*(HA^{(r)}; \mathbb{F}_p) = \pi_*(Hk; \mathbb{F}_p)[\![x]\!]^{(r)} \not\cong \pi_*(Hk; \mathbb{F}_p)[\![x_1, \dots, x_r]\!] \cong \lim_{\substack{\longleftarrow \\ n}} \pi_*(HA_n^{(r)}\ \mathbb{F}_p).$$

6. Pointed Monoids

6.1. By a *pointed monoid* we mean a monoid in the monoidal category of based spaces and smash product, that is, a based space Π equipped with a multiplication and unit

$$\mu^\Pi \colon \Pi \wedge \Pi \to \Pi, \qquad \mathbf{1}^\Pi \colon S^0 \to \Pi$$

satisfying associativity and unit laws up to coherent isomorphism. The *cyclic bar construction* of Π is the cyclic space $N^{\mathrm{cy}}_\wedge(\Pi)$ whose k-simplices are the $(k+1)$-fold smash product

$$N^{\mathrm{cy}}_{\wedge,k}(\Pi) = \Pi^{\wedge(k+1)}$$

with the Hochschild-like structure maps

$$(6.1.1) \qquad \begin{aligned} d_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \pi_{i+1} \wedge \dots \wedge \pi_k & , 0 \le i < k \\ &= \pi_k \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1} & , i = k \\ s_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \wedge 1 \wedge \pi_{i+1} \wedge \dots \wedge \pi_k & , 0 \le i \le k \\ \tau_k(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_k \wedge \pi_0 \wedge \dots \wedge \pi_{k-1}. & \end{aligned}$$

Since it is a cyclic space the C'th edgewise subdivision $\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(\Pi)$ has a simplicial action by the cyclic group C and completely analogous to [BHM; §2] we have an isomorphism of cyclic spaces

$$\Delta_C \colon N^{\mathrm{cy}}_\wedge(\Pi) \to (\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(\Pi))^C.$$

If Γ is an ordinary monoid then Γ_+ is a pointed monoid and $N^{\mathrm{cy}}_\wedge(\Gamma_+) = N^{\mathrm{cy}}(\Gamma)_+$. Conversely a pointed monoid, for which $\mu^M(x \wedge y) = *$ implies that $x \wedge y = *$, is of this form. We define for each $n \ge 1$ a pointed monoid

$$\Pi_n = \{0, 1, v, v^2, \dots, v^{n-1}\}$$

with 0 as basepoint and the multiplication determined by the rule $v^n = 0$. These are not of the form Γ_+. In the pointed situation we have no analog of the (usual) bar construction since in general we lack the projections $\mathrm{pr}_i \colon \Pi \wedge \Pi \to \Pi$.

Suppose A is a ring and Π is a discrete pointed monoid. Then we can give the quotient $A[\Pi] = A\langle\Pi\rangle/A\langle *\rangle$ the structure of a ring with multiplication and unit

$$\mu\colon A[\Pi]\otimes A[\Pi] \to A[\Pi\wedge\Pi] \xrightarrow{A[\mu^\Pi]} A[\Pi], \qquad \eta\colon \mathbb{Z}\to A[S^0] \xrightarrow{A[\mathbf{1}^\Pi]} A[\Pi].$$

If $\Pi = \Gamma_+$ for a discrete group Γ and A is commutative, then $A[\Pi]$ is the usual group algebra $A[\Gamma]$. Note also that $A[\Pi_n]$ is the truncated poloynomial algebra $A[v]/(v^n)$. Moreover, $A[N^{\rm cy}_\wedge(\Pi)] \cong \mathrm{HH}(A[\Pi])$, provided that the multiplication map $A\otimes A\to A$ is an isomorphism, so in this case

$$\widetilde{H}_*(|N^{\rm cy}_\wedge(\Pi)|;A) \cong \mathrm{HH}_*(A[\Pi]). \tag{6.1.1}$$

We want to replace the coefficient ring by an *FSP*.

Definition 6.1. Let L be an *FSP* and Π a pointed monoid. We define a new *FSP* denoted $L[\Pi]$ by

$$L[\Pi](X) = L(X)\wedge\Pi$$

with the structure maps $\mu^{L[\Pi]}_{X,Y} = (\mu^L_{X,Y}\wedge\mu^\Pi)\circ(\mathrm{id}\wedge\mathrm{tw}\wedge\mathrm{id})$ and $\mathbf{1}^{L[\Pi]}_X = \mathbf{1}^L_X\wedge\mathbf{1}^\Pi$.

Let us write $\tilde{A}$ for the *FSP* associated with the ring A, *cf.* section 4.1.

Proposition 6.1. *Let A be a ring and Π a discrete pointed monoid. There is a natural transformation $b\colon\tilde{A}[\Pi]\to\widetilde{A[\Pi]}$ which induces an equivalence of cyclotomic spectra*

$$T(\tilde{A}[\Pi]) \simeq_G T(\widetilde{A[\Pi]}).$$

Proof. Let R be any ring. The multiplicative monoid $(R,\cdot)$ acts on the functor $\tilde{R}$. Indeed $R = \tilde{R}(S^0)$ and the action is given by the composition

$$R\wedge\tilde{R}(X) = \tilde{R}(S^0)\wedge\tilde{R}(X) \xrightarrow{\mu^{\tilde{R}}} \tilde{R}(S^0\wedge X) = \tilde{R}(X).$$

Hence $\Pi\subset A[\Pi]$ acts on $\widetilde{A[\Pi]}$. Now $b(X)$ is the adjoint of the map

$$\Pi\to F(\tilde{A}(X),\widetilde{A[\Pi]}); \qquad \pi\mapsto\pi\cdot\eta(X).$$

Note that $b(X)$ is the inclusion of a wedge of copies of $\tilde{A}(X)$ indexed by $\Pi - *$ in the corresponding weak product. The proof that $T(b)$ is a G-equivalence, is completely analogous to the proof of the theorem below. □

If t is a cyclotomic prespectrum, the smash product G-prespectrum $t\wedge|N^{\rm cy}_\wedge(\Pi)|$ may be given the structure of a cyclotomic prespectrum. Indeed, the composition

$$\begin{aligned}\rho^*_C t(V)^C\wedge\rho^*_C|N^{\rm cy}_\wedge(\Pi)|^C &\xrightarrow{1\wedge D^{-1}} \rho^*_C t(V)^C\wedge\rho^*_C|\,\mathrm{sd}_C\,N^{\rm cy}_\wedge(\Pi)|^C\\ &\xrightarrow{\varphi(V)\wedge\Delta_C^{-1}} t(\rho^*_C V^C)\wedge|N^{\rm cy}_\wedge(\Pi)|\end{aligned}$$

is G-equivariant, and conditions i), ii) and iii) in lemma 1.2 are trivially satisfied. The spectrification $T\wedge|N^{\rm cy}_\wedge(\Pi)|$ is a cyclotomic spectrum by the remark following theorem 1.2.

Theorem 6.1. *Let L be an FSP and Π a pointed monoid. Then there is a natural equivalence of cyclotomic spectra*

$$T(L[\Pi]) \simeq_G T(L)\wedge|N^{\rm cy}_\wedge(\Pi)|.$$

Here the smash product on the right has $\varinjlim_{V\subset U} \Omega^V(t^\tau(L)(V)\wedge|N^{\mathrm{cy}}_\wedge(\Pi)|)$ *as its 0'th space.*

Proof. We define a map $f(\underline{i},k,V)$ as the composition

$$\begin{aligned}&F(S^{i_0}\wedge\ldots\wedge S^{i_k}, S^V\wedge L(S^{i_0})\wedge\ldots\wedge L(S^{i_k}))\wedge N^{\mathrm{cy}}_{\wedge,k}(\Pi)\\ &\to F(S^{i_0}\wedge\ldots\wedge S^{i_k}, S^V\wedge L(S^{i_0})\wedge\ldots\wedge L(S^{i_k})\wedge N^{\mathrm{cy}}_{\wedge,k}(\Pi))\\ &\to F(S^{i_0}\wedge\ldots\wedge S^{i_k}, S^V\wedge L[\Pi](S^{i_0})\wedge\ldots\wedge L[\Pi](S^{i_k})).\end{aligned}$$

The first map is the adjoint of $\mathrm{ev}\wedge\mathrm{id}$ while the second map is a 'twist' map.

The maps $f(\underline{i},k,V)$ for different $\underline{i}$'s in the indexing category I^{k+1} are compatible so we obtain maps $f(k,V)$ on the homotopy colimits. It is straight forward to check that these commute with the face and degeneracy maps such that we get maps of the geometric realizations. The maps $f(V)$ which result form a map of cyclotomic prespectra and we obtain a map of the associated cyclotomic spectra

$$f\colon T(L)\wedge|N^{\mathrm{cy}}_\wedge(\Pi)|\to T(L[\Pi]).$$

In order to prove that f is a G-equivalence, we apply lemma 1.5 with $j(V)$ the homotopy fiber of $f(V)$. We claim that f induces an equivalence on C-fixed points for any finite subgroup $C\subset S^1$. Indeed let $R=\mathbb{R}C$ be the regular representation of C. It follows from (the proof of) [BHM] 6.10 that $\mathrm{sd}_C\, f(\underline{i},k,lR)^C$ is $2l-1$ connected. By the approximation lemma, [B] 1.5, the same holds for $\mathrm{sd}_C\, f(k,lR)$. Now since the C-action is simplicial the C-fixed points of the realization is the realization of the C-fixed points. Therefore the spectral sequence of [S] shows that $f(lR)^C$ is homology $2l-1$ connected. But when $l\geq 1$ the domain and codomain for $f(lR)^C$ are both simply connected and consequently $f(lR)^C$ is $2l-1$ connected. Hence $J^C\simeq_G *$ To see that $j(V)^G\simeq_G *$ note that the G-fixed set of $t(L)(V)$ is S^{V^G}. Indeed it is those 0-simplices $x\in t(L)(V)_0$ for which $s_0x=t_1s_0x$. □

Remarks. (i) We can write the theorem as a statement for $\mathrm{RO}(G)$-graded homology theories,

$$T(L[\Pi])_*(X)\cong T(L)_*(X\wedge|N^{\mathrm{cy}}_\wedge(\Pi)|)$$

for any G-space X

(ii) The theorem shows in particular that the underlying non-equivariant spectra are equivalent. Combined with Bökstedt's calculation of $T(\mathbb{F}_p)$ and $T(\mathbb{Z})$, *cf.* 4.2 and [Br], we obtain ($\mathbb{Z}$-graded)

$$\begin{aligned}T(\mathbb{F}_p[\Pi])_* &= \bigoplus_{i\geq 0}\mathrm{HH}_{*-2i}(\mathbb{F}_p[\Pi]),\\ T(\mathbb{Z}[\Pi])_* &= \mathrm{HH}_*(\mathbb{Z}[\Pi])\oplus\bigoplus_{i\geq 1}\mathrm{HH}_{*-2i+1}(\mathbb{Z}/i[\Pi]).\end{aligned}$$

Results of this form has also been proved by T.Pirashvili and A.Lindenstrauss by different methods.

6.2. We evaluate the cyclic bar construction of the pointed monoid Π_2, which in view of the above corresponds to dual numbers. First we need a description of the cyclic n-simplex Λ^n.

Recall from [G], [J] the isomorphism $\Lambda^\bullet\cong S^1\times\Delta^\bullet$ of cocyclic spaces. It is chosen such that on the right the cosimplicial structure is simply the product of that on $\Delta^\bullet$ and the identity map on S^1. As a consequence the action of τ_n is complicated; let $\mathbb{R}/\mathbb{Z}$ be our model of S^1 and identify Δ^n with the convex hull of the standard basis in $\mathbb{R}^{n+1}$, then

$$\tau_n(x;u_0,\ldots,u_n)=(x-u_0;u_1,\ldots,u_n,u_0).$$

We want, however, to choose the isomorphism $\Lambda^\bullet\cong S^1\times\Delta^\bullet$ differently so that the action by τ_n becomes diagonal

$$\tau_n(x;u_0,\ldots,u_n)=(x-1/(n+1);u_1,\ldots,u_n,u_0).$$

Let us write Λ^n for $S^1 \times \Delta^n$ with the C_{n+1}-action which is given by Jones' isomorphism and let $S^1 \times \Delta^n$ have the diagonal C_{n+1}-action. Then we want a $G \times C_{n+1}$-equivariant homeomorphism $F_n \colon \Lambda^n \to S^1 \times \Delta^n$, which covers the identity on Δ^n. We introduce an auxiliary function $f_n \colon \Delta^n \to \mathbb{R}$ and write

$$F_n(x; u_0, \dots, u_n) = (x - f_n(u_0, \dots, u_n); u_0, \dots, u_n).$$

We obtain the following equation for f_n,

$$f_n(u_1, \dots, u_n, u_0) - f_n(u_0, \dots, u_n) = 1/(n+1) - u_0.$$

For each choice of $f_n(1, 0, \dots, 0)$, the equation has a unique affine solution f_n; we choose the affine function f_n whose value on $(1, 0, \dots, 0)$ is 0. This gives us the desired isomorphism $\Lambda^\bullet \cong S^1 \times \Delta^\bullet$. Of course in this description the cosimplicial structure on the right is no longer a product.

We say that a k-simplex $v^{i_0} \wedge \dots \wedge v^{i_k}$ in $N^{\mathrm{cy}}_\wedge(\Pi_n)$ has degree $i_0 + \cdots + i_k$ and that the basepoint 0 has all degrees. The cyclic structure maps preserve degree, so the simplices of degree s form a cyclic subset $N^{\mathrm{cy}}_\wedge(\Pi_n; s)$ and we get a splitting

(6.2.1) $$N^{\mathrm{cy}}_\wedge(\Pi_n) = \bigvee_{s \geq 0} N^{\mathrm{cy}}_\wedge(\Pi_n; s)$$

as cyclic sets.

Lemma 6.2. *As G-spaces $|N^{\mathrm{cy}}_\wedge(\Pi_2; s)| \cong S^1_+ \wedge_{C_s} S(\mathbb{R}C_s)$, with G acting by multiplication in the first variable.*

Proof. Let us write $\Pi_2 = \{0, 1, \epsilon\}$, with $\epsilon^2 = 0$, and $N(s)$ instead $N^{\mathrm{cy}}_\wedge(\Pi_2; s)$. The degree counts the number of ϵ's in a simplex, so the k-simplices in $N(s)$ different from 0 has exactly s ϵ's. Thus when $k \leq s-2$ there is only one k-simplex 0 whereas for $k = s-1$ there is also the simplex $\epsilon \wedge \dots \wedge \epsilon$ (s times) and this generates $N(s)$ as a cyclic set. It follows that the realization of $N(s)$ is a quotient of Λ^{s-1} and in fact that

$$|N(s)| \cong (\Lambda^{s-1}/\partial\Lambda^{s-1})/C_s \cong (\Lambda^{s-1}/C_s)/(\partial\Lambda^{s-1}/C_s).$$

In view of the above description of Λ^{s-1} the claimed homeomorphism is evident. □

For $s = 2r$ even we define an equivariant version of $\mathbb{R}\mathrm{P}^2$ defined as the mapping cone

(6.2.2) $$S^1/C_{r+} \xrightarrow{\pi^s_r} S^1/C_{s+} \to \mathbb{R}\mathrm{P}^2(s).$$

The regular representation $\mathbb{R}C_s$ splits as $\mathbb{R} \oplus W_s$ if s is odd and $\mathbb{R} \oplus \mathbb{R}_- \oplus W_s$ if s is even, where W_s is the maximal complex subrepresentation. We have the

Corollary 6.2. *There are G-equivariant homeomorphisms*

$$\begin{aligned} |N^{\mathrm{cy}}_\wedge(\Pi_2; s)| &\cong S^1/C_{s+} \wedge S^{W_s}, \quad \textit{if } s \textit{ is odd} \\ &\cong \mathbb{R}\mathrm{P}^2(s) \wedge S^{W_s}, \quad \textit{if } s \textit{ is even,} \end{aligned}$$

with G acting diagonally on the spaces on the right.

Proof. When s is odd, $S(\mathbb{R}C_s) = S(\mathbb{R} \oplus W_s) = S^{W_s}$ and since W_s is a complex representation we have the usual G-homeomorphism

$$S^1_+ \wedge_{C_s} S^{W_s} \xrightarrow{\zeta} (S^1/C_s)_+ \wedge S^{W_s}, \qquad \zeta(u, w) = (u, uw),$$

where G acts diagonally on the target. When $s = 2r$ is even, we get similarly

$$S^1_+ \wedge_{C_s} S(\mathbb{R}C_s) \cong_G (S^1_+ \wedge_{C_s} S^{\mathbb{R}_-}) \wedge S^{W_s},$$

and C_s acts on $\mathbb{R}_-$ through the quotient $C_s \to C_s/C_r$, so

$$S^1_+ \wedge_{C_s} S^{\mathbb{R}_-} \cong (S^1/C_r)_+ \wedge_{C_s/C_r} S^{R_-}.$$

Finally, the right hand side is the Thom space

$$\mathrm{Th}(S^1/C_r \times_{C_s/C_r} \mathbb{R}_- \to S^1/C_s) = (S^1/C_s) \cup_{\pi^s_r} CS^1/C_r = \mathbb{R}\mathrm{P}^2(s). \qquad □$$

6.3. We end this paragraph with a partial description of the (realization of the) cyclic sets $N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)$ for $n > 2$. In particular we calculate their singular homology.

Let R be a commutative ring and suppose $A = R[v]/(f(v))$, where $f(x)$ is monic. We write $x = v \otimes 1$, $y = 1 \otimes v$ and $\Delta = (f(x) - f(y))/(x - y)$. Then there is the following free resolution of A as an A-A-bimodule

$$0 \leftarrow A \xleftarrow{\mu} A \otimes A \xleftarrow{x-y} A \otimes A \xleftarrow{\Delta} A \otimes A \xleftarrow{x-y} A \otimes A \xleftarrow{\Delta} \dots,$$

see *e.g.* [LL]. The Hochschild homology of A is now immediately calculated from the complex

$$0 \leftarrow A \xleftarrow{0} A \xleftarrow{f'(x)} A \xleftarrow{0} A \xleftarrow{f'(x)} \dots.$$

Combined with (6.1.1) we get

$$\tilde{H}_i(|N^{\mathrm{cy}}_{\wedge}(\Pi_n)|; R) \cong \begin{cases} R[v]/(v^n), & \text{if } i = 0 \\ {}_nR\langle 1\rangle \oplus R\langle v, \dots, v^{n-1}\rangle, & \text{if } i > 0 \text{ is even} \\ R\langle 1, v, \dots, v^{n-2}\rangle \oplus R/nR\langle v^{n-1}\rangle, & \text{if } i \text{ is odd} \end{cases}$$

Recall from 6.2 the splitting of $N^{\mathrm{cy}}_{\wedge}(\Pi_n)$ as a cyclic set. It induces a splitting of the realization and we want to calculate the homology of the individual wedge summands $|N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)|$. We compare the resolution above for $A = R[v]/(v^n)$ with the bar-resolution and choose a chain equivalence f_*,

$$\begin{array}{ccccccccc} \dots \longrightarrow & A \otimes A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{x-y} & A \otimes A & \xrightarrow{\mu} & A & \longrightarrow 0 \\ & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \| & \\ \dots \xrightarrow{b'} & A^{\otimes 4} & \xrightarrow{b'} & A^{\otimes 3} & \xrightarrow{b'} & A^{\otimes 2} & \xrightarrow{\mu} & A & \longrightarrow 0. \end{array}$$

We will not need explicit formulas for f_i. The degree defined on $N^{\mathrm{cy}}_{\wedge}(\Pi_n)$ extends such that A, and therefore also $A^{\otimes s}$, become graded rings. Moreover

$$\deg(x - y) = 1, \quad \deg \Delta = n - 1, \quad \deg b' = 0,$$

and we immediately get

$$\deg f_{2j} = jn, \quad \deg f_{2j+1} = jn + 1.$$

Next we form the tensorproduct with the A-A-bimodule A. Since the multiplication $\mu\colon A \otimes A \to A$ has degree 0 the induced chain map $\bar{f}_*$ has $\deg \bar{f}_i = \deg f_i$. We compare with the homology calculation above and get

Lemma 6.3. (i) *If* $(j-1)n < s < jn$ *then*

$$\tilde{H}_{2j-1}(|N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)|; R) \cong \tilde{H}_{2j-2}(|N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)|; R) \cong R,$$

(ii) *if* $s = jn$ *then there is an exact sequence*

$$0 \to \tilde{H}_{2j}(|N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)|; R) \to R \xrightarrow{n} R \to \tilde{H}_{2j-1}(|N^{\mathrm{cy}}_{\wedge}(\Pi_n; s)|; R) \to 0$$

and these are the only non-zero reduced homology groups.

7. A formula for $\mathrm{TC}(L[\epsilon])$

7.1. In §6 we evaluated $T(L[\epsilon])$, the topological Hochschild spectrum. We now determine its fix point structure and give a formula for $\mathrm{TC}(L[\epsilon])$. In the first section we recall some equivariant duality theory, and here G may be any compact Lie group.

For any finite subgroup $H \subset G$ and any G-spectrum T indexed on a complete G-universe $\mathcal{U}$ we have the following duality, natural in T,

$$\Sigma^{\mathrm{Ad}(G)} F(G/H_+, T) \simeq_G T \wedge G/H_+. \tag{7.1.1}$$

Here $\mathrm{Ad}(G)$ denotes the adjoint representation of G on its Lie algebra and the smash product on the right takes place in $G\mathcal{S}\mathcal{U}$.

To define the duality map we choose an embedding of G/H in an orthogonal G-representation V and consider the normal bundle ν. As an H-representation $V = L \oplus L^\perp$, where $L = T_H(G/H)$ is the tangent space. Indeed, H acts by left translation on G/H and hence on L and the embedding identifies L as a sub H-representation of V. Therefore the normal bundle is $G \times_H L^\perp \to G/H$. In general this is non-trivial.

When H is finite we may identify L with $\mathrm{Ad}(G)$. Indeed, left translation by h on G/H coincides with conjugation by h and the projection $G \to G/H$ is a local diffeomorphism. Now G/H embeds in $V \oplus L$ with normal bundle $G \times_H (L^\perp \oplus L) \cong G \times_H V$. The action by G on V gives a trivialization of the normal bundle. Thus the Thom-Pontryagin construction yields a G-map

$$(\pi_H^G)^! \colon S^{L \oplus V} \to G/H_+ \wedge S^V, \tag{7.1.2}$$

and the duality map in (7.1.1) is then given by the composite

$$F(G/H_+, T) \wedge S^{L\oplus V} \xrightarrow{1\wedge t} F(G/H_+, T) \wedge G/H_+ \wedge S^V \xrightarrow{(\mathrm{ev},1)} T \wedge G/H_+ \wedge S^V.$$

We refer to [LMS] p. 89 for the proof that this is a G-equivalence. We shall need the

Lemma 7.1. *Let $H \subset K$ be finite subgroups of G, let $\pi_H^K \colon G/H \to G/K$ be the projection and let $(\pi_H^K)^! \colon \Sigma_G^\infty G/K_+ \to \Sigma_G^\infty G/H_+$ be the associated equivariant transfer. Then the diagram*

$$\begin{array}{ccc} \Sigma^{\mathrm{Ad}(G)} F(G/H_+, T) & \longrightarrow & T \wedge G/H_+ \\ \big\downarrow {\scriptstyle (\pi_H^K)^{!*}} & & \big\downarrow {\scriptstyle 1\wedge \pi_H^K} \\ \Sigma^{\mathrm{Ad}(G)} F(G/K_+, T) & \longrightarrow & T \wedge G/K_+ \end{array}$$

is G-homotopy commutative.

Proof. We may write (7.1.1) as the composite

$$\Sigma^{\mathrm{Ad}(G)} \Sigma_G^\infty G/G_+ \wedge F(\Sigma_G^\infty G/H_+, T) \xrightarrow{(\pi_H^G)^! \wedge 1} \Sigma_G^\infty G/H_+ \wedge F(\Sigma_G^\infty G/H_+, T) \xrightarrow{(1,\mathrm{ev})} \Sigma_G^\infty G/H_+ \wedge T$$

where $(\pi_H^G)^!$ is the map of equivariant suspension spectra induced from (7.1.2). The transitivity triangle

$$\begin{array}{ccc} & & \Sigma_G^\infty G/K_+ \\ & \nearrow {\scriptstyle (\pi_K^G)^!} & \\ \Sigma^{\mathrm{Ad}(G)} \Sigma_G^\infty G/G_+ & & \big\downarrow {\scriptstyle (\pi_H^K)^!} \\ & \searrow {\scriptstyle (\pi_H^G)^!} & \\ & & \Sigma_G^\infty G/H_+ \end{array}$$

is G-homotopy commutative and reduces us to prove the following kind of Frobenius reciprocity: The diagram

$$\begin{array}{ccc} \Sigma_G^\infty G/H_+ \wedge F(\Sigma_G^\infty G/H_+, T) & \xrightarrow{(\mathrm{ev},1)} & \Sigma_G^\infty G/H_+ \wedge T \\ \big\uparrow {\scriptstyle (\pi_H^K)^! \wedge 1} & & \\ \Sigma_G^\infty G/K_+ \wedge F(\Sigma_G^\infty G/H_+, T) & & \big\downarrow {\scriptstyle \pi_H^K \wedge 1} \\ \big\downarrow {\scriptstyle 1\wedge ((\pi_H^K)^!)^*} & & \\ \Sigma_G^\infty G/K_+ \wedge F(\Sigma_G^\infty G/K_+, T) & \xrightarrow{(\mathrm{ev},1)} & \Sigma_G^\infty G/K_+ \wedge T. \end{array}$$

is G-homotopy commutative. This in turn is a straight forward consequence of the standard fact that the square

$$\begin{array}{ccc} \Sigma_G^\infty G/H_+ & \xrightarrow{(\pi_H^K,1)} & \Sigma_G^\infty(G/K_+ \wedge G/H_+) \\ \uparrow{\scriptstyle (\pi_H^K)^!} & & \uparrow{\scriptstyle 1\wedge(\pi_H^K)^!} \\ \Sigma_G^\infty G/K_+ & \xrightarrow{\Delta} & \Sigma_G^\infty(G/K_+ \wedge G/K_+) \end{array}$$

is G-homotoy commutative. □

7.2. We return to the calculation of $\mathrm{TC}(L[\epsilon])$. Again G will be the circle group. Let $\widetilde{T}(L[\epsilon])$ be the reduced topological Hochschild homology of $L[\epsilon]$, *i.e.* the homotopy fiber

$$\widetilde{T}(L[\epsilon]) = \mathrm{hofiber}(T(L[\epsilon]) \to T(L)), \qquad \epsilon \mapsto 0.$$

Recall that for any representation $W \subset \mathcal{U}$ we write T_W for the smash product G-spectrum $T \wedge S^W$. Then from §6 we have the cofibration sequence of G-spectra

$$\bigvee_{r\geq 1} T(L)_{W_{2r}} \wedge S^1/C_{r+} \to \bigvee_{s\geq 1} T(L)_{W_s} \wedge S^1/C_{s+} \to \widetilde{T}(L[\epsilon]),$$

where the first maps takes the summand r to the summand $s = 2r$ by the map induced from the projection $\pi_r^s\colon S^1/C_r \to S^1/C_s$. If we take C_n-fixed points we still get a cofibration sequence. Moreover, we may replace the wedge sums by the corresponding products and get

$$\prod_{r\geq 1} \left(T(L)_{W_{2r}} \wedge S^1/C_{r+}\right)^{C_n} \to \prod_{s\geq 1} \left(T(L)_{W_s} \wedge S^1/C_{s+}\right)^{C_n} \to \widetilde{T}(L[\epsilon])^{C_n}. \tag{7.2.1}$$

This is because $T(L)_{W_s} \wedge S^1/C_{s+}$ is $(s-2)$-connected and hence by theorem 1.2 so is its C_n-fixed sets.

Lemma 7.2. *For any G-spectrum T indexed on $\mathcal{U}$ the inclusion of the G-fixed set induces a natural map*

$$(T \wedge S^1/C_{s+})^G \to \operatorname*{holim}_{\overleftarrow{F}} (T \wedge S^1/C_{s+})^{C_n}$$

which becomes an equivalence after profinite completion. Here the limit on the right runs over the inclusion maps and the smash products are taken in $GS\mathcal{U}$.

Proof. The adjoint representation of G is trivial so the duality of (7.1.1) becomes

$$(T \wedge S^1/C_{s+})^{C_n} \simeq \Sigma F(S^1/C_{s+}, T)^{C_n}$$

For $C_n \supset C_s$ we have a cofibration sequence of C_n-spaces

$$C_n/C_{s+} \to S^1/C_{s+} \to |S^1/C_n| \wedge C_n/C_{s+},$$

where the bars on the right indicate S^1/C_n with trivial G-action. This implies a cofibration sequence of function spectra

$$F(|S^1/C_n| \wedge C_n/C_{s+}, T)^{C_n} \to F(S^1/C_{s+}, T)^{C_n} \to F(C_n/C_{s+}, T)^{C_n},$$

or equivalently the cofibration sequence

$$F(S^1/C_n, T^{C_s}) \to F(S^1/C_{s+}, T)^{C_n} \xrightarrow{\mathrm{ev}_1} T^{C_s}$$

and one readily verifies commutativity in the diagram

$$\begin{array}{ccccc} F(S^1/C_{nr}, T^{C_s}) & \longrightarrow & F(S^1/C_{s+}, T)^{C_{nr}} & \xrightarrow{\mathrm{ev}_1} & T^{C_s} \\ \downarrow{\scriptstyle (\pi_n^{nr})^*} & & \downarrow{\scriptstyle F_r} & & \| \\ F(S^1/C_n, T^{C_s}) & \longrightarrow & F(S^1/C_{s+}, T)^{C_n} & \xrightarrow{\mathrm{ev}_1} & T^{C_s}. \end{array} \tag{7.2.2}$$

The homotopy limit of the left hand term is

$$\operatorname{holim}_{\overleftarrow{n}} F(S^1/C_n, T^{C_s}) = F(\operatorname{holim}_{\overrightarrow{n}} S^1/C_n, T^{C_s}) = F(S^1\mathbb{Q}, T^{C_s}),$$

where $S^1\mathbb{Q}$ is a Moore space with integral homology $\mathbb{Q}$, concentrated in degree one. It vanishes after profinite completion:,

$$F(S^1\mathbb{Q}, T)^\wedge = F(S^{-1}\mathbb{Q}/\mathbb{Z}, F(S^1\mathbb{Q}, T)) = F(S^{-1}\mathbb{Q}/\mathbb{Z} \wedge S^1\mathbb{Q}, T) \simeq *.$$

Finally, the evaluation maps in (7.2.2) are split by the inclusion of the G-fixed set,

$$T^{C_s} = F(S^1/C_{s+}, T)^G \to F(S^1/C_{s+}, T)^{C_s}$$

and the lemma follows by one more application of (7.1.1). □

Proposition 7.2. *After profinite completion there is a cofibration sequence of spectra*

$$\Sigma \operatorname{holim}_{\overleftarrow{R}} T(L)^{C_{s/2}}_{W_s} \xrightarrow{V_2} \Sigma \operatorname{holim}_{\overleftarrow{R}} T(L)^{C_s}_{W_s} \to \widetilde{\mathrm{TC}}(L[\epsilon]),$$

where the homotopy limits runs over the natural numbers ordered by division and where $T(L)^{C_{s/2}}_{W_s}$ *is a point when* s *is odd.* □

Proof. The lemma gives us a cofibration sequence for $\widetilde{\mathrm{TF}}(L[\epsilon]) = \operatorname{holim}_{\overleftarrow{F}} \widetilde{T}(L[\epsilon])^{C_n}$. Indeed, from lemma 7.1 we have the commutative square

$$\begin{array}{ccc} \Sigma T^{C_r} & \xrightarrow{\simeq} & (T \wedge S^1/C_{r+})^G \\ \downarrow V_2 & & \downarrow (\pi_r^{2r})_* \\ \Sigma T^{C_{2r}} & \xrightarrow{\simeq} & (T \wedge S^1/C_{2r+})^G, \end{array}$$

where, we remember, $V_2 = ((\pi_r^{2r})^!)^*$. Therefore, upon taking homotopy limits over the inclusion maps in (7.2.1), we get the cofibration sequence

$$\prod_{r\geq 1} \Sigma T(L)^{C_r}_{W_{2r}} \xrightarrow{V_2} \prod_{s\geq 1} \Sigma T(L)^{C_s}_{W_s} \to \widetilde{\mathrm{TF}}(L[\epsilon]), \tag{7.2.3}$$

where the first map takes the factor r to the factor $s = 2r$ by the map V_2.

The restriction maps

$$R_n\colon \widetilde{T}(L[\epsilon])^{C_s} \to \widetilde{T}(L[\epsilon])^{C_{s/n}}$$

induce self maps of $\widetilde{\mathrm{TF}}(L[\epsilon])$, again denoted R_n, and

$$\widetilde{\mathrm{TC}}(L[\epsilon]) = \widetilde{\mathrm{TF}}(L[\epsilon])^{hR},$$

the homotopy fixed points of the multiplicative monoid of natural numbers acting through the maps R_n, $n \geq 1$. When n divides s,

$$\rho_{C_n}\colon \rho^*_{C_n} S^1/C_s \xrightarrow{\cong} S^1/C_{s/n}, \qquad \rho^*_{C_n} W_s^{C_n} = W_{s/n}$$

and R_n maps a factor s (resp. r) in (7.2.1) to the factor s/n (resp. r/n). The factors with s not divisible by n are annihilated by R_n. In fact, we have

$$R_n = \Sigma R_{n,W_s}\colon \Sigma T(L)^{C_s}_{W_s} \to \Sigma T(L)^{C_{s/n}}_{W_{s/n}},$$

where R_{n,W_s} are the restriction maps of (1.2.3) associated with $T(L)$. This is direct from the discussion of $N^{\mathrm{cy}}_\wedge(\Pi_2)$ in 6.1. Hence the claim. □

Addendum 7.2. *After p-completion there are equivalences of spectra*
(i) *For p odd*

$$\widetilde{\mathrm{TC}}(L[\epsilon]) \simeq \prod_{(d,2p)=1} \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_{p^n}}_{W_{p^n d}}$$

(ii) *For $p = 2$*

$$\widetilde{\mathrm{TC}}(L[\epsilon]) \simeq \prod_{(d,2)=1} \Sigma \operatorname{cofiber}(V_2\colon \operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_{2^{n-1}}}_{W_{2^n d}} \to \operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_{2^n}}_{W_{2^n d}}).$$

Here $W_s \subset \mathbb{R}C_s$ is the maximal complex subrepresentation. Moreover, the projection map

$$\operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_{p^n}}_{W_{p^n d}} \to T(L)^{C_{p^m}}_{W_{p^m d}}$$

is $(p^{m+1}d-1)$-connected for p odd and $(2^{m+1}d-2)$-connected for $p=2$.

Proof. For every k prime to p the map

$$\prod_{d|k} R_{k/d}F_d\colon T(L)^{C_{p^n k}}_{W_{p^n k}} \to \prod_{d|k} T(L)^{C_{p^n}}_{W_{p^n d}} \tag{7.2.4}$$

becomes an equivalence after p-completion. This follows from the proof of lemma 3.3. Note that (7.2.4) induces an equivalence after p-completion

$$\operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_s}_{W_s} \xrightarrow{\simeq} \prod_{(d,p)=1} \operatorname*{holim}_{\overleftarrow{R}} T(L)^{C_{p^n}}_{W_{p^n d}}.$$

We evaluate the cofiber of the map

$$V_2\colon T(L)^{C_{s/2}}_{W_s} \to T(L)^{C_s}_{W_s}$$

under the equivalence of (7.2.4).

First, suppose that p is an odd prime. The composition

$$T(L)^{C_{s/2}}_{W_s} \xrightarrow{V_s} T(L)^{C_s}_{W_s} \xrightarrow{F_2} T(L)^{C_{s/2}}_{W_s}$$

induces multiplication by 2 on homotopy groups. Hence the map from the cofiber of V_2 to the homotopy fiber of F_2 becomes an equivalence after p-completion. We write $s = p^n 2k$ with $(k,p)=1$ and consider the commutative square

$$\begin{array}{ccc}
T(L)^{C_{p^n 2k}}_{W_{p^n 2k}} & \xrightarrow{F_2} & T(L)^{C_{p^n 2k}}_{W_{p^n k}} \\
\downarrow{\scriptstyle \prod R_{2k/d}F_d} & & \downarrow{\scriptstyle \prod R_{2k/d}F_d} \\
\prod_{d|2k} T(L)^{C_{p^n}}_{W_{p^n d}} & \xrightarrow{\mathrm{pr}_{\mathrm{even}}} & \prod_{d|2k,\ d \text{ even}} T(L)^{C_{p^n}}_{W_{p^n d}}.
\end{array}$$

It shows that after p-completion

$$\operatorname{cofiber}(T(L)^{C_{s/2}}_{W_s} \to T(L)^{C_s}_{W_s}) \simeq \prod_{d|2k,\ d \text{ odd}} T(L)^{C_{p^n}}_{W_{p^n d}}.$$

Taking homotopy limits over the restriction maps as s runs through the natural numbers we get (i).

For $p=2$ we have a commutative square

$$\begin{array}{ccc}
T(L)^{C_{2^{n-1}k}}_{W_{2^n k}} & \xrightarrow{V_2} & T(L)^{C_{2^n k}}_{W_{2^n k}} \\
\downarrow{\scriptstyle \prod R_{k/d}F_d} & & \downarrow{\scriptstyle \prod R_{k/d}F_d} \\
\prod_{d|k} T(L)^{C_{2^{n-1}}}_{W_{2^n d}} & \xrightarrow{V_2} & \prod_{d|k} T(L)^{C_{2^n}}_{W_{2^n d}}
\end{array}$$

from which (ii) follows by taking homotopy limits over the restriction maps. Finally, the claimed connectivity of the projection map follows from theorem 1.2 since taking homotoy orbits preserves connectivity. □

8.1. We use the scheme set up in §4 to evalulate the fixed point spectra $T(k)_W^{C_{p^n}}$ for any complex representation $W \subset \mathcal{U}$. We first consider the case $k = \mathbb{F}_p$ where we use that (4.1.2) gives a diagram of cofibration sequences

$$\begin{array}{ccccc} (T(\mathbb{F}_p)_W)_{hC_{p^n}} & \xrightarrow{N} & T(\mathbb{F}_p)_W^{C_{p^n}} & \xrightarrow{R} & T(\mathbb{F}_p)^{C_{p^{n-1}}}_{\rho^*_{C_p} W^{C_p}} \\ \| & & \downarrow{\scriptstyle \Gamma_{n,W}} & & \downarrow{\scriptstyle \hat{\Gamma}_{n,W}} \\ (T(\mathbb{F}_p)_W)_{hC_{p^n}} & \xrightarrow{N} & T(\mathbb{F}_p)_W^{hC_{p^n}} & \xrightarrow{R} & \hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)_W). \end{array} \tag{8.1.1}$$

Indeed, lemma 1.1 and (1.2.2) gives us the G-equivalences

$$\rho_{C_p}^{\#} \Phi_{C_p} T(\mathbb{F}_p)_W \simeq_G \rho_{C_p}^{\#} \Phi^{C_p} T(\mathbb{F}_p)_{W^{C_p}} \simeq_G T(\mathbb{F}_p)_{\rho^*_{C_p} W^{C_p}}.$$

We start we the following

Lemma 8.1. *Let T be a C-spectrum and X be a finite C-CW-complex. Then the inclusion of the C-singular set $X^{\mathrm{sing}} \subset X$ induces an equivalence*

$$\hat{\mathbb{H}}(C; T \wedge X) \simeq \hat{\mathbb{H}}(C; T \wedge X^{\mathrm{sing}}).$$

Proof. Recall from 4.1 that $\hat{\mathbb{H}}(C;T)$ is the C-fixed point spectrum of the C-equivariant spectrum

$$\kappa_C(T) = \widetilde{E}C_+ \wedge F(EC_+, T).$$

We prove by induction over the C-cells that $\kappa_C(T \wedge (X/X^{\mathrm{sing}}))$ is C-contractible. Since X/X^{sing} is a free C-CW-complex in the based sense, it is enough to show that $\kappa_C(T \wedge C_+)$ is C-contractible. Now by (7.1.1)

$$F(EC_+, T \wedge C_+) \simeq_C F(EC_+, F(C_+, T)) \cong F(EC_+ \wedge C_+, T),$$

and $EC_+ \wedge C_+$ is C-contractible. Hence $\kappa_C(T \wedge C_+)$ is C-contractible. □

Corollary 8.1. *The map $\hat{\Gamma}_{1,W}$ induces isomorphisms on homotopy groups in dimensions greater than or equal to* $\dim W^{C_p}$.

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} (\Phi^{C_p} T(\mathbb{F}_p))_{W^{C_p}} & \longrightarrow & \Phi^{C_p}(T(\mathbb{F}_p)_{W^{C_p}}) & \longrightarrow & \Phi^{C_p} T(\mathbb{F}_p)_W \\ \downarrow{\scriptstyle \hat{\gamma}\wedge 1} & & \downarrow{\scriptstyle \hat{\gamma}} & & \downarrow{\scriptstyle \hat{\gamma}} \\ \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))_{W^{C_p}} & \longrightarrow & \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)_{W^{C_p}}) & \longrightarrow & \hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)_W). \end{array}$$

The right hand horizontal maps are equivalences by lemma 1.1 and lemma 8.1, respectively, and the left hand horizontal maps are equivalences because $S^{W^{C_p}}$ is a C_p-trivial finite C_p-CW-complex. Now proposition 4.3 shows that the left hand vertical map induces an isormorphism on $\pi_i(-)$ when $i \geq \dim W^{C_p}$, and the corollary follows. □

We next consider the spectral sequence of 4.1 for $\pi_*(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)_W); \mathbb{F}_p)$. It has E^2-term

$$\hat{E}^2_W = (\Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes \Lambda_{\mathbb{F}_p}\{e_1\} \otimes S_{\mathbb{F}_p}\{\bar{\sigma}\})[W],$$

where the decoration $[W]$ indicates that the bidegrees are shifted $(0, \dim W)$. The spectral sequence is a module over the spectral sequence $\hat{\boldsymbol{E}}^r$ of (4.3.2), and one may repeat the proof of lemma 4.3 and show that the differentials are generated from $d^2 e_1[W] = t\bar{\sigma}[W]$ in the module structure over $\hat{\boldsymbol{E}}^r$. It follows that

$$\pi_*(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)_W); \mathbb{F}_p) \cong (\Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\})[W], \tag{8.1.2}$$

where again $[W]$ indicates that the degrees are shifted up by $\dim W$. Note also that the proof of corollary 4.3 shows that the integral homotopy groups of $\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)_W)$ are cyclic $\mathbb{Z}_p$-modules.

Addendum 8.1. *The maps* $\Gamma_{n,W}$ *and* $\hat{\Gamma}_{n,W}$ *of* (8.1.1) *induces isomorphisms on homotopy groups in dimensions greater than or equal to* $\dim W^{C_p}$.

Proof. We prove the claim by induction over n starting from the case $n = 1$, which was proved in corollary 8.1. The induction step uses the diagram

$$\begin{array}{ccc} T(\mathbb{F}_p)^{C_{p^{n-1}}}_{\rho^*_{C_p} W^{C_p}} & \xrightarrow{\hat{\Gamma}_{n,W}} & \rho^*_{C_p}\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)_W)^{C_{p^{n-1}}} \\ \downarrow{\scriptstyle \Gamma_{n-1,\rho^*_{C_p} W^{C_p}}} & & \downarrow{\scriptstyle G_{n,W}} \\ T(\mathbb{F}_p)^{hC_{p^{n-1}}}_{\rho^*_{C_p} W^{C_p}} & \xrightarrow{\hat{\Gamma}_{n,W}} & \rho^*_{C_p}\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p)_W)^{hC_{p^{n-1}}} \end{array}$$

By induction the left hand vertical map induces isomorphism on $\pi_i(-)$ for $i \geq \dim W^{C_{p^2}}$. Moreover, since taking homotopy fixed sets preserves connectivity, it follows from corollary 8.1 that the lower horizontal map induces isomorphism on $\pi_i(-)$ for $i \geq \dim W^{C_p}$. Finally, $G_{n,W}$ is an equivalence. Indeed, when $W = 0$ this was proved in 4.3, and given (8.1.2), the argument of 4.3 extends verbatim to the case of a general W. This proves the induction step, and hence the addendum. □

We can now repeat the proof of lemma 4.4 and solve the spectral sequence

$$\hat{\boldsymbol{E}}^2_W = (\Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\})[W] \Rightarrow \pi_*\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)).$$

It is a module over the spectral sequence $\hat{\boldsymbol{E}}^r$ of (4.3.2) and the differentials are generated from $d^{2n-1}u_n[W] = t^{n+1}\sigma^n[W]$. The extensions in the passage from $\hat{\boldsymbol{E}}^\infty_W$ to the actual homotopy groups are maximally non-trivial so we obtain

$$\pi_*\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{F}_p)_W) \cong S_{\mathbb{Z}/p^n}\{\hat{\sigma}, \hat{\sigma}^{-1}\}[W]. \tag{8.1.3}$$

We can now evaluate the promised the homotopy groups.

Proposition 8.1. *Let k be a perfect field of positive characteristic and let $W \subset \mathcal{U}$ be a complex representation. The non-zero integral homotopy groups of $T(k)^{C_{p^n}}_W$ are concentrated in even degrees greater than or equal to $\dim W^{C_{p^n}}$. They are given by*

$$\pi_{2i}T(k)^{C_{p^n}}_W = \begin{cases} W_s(k), & \dim W^{C_{p^{n-(s-1)}}} \leq 2i < \dim W^{C_{p^{n-s}}}, s = 1, \dots, n \\ W_{n+1}(k), & 2i \geq \dim W. \end{cases}$$

Moreover, the maps

$$F\colon \pi_{2i}T(k)^{C_{p^n}}_W \to \pi_{2i}T(k)^{C_{p^{n-1}}}_W, \qquad V\colon \pi_{2i}T(k)^{C_{p^{n-1}}}_W \to \pi_{2i}T(k)^{C_{p^n}}_W$$

are the Frobenius $F\colon W_s(k) \to W_{s-1}(k)$ and the Verschiebung $V\colon W_{s-1}(k) \to W_s(k)$, respectively.

Proof. First, suppose $k = \mathbb{F}_p$. We let $\widetilde{W}$ denote the representation of $C_{p^{n+1}}$ on W through the reduction map $C_{p^{n+1}} \to C_{p^n}$. Then $W = \rho^{\#}_{C^p}\widetilde{W}^{C_p}$ and addendum 8.1 and (8.1.3) shows that

$$\pi_i T(\mathbb{F}_p)^{C_{p^n}}_W = \pi_i\hat{\mathbb{H}}(C_{p^{n+1}}; T(\mathbb{F}_p)) = \mathbb{Z}/p^{n+1},$$

when $i \geq \dim W$ and even. By theorem 1.2 the restriction map

$$R_{n,W}\colon T(\mathbb{F}_p)^{C_{p^n}}_W \to T(\mathbb{F}_p)^{C_{p^{n-1}}}_{\rho^*_{C_p} W^{C_p}}$$

is $(\dim W - 1)$-connected, and hence a downward induction on n gives the claimed homotopy groups. One may repeat the proof of proposition 4.4 to see that F and V are as claimed.

Next, let k be any perfect field with $\operatorname{char} k = p$. The proof of theorem 4.5 shows that

$$\pi_* T(k)_W^{C_{p^n}} \cong \mathbf{W}_{n+1}(k) \otimes \pi_* T(\mathbb{F}_p)_W^{C_{p^n}}.$$

Indeed, $T(k)_W^{C_{p^n}}$ is a $T(k)^{C_{p^n}}$-module spectrum, so in particular, the homotopy groups are $W_{n+1}(k)$-modules. Since $W_{n+1}(k) \otimes W_s(\mathbb{F}_p) \cong W_s(k)$ we see that the homotopy groups of $T(k)_W^{C_{p^n}}$ are as stated. Finally, the diagram

$$\begin{array}{ccc} W_{n+1}(k) \otimes \pi_* T(\mathbb{F}_p)_W^{C_{p^n}} & \xrightarrow{\cong} & \pi_* T(k)_W^{C_{p^n}} \\ {\scriptstyle V\otimes V}\uparrow \ \downarrow{\scriptstyle F\otimes F} & & {\scriptstyle V}\uparrow \ \downarrow{\scriptstyle F} \\ W_n(k) \otimes \pi_* T(\mathbb{F}_p)_W^{C_{p^{n-1}}} & \xrightarrow{\cong} & \pi_* T(k)_W^{C_{p^{n-1}}} \end{array}$$

commutes, and the proposition follows. □.

8.2. In this section k is a perfect field of characteristic $p > 0$. Let $n = n(i,d)$ be the unique positive integer with $p^{n-1}d \le i < p^n d$.

Theorem 8.2. *The homotopy groups of* $\widetilde{\mathrm{TC}}(k[\epsilon])$ *are concentrated in odd positive degrees. If* $\operatorname{char} k$ *is odd, then*

$$\widetilde{\mathrm{TC}}_i(k[\epsilon]) \;\cong \bigoplus_{\substack{(d,2p)=1 \\ 1\le d\le i}} W_{n(i,d)}(k), \qquad i \text{ odd}$$

and if $\operatorname{char} k = 2$, *then*

$$\widetilde{\mathrm{TC}}_i(k[\epsilon]) \;\cong\; k^{\oplus (i+1)/2}, \qquad i \text{ odd.}$$

Proof. Since k is an $\mathbb{F}_p$-algebra $\mathrm{TC}(k[\epsilon]) \simeq \mathrm{TC}(k[\epsilon])_p^\wedge \simeq \mathrm{TC}(k[\epsilon];p)_p^\wedge$ and we use addendum 7.2 with $L = \tilde{k}$: for p odd,

$$\widetilde{\mathrm{TC}}(k[\epsilon]) \simeq \prod_{(d,2p)=1} \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{p^m d}}^{C_{p^m}}, \tag{8.2.1}$$

and by theorem 1.2,

$$\pi_i \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{p^m d}}^{C_{p^m}} \cong \pi_{i-1} T(k)_{W_{p^n d}}^{C_{p^n}}, \qquad \text{for } i < \dim W_{p^{n+1}d} + 1.$$

On the other hand, if further $i - 1 \ge \dim W_{p^n d}^{C_p} = p^n d - 1$ then by proposition 8.1

$$\pi_{i-1} T(k)_{W_{p^n d}}^{C_{p^n}} \cong W_n(k)$$

when i is odd, and the groups vanish when i is even. Thus for $p^n d \le i < p^{n+1} d$ and i odd, the d'th factor in (8.2.1) contribute one copy of $W_n(k)$. For $i < d - 1$ the d'th factor does not contribute. This finishes the proof when $\operatorname{char} k$ is odd.

Assume now that $\operatorname{char} k = 2$, where by addendum 7.2 (ii),

$$\widetilde{\mathrm{TC}}(k[\epsilon]) \;\simeq\; \prod_{(d,2)=1} \Sigma \operatorname{cofiber}\big(V_2 \colon \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{2^m d}}^{C_{2^{m-1}}} \to \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{2^m d}}^{C_{2^m}}\big). \tag{8.2.2}$$

This time $\dim W_{2^n d} = 2^n d - 2$, and the projections

$$\pi_{i-1} \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{2^m d}}^{C_{2^m}} \to \pi_{i-1} T(k)_{W_{2^n d}}^{C_{2^n}},$$

$$\pi_{i-1} \operatorname*{holim}_{\overleftarrow{R}} T(k)_{W_{2^m d}}^{C_{2^{m-1}}} \to \pi_{i-1} T(k)_{W_{2^n d}}^{C_{2^{n-1}}},$$

are isomorphisms when $i < \dim W_{2^{n+1}d} - 1$. We have left to evaluate the Verschiebung map

$$V_2\colon \pi_{i-1}T(k)^{C_{2^{n-1}}}_{W_{2^n d}} \to \pi_{i-1}T(k)^{C_{2^n}}_{W_{2^n d}}.$$

By proposition 8.1

$$\pi_{i-1}T(k)^{C_{2^{n-1}}}_{W_{2^n d}} \cong W_n(k),$$
$$\pi_{i-1}T(k)^{C_{2^n}}_{W_{2^n d}} \cong W_{n+1}(k),$$

for $i \geq \dim W_{2^n d} + 1$ and i odd, and they vanish for i even. Moreover, the Verschiebung map on the left corresponds to the Verschiebung map on Witt vectors, *cf.* 2.1. This in an injection with cokernel $k \cong W_{n+1}(k)/W_n(k)$. Hence for $n \geq 1$ and an odd i with $2^n d - 1 \leq i < 2^{n+1}d - 1$, the d'th factor in (8.2.2) contributes one copy of k to $\widetilde{\mathrm{TC}}(k[\epsilon])$. For $d \leq i < 2d - 1$

$$\pi_i \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(k)^{C_{2^m}}_{W_{2^m d}} \cong \pi_{i-1}T(k)_{W_d} \cong k,$$
$$\pi_i \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(k)^{C_{2^{m-1}}}_{W_{2^m d}} = 0$$

which gives one copy of k in the d'th factor of (8.2.2) when i is odd. Finally, for $i < d$ there is no contribution from the d'th factor. This proves the case $\operatorname{char} k = 2$. □

We are now ready to prove theorem E of the introduction.

Proof of theorem E. In view of theorem 8.2 above it suffices to show for $\operatorname{char} k = p$, an odd prime, that

(8.2.3) $$\mathbf{W}_i(k)^{\langle -1 \rangle} \cong \bigoplus_{\substack{(d,2p)=1 \\ 1 \leq d \leq i}} W_{n(i,d)}(k).$$

For any $\mathbb{Z}_{(p)}$-algebra R, and in particular for $R = k$, we have the Artin-Hasse exponential

$$E(X) = \exp(\sum_{s=0}^{\infty} X^{p^s}/p^s) = \prod_{(d,p)=1} (1 - X^d)^{-\mu(d)/d} \in \mathbf{W}(R),$$

where μ is the Möbius function given by $\mu(d) = 0$ if d is divisible by a prime square, $\mu(p_1 \cdots p_r) = (-1)^r$ if $p_1, \dots, p_r$ are distinct primes, and $\mu(1) = 1$. It gives rise to an injective map of sets

$$\hat{E}\colon \prod_{n=0}^{\infty} R \to \mathbf{W}(R); \qquad \hat{E}(a_0, a_1, \dots)(X) = \prod_{s=0}^{\infty} E(a_s X^{p^s}),$$

whose image is a (non-unital) subring of $\mathbf{W}(R)$, isomorphic to the ring of p-typical Witt vectors $W(R)$ (in the induced ring structure).

For any $d \geq 1$ with $(d,p) = 1$ we consider the following slight modification of $\hat{E}$,

$$\hat{E}_d(a_0, a_1, \dots)(X) = \prod_{s=0}^{\infty} E(a_s X^{p^s d})^{1/d},$$

which again is a (non-unital) ring homomorphism $\hat{E}_d\colon W(R) \to \hat{W}(R)$. Is is not hard to see that any $p(X) \in \mathbf{W}(R)$ can be written uniquely as

$$p(X) = \prod_{n=1}^{\infty} E(a_n X^n),$$

so using all $\hat{E}_d$ we get a decomposition of the ring $\mathbf{W}(R)$ as a product of rings

$$\mathbf{W}(R) \cong \prod_{(d,p)=1} W(R). \tag{8.2.4}$$

The V-filtration of $W(R)$ can be compared to the obvious filtration of $\mathbf{W}(R)$,

$$F^i\mathbf{W}(R) = (1 + X^{i+1}R[\![X]\!])^\times,$$

through $\hat{E}_d$. One finds that

$$\hat{E}_d(V^nW(R)) \subset F^i\mathbf{W}(R) \;\Leftrightarrow\; i < p^n d.$$

We write $\mathbf{W}_i(R) = \mathbf{W}(R)/F^{i+1}\mathbf{W}(R)$ and call it the big Witt vectors in R of length i. When p is odd the involution τ on $\mathbf{W}(\mathbb{F})$ given by $X \mapsto -X$ can be compared with the splitting (8.2.4). The fixed set of τ corresponds to the factors $W(R)$ with d even and the (-1)-eigenspace corresponds to the factors $W(R)$ with d odd. When $R = k$ this gives us (8.2.3) and hence theorem E. □

We owe to M. Bökstedt the formula (8.2.3).

Appendix A: Spectra and prespectra

A.1. This appendix concerns the passage from G-prespectra to G-spectra. We introduce a class of *good* G-prespectra and a functor which replaces a G-prespectrum by one which is good.

The forgetful functor $l\colon G\mathcal{SU} \to G\mathcal{PU}$ has a left adjoint $L\colon G\mathcal{PU} \to G\mathcal{SU}$, which to a G-prespectrum t associates a G-spectrum Lt, see [LMS]. The need for such a functor comes from the fact that many spacewise constructions leaves the subcategory of G-spectra. As an example let T be a G-spectrum and X a G-space, then the obvious map

$$X \wedge T(V) \to \Omega^{W-V}(X \wedge T(W))$$

is not in general a homeomorphism. Similar a spacewise (homotopy) colimit of G-spectra is not in general a G-spectrum. However, for general G-prespectra the functor L is rather badly behaved; for example one might very well have

$$\pi_n Lt(V) \neq \varinjlim_{W \subset U} \pi_n \Omega^{W-V} t(W).$$

We call a G-prespectrum t *good* if the structure maps

$$\tilde{\sigma}\colon \Sigma^{W-V} t(V) \to t(W)$$

are all closed inclusions. Goodness is preserved by smash products and homotopy colimits, and since the adjoints $\sigma\colon t(V) \to \Omega^{W-V} t(W)$ are inclusions, the spectrification functor takes the simple form,

$$Lt(V) = \varinjlim_{W \subset U} \Omega^{W-V} t(W).$$

In particular the homotopy groups are what one expects.

Now let t be any G-prespectrum indexed on U and let $V \subset U$ be a f.d. sub inner product space. The sub inner product spaces $Z \subset V$ form a poset and hence a category, and for $Z_1 \subset Z_2 \subset V$ we have a map of G-spaces

$$\Sigma^{V-Z_2}\tilde{\sigma}\colon \Sigma^{V-Z_1} t(Z_1) \to \Sigma^{V-Z_2} t(Z_2).$$

These data specifies a functor and we define

Definition A.1. The thickening t^τ of a G-prespectrum t, is the G-prespectrum with V'th space the homotopy colimit

$$t^\tau(V) = \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\, \Sigma^{V-Z} t(Z)$$

and structure maps the compositions

$$\Sigma^{W-V} \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\, \Sigma^{V-Z} t(Z) \cong \underset{Z \subset V}{\underrightarrow{\mathrm{holim}}}\, \Sigma^{W-Z} t(Z) \to \underset{Z \subset W}{\underrightarrow{\mathrm{holim}}}\, \Sigma^{V-Z} t(Z),$$

where the last map is induced by the inclusion of the category of sub inner product spaces of V in that of W.

Lemma A.1. *t^τ is good and comes with a map $\pi: t^\tau \to t$ of G-prespectra, which is a spacewise G-equivalence.*

Proof. The map on homotopy colimits induced by the inclusion of a subcategory is always a closed G-cofibration, hence $\widetilde{\sigma}^\tau: \Sigma^{W-V} t^\tau t(V) \to t(W)$ is a cofibration. Since the category of sub inner product spaces of V has V as terminal object, there is a natural G-map $\pi(V): t^\tau(V) \to t(V)$, with $\iota(V): t(V) \to t^\tau(V)$ as G-homotopy inverse. Finally the maps $\pi(V)$ form a map of G-prespectra. □

Note that the functor $(-)^\tau$ produces extremely large spaces, because we use all sub inner product spaces of V. A smaller version is considered in [LMS] p. 37. Alternatively one could topologize the index category.

We call a G-spectrum *good* if it is the spectrification of a good G-prespectrum, *e.g.*

$$T(V) = \varinjlim_{W \subset U} \Omega^{W-V} t^\tau(W).$$

Let us note that a good G-spectrum is not good regarded as a G-prespectrum. We claim that smashing with a G-space X and taking homotopy colimits preserve good G-spectra. To see this we recall that if $a: G\mathcal{PU} \to G\mathcal{PU}$ is a functor, then the associated functor $A: G\mathcal{SU} \to G\mathcal{SU}$ is the composite Lal. If a has a right adjoint b, then B is the right adjoint of A, and if moreover b preserves G-spectra, *i.e.* $b(lT) \cong lB(T)$ for any $T \in G\mathcal{SU}$, then

$$A(Lt) \cong La(t).$$

Smash products and homotopy colimits are examples of such functors a. Moreover they both preserve good G-prespectra, and the claim follows.

Appendix B: Continuity properties of K-theory

In this appendix we prove theorem C (iii) of the introduction. The proof amounts to a recollection of facts due primarily to Suslin and coworkers, [Su], [SuY].

Let k be a perfect field of positive characteristic p, and $W(k)$ its Witt-vectors. We consider finite $W(k)$-algebras, *i.e.* $W(k)$-algebras whose underlying $W(k)$-module is finitely generated.

Theorem B.1. *For a finite $W(k)$-algebra A,*

$$K(A)^\wedge_p \simeq \mathrm{TC}(A)^\wedge_p,$$

where $p = \mathrm{char}(k)$.

In view of theorem C (i), (ii) the statement is equivalent to the continuity statement that

$$K(A)^\wedge_p \simeq K^{\mathrm{top}}(A)^\wedge_p, \tag{B.2}$$

where the right-hand side is the homotopy limit of $K(A/p^s A)^\wedge_p$. We begin by reducing to a special case. Let F denote the fraction field of the local ring $W(k)$, and let $E = A \otimes_{W(k)} F$.

Lemma B.3. *If theorem B.1 is true when E is semisimple then it it true in general.*

Proof. Let $J(E)$ be the radical of E. It is nilpotent since E is finite dimensional over F, hence artenian. Then $J = J(E) \cap A$ is a nilpotent ideal of A, so by theorem A of the introduction the diagram

$$\begin{array}{ccc} K(A) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(A/J) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(A/J) \end{array}$$

is homotopy cartesian after p-completion. But A/J is finite over $W(k)$ and

$$A/J \otimes_{W(k)} F = E/J(E)$$

is semisimple. □

So from now on we assume that $E = A \otimes_{W(k)} F$ is semisimple, and hence

$$E = \prod_{i=1}^{t} M_{l_i}(D_i) \tag{B.4}$$

for certain division algebras whose centers F_i are finite extensions of F. If $\Delta_i \subset D_i$ is the maximal order of D_i, *cf.* [R], ch. 5, then

$$B = \prod_{i=1}^{t} M_{l_i}(\Delta_i)$$

is the maximal order in E, and $A \subset B$. As F comes from $W(k)$ by inverting p and $A \otimes_{W(k)} F = B \otimes_{W(k)} F$, $p^s B \subset A$ for some integers. We give E the topology whose neighborhoods of 0 has $\{p^i A\}$ or equivalently $\{p^i B\}$ as a basis. Let $GL_n(A, p^i A)$ be the kernel of the reduction map

$$GL_n(A) \to GL_n(A/p^i A).$$

Then $\{GL_n(A, p^i A)\}$ is a basis of the neighborhoods of 1 in $GL_n(E)$.

Suppose now first that A is *commutative*, and consider the variety

$$X_{i,j}(E) = GL_n(E) \times \cdots \times GL_n(E), \qquad i \text{ factors}.$$

Let $\mathcal{O}_{n,i}(E)$ and $\mathcal{O}^c_{n,i}(E)$ denote the germs at 1 of rational and continuous E-valued functions on $X_{n,i}(E)$, and let $\mathcal{M}_{n,i}(E)$ and $\mathcal{M}^c_{n,i}(E)$ be the maximal ideals of functions which vanish at 1. To prove (B.2) it suffices to show that the natural map

$$H_k(GL(A); \mathbb{F}_p) \to \varprojlim H_k(GL(A/p^i A); \mathbb{F}_p) \tag{B.5}$$

is an isomorphism (Here $GL(A)$ is considered as a discrete group). Indeed if this is true with $\mathbb{F}_p$ coefficients then it is true for p-adic coefficients, and the pro-Hurewicz theorem of [P] supplies the corresponding theorem for p-completed K-theory.

In section 3 of [Su], (B.5) is derived from the following two statements

(B.6)

- (i) $\widetilde{H}_k(GL(\mathcal{O}^c_{n,i}(E), \mathcal{M}^c_{n,i}(E)); \mathbb{F}_p) = 0$
- (ii) $H_k(GL_n(A/p^\sigma); \mathbb{F}_p) \to H_k(GL(A/p^\sigma); \mathbb{F}_p)$ are isomorphisms for $n \gg k$ and $1 \le \sigma \le \infty$ ($A/p^\infty A = A$).

A few words of explanation is in order. Write $G = GL(\mathcal{O}^c_{n,i}(E), \mathcal{M}^c_{n,i}(E))$. An element $g \in G$ lies in $GL_r(\mathcal{O}^c_{n,i}(E), \mathcal{M}^c_{n,i}(E))$ for some $r \ge n$, say, and g amounts to a continuous germ from $(GL_n(E)^i, 1)$ to $(GL_r(E), 1)$. Thus for each $\sigma > 0$ there exists a $\tau \ge \sigma$ so that the germ g induces a map

$$g_\#\colon GL_n(A, p^\tau A) \to GL_r(A, p^\sigma A).$$

A (finite) chain $c \in C_{i+1}(G; \mathbb{F}_p) = \mathbb{F}_p[G^{i+1}]$ in the bar construction then induces a homomorphism

$$c_\#\colon C_i(GL_n(A, p^\tau A); \mathbb{F}_p) \to C_{i+1}(GL_r(A, p^\sigma A); \mathbb{F}_p).$$

Using (B.6) (i), [Su], proposition 2.2 exhibits chains $c_{n,i} \in C_{i+1}(G; \mathbb{F}_p)$ such that $(c_{n,i})_\#$ becomes a contracting chain homotopy of the natural inclusion of $C_i(GL_n(A, p^\tau A); \mathbb{F}_p)$ in $C_i(GL_r(A, p^\sigma A); \mathbb{F}_p)$. Hence for given n, σ, i, there exists $r \ge n$, $\tau \ge \sigma$ such that the natural inclusion induces the zero homomorphism

$$H_i(GL_n(A, p^\tau A); \mathbb{F}_p) \to H_i(GL_r(A, p^\tau A); \mathbb{F}_p). \tag{B.7}$$

Finally, in theorem 3.6 and corollary 3.7 of [Su] it is shown, via a study of the Hochschild-Serre spectral sequence of

$$BGL_n(A, p^\sigma A) \to BGL_n(A) \to BGL_n(A/p^\sigma A),$$

that (B.6) (ii) and (B.7) implies (B.5).

It remains to discuss (B.6). The first part of the statement follows from [Gab]. Indeed, as A was assumed commutative, E is a product of fields F_j, and $\mathcal{O}^c_{n,i}(E)$ is a product of $\mathcal{O}^c_{n,i}(F_j)$, the germs of F_j-valued functions on $X_{n,i}(E)$. Then

$$GL(\mathcal{O}^c_{n,i}(E), \mathcal{M}^c_{n,i}(E)) = \prod_{j=1}^{t} GL(\mathcal{O}^c_{n,i}(F_j), \mathcal{M}^c_{n,i}(F_j)).$$

Since $(\mathcal{O}^c_{n,i}(E), \mathcal{M}^c_{n,i}(E))$ is a henselian pair, [Gab], theorem 1 implies that the reduced homology of each of the t factors above is trivial. Then use the Kunneth theorem.

The second part of (B.6) follows from van der Kallens work on stability, and does *not* use the fact that A is commutative, *cf.* [vdK], (2.2) and theorem 4.11.

The general case where A is not commutative is quite similar, only the argument for producing the contracting homotopy $(c_{n,i})_\#$ is different.

Let $\mathcal{O}^h_{n,i}(F_j)$ denote the henselization of $\mathcal{O}_{n,i}(F_j)$. It is proved in [SuY], that

$$GL(\mathcal{O}^h_{n,i}(F_j) \otimes_{F_j} D_j, \mathcal{M}^h_{n,i}(F_j) \otimes_{F_j} D_j) = GL(\mathcal{O}^h_{n,i}(F_j) \otimes_{F_j} M_{l_j}(D_j), \mathcal{M}^h_{n,i}(F_j) \otimes_{F_j} M_{l_j}(D_j))$$

has vanishing homology, and universal chains $c^h_{n,i}$ are exhibited. But $(\mathcal{O}^h_{n,i}(F_j), \mathcal{M}^h_{n,i}(F_j))$ maps into $(\mathcal{O}^c_{n,i}(F_j), \mathcal{M}^c_{n,i}(\mathbb{F}_i))$ by the universal properties of henselizations, and the images of the chains $c^h_{n,i}$ give the required chains $c_{n,i}$, hence the contracting chain homotopy.

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