

Divisible homology classes
in the special linear group of a number field

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Introduction

Let F be a number field and $SL(F)$ denote the infinite special linear group over F . The integral homology groups of $SL(F)$ are in general not finitely generated, but it was shown by the first author in Section 2 of [A1] that, for all integers $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. The next interesting problem consists in understanding the structure of this torsion subgroup.

Recently, G. Banaszak looked at the corresponding question for the algebraic K -theory of number fields. The localization exact sequence in algebraic K -theory (see Section 5 of [Q2], Theorem 8 of [Q4] and Théorème 1 of [S])

$$\cdots \longrightarrow K_i O \xrightarrow{r_*} K_i F \longrightarrow \bigoplus_m K_{i-1}(O/m) \longrightarrow \cdots,$$

where O is the ring of algebraic integers in F , r_* the homomorphism induced by the inclusion $r : O \hookrightarrow F$, and where m runs over the set of maximal ideals of O , shows immediately that $K_i F$ is a finitely generated group if i is odd and a torsion group if i is even, because $K_i O$ is finitely generated for all positive i 's and moreover finite for even i 's (see [Q3] and [Bo]), and because $K_{i-1}(O/m)$ is trivial for odd i 's and finite cyclic for even i 's by [Q1].

In [B1], Chapter VIII, and [B2], Section II, G. Banaszak investigated the subgroup $D(i)$ of divisible elements in $K_i F$ (notice that a subgroup of divisible elements in an ambient abelian group is not necessarily a divisible group; here for instance, $D(i)$ is finite).

The second author wishes to thank the Swiss National Science Foundation and the Leverhulme Foundation for financial support while this research was being carried out.

It follows from the above information on the structure of $K_i F$ that $D(i)$ is trivial if i is odd and a torsion group if i is even. Moreover, since for i even, $\bigoplus_m K_{i-1}(O/m)$ is a direct sum of cyclic groups, hence, it contains no non-trivial divisible elements. Consequently, $D(i)$ is a subgroup of the image of r_* , hence, it is isomorphic to a subgroup of $K_i O$. Thus,

$$D(i) = 0 \text{ if } i \text{ is odd and } D(i) \text{ is a finite group if } i \text{ is even.}$$

For any prime number ℓ , let $D(i)_\ell$ denote the ℓ -torsion subgroup of $D(i)$ (in other words, the subgroup of ℓ -divisible ℓ -torsion elements in $K_i F$). For $i = 2n$, n odd, G. Banaszak deduced that $D(2n)_\ell$ is in general non-trivial. Subsequently, together with M. Kolster, he obtained the following description (see [B2], Theorem 3): if F is totally real, n an odd positive integer and ℓ an odd prime, the order of $D(2n)_\ell$ is exactly given by the ℓ -adic absolute value of

$$\frac{w_{n+1}(F) \zeta_F(-n)}{\prod_{\nu|\ell} w_n(F_\nu)},$$

where $\zeta_F(-)$ is the Dedekind zeta function of F , $w_n(k)$ the biggest integer s such that the exponent of the Galois group $\text{Gal}(k(\mu_s)/k)$ divides n for a field k (here μ_s is an s -th primitive root of unity), and F_ν the completion of F at ν . For instance, if F is the field of rationals \mathbb{Q} , n an odd integer and ℓ an odd prime, the order of $D(2n)_\ell$ is equal to the ℓ -adic absolute value of the numerator of $\frac{B_{n+1}}{n+1}$, where B_{n+1} is the $(n+1)$ -st Bernoulli number. Notice that the knowledge of $D(2n)$ is of particular interest since it is related to the Lichtenbaum-Quillen conjecture (see [B2], Section II.2) and to étale K-theory (see [BZ]).

The purpose of the present paper is to study the *divisible elements in homology* of the infinite special linear group of a number field. Denote by $\overline{D}(i)$ the subgroup of divisible elements in $H_i(SL(F); \mathbb{Z})$, and for a prime ℓ , by $\overline{D}(i)_\ell$ the ℓ -torsion subgroup of $\overline{D}(i)$ (observe that $\overline{D}(i)$ is a torsion group because of the result of [A1] mentioned above). In the first section (see Theorem 1.1), we prove that

$$\overline{D}(i) \text{ is a finite group for any } i \geq 0.$$

In Section 2, we use the fact that the group $SL(F)$ has the same homology as the simply connected infinite loop space $BSL(F)^+$ obtained by performing the plus construction on the classifying space of $SL(F)$ and consider the Hurewicz homomorphism

$$h_i : K_i F \cong \pi_i BSL(F)^+ \longrightarrow H_i(BSL(F)^+; \mathbb{Z}) \cong H_i(SL(F); \mathbb{Z})$$

for $i \geq 2$. We concentrate our attention to its restriction $h_i : D(i) \rightarrow \overline{D}(i)$ for $i = 2n$ and show the following assertion (see Corollary 2.5):

For any $n \geq 1$, $h_{2n} : D(2n)_\ell \longrightarrow \overline{D}(2n)_\ell$ is a split injection if $\ell > n$.

We also observe that $\overline{D}(i)_\ell$ may contain non-trivial elements which do not belong to the image of $h_i : K_i F \rightarrow H_i(SL(F); \mathbb{Z})$; for example, it is possible that $\overline{D}(i)_\ell$ is non-trivial even if i is odd or if $i = 2n$ with n even. The last section is devoted to the following vanishing result (see Theorem 3.1):

If N is a positive integer and ℓ a prime number $> N$ such that $D(2n)_\ell = 0$ for $1 \leq n \leq N$, then $\overline{D}(i)_\ell = 0$ for $1 \leq i \leq 2N$.

Let us finally mention that the structure of the integral homology groups of the infinite *general* linear group $GL(F)$ may be deduced from the knowledge of the integral homology of $SL(F)$ by the Künneth formula, because of the homotopy equivalence $BGL(F)^+ \simeq BSL(F)^+ \times BF^\times$ which follows from the fact that $BSL(F)^+$ is the universal cover of $BGL(F)^+$ (see [A1], proof of Corollary 9).

1. A finiteness theorem

The first result on the structure of the integral homology groups of $SL(F)$ is given by Theorem 7 of [A1]: for any $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. This has the following consequence for the integral *cohomology* of $SL(F)$: for any $i \geq 0$, $H^i(SL(F); \mathbb{Z})$ contains *no divisible elements* except 0 (see [A1], Corollary 8). The purpose of this section is to investigate the subgroup $\overline{D}(i)$ of divisible elements in $H_i(SL(F); \mathbb{Z})$.

Theorem 1.1. *For any $i \geq 0$, $\overline{D}(i)$ is a finite group.*

Proof. According to [Q4], Theorem 4 or [Q2], Section 7, Proposition 3.2, there is a fibration

$$\prod_m BQP(O/m) \longrightarrow BQP(O) \longrightarrow BQP(F),$$

where $BQP(A)$ denotes the classifying space of the Q -construction over the category of finitely generated projective A -modules for a ring A , \prod the weak product (i.e., the

direct limit of cartesian products with finitely many factors), and where m runs over the set of maximal ideals of O . By looping its base space, we obtain the fibration (see [Q4], Theorem 1)

$$\Omega BQP(F) \simeq BGL(F)^+ \times K_0F \xrightarrow{g} \prod_m BQP(O/m) \longrightarrow BQP(O).$$

Since $K_iO \cong \pi_{i+1}BQP(O)$ is finitely generated for all $i \geq 0$ [Q3], the homotopy exact sequence of this fibration shows that, for all $i \geq 1$, the map g induces a \mathcal{C} -isomorphism $g_* : K_iF \rightarrow \pi_i(\prod_m BQP(O/m))$, where \mathcal{C} is the Serre class of all finitely generated abelian groups. However, g may be lifted to a map on the universal covers:

$$f : BSL(F)^+ \simeq \widetilde{BGL}(F)^+ \longrightarrow \prod_m \widetilde{BQP}(O/m),$$

which induces again a \mathcal{C} -isomorphism on π_i for all $i \geq 2$. Consequently, the induced homomorphism $f_{\sharp} : H_i(BSL(F)^+; \mathbb{Z}) \rightarrow H_i(\prod_m \widetilde{BQP}(O/m); \mathbb{Z})$ is also a \mathcal{C} -isomorphism for $i \geq 2$. But by the Künneth formula, $H_i(\prod_m \widetilde{BQP}(O/m); \mathbb{Z})$ is a direct sum of finitely generated abelian groups, since the integral homology groups of $\widetilde{BQP}(O/m)$ are finitely generated for all m , and therefore has no divisible elements except 0. This implies that $\overline{D}(i)$ is contained in the kernel of f_{\sharp} , hence, it is finitely generated. Finally, the fact that $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group (see Section 2 of [A1]) implies that $\overline{D}(i)$ is a torsion group: consequently, it is finite.

We shall check that $\overline{D}(i)$ is in general *non-trivial* (see Corollary 2.5).

Remark 1.2. The same argument proves that the subgroup of divisible elements in $H_i(\Omega^s BSL(F)^+; \mathbb{Z})$ is also finite for all $i \geq 0$ and $s \geq 0$.

2. The Hurewicz homomorphism

Denote by X_F a 1-connected Ω -spectrum whose 0-th space is the infinite loop space $BSL(F)^+$: the homotopy groups of X_F are the K -groups of F in dimensions ≥ 2 . This spectrum is of interest for algebraic K -theory because of the following result.

Theorem 2.1. *For $i \geq 2$, the Hurewicz homomorphism with coefficients in $\mathbb{Z}_{(\ell)}$, the integers localized at ℓ , $\tilde{h}_i : K_i(F; \mathbb{Z}_{(\ell)}) \rightarrow H_i(X_F; \mathbb{Z}_{(\ell)})$ is an isomorphism if ℓ is a prime number $> \frac{i+1}{2}$.*

Proof. Since the spectrum X_F is 1-connected, its Postnikov k -invariants $k^{i+1}(X_F)$ are cohomology classes of finite order ρ_i for $i \geq 3$, and ρ_i is only divisible by primes $p \leq \frac{i+1}{2}$ (see [A3], Theorem 1.5). Now, let us write $X_F[i]$ for the i -th Postnikov section of X_F (i.e., $X_F[i]$ is a spectrum with $\pi_j X_F[i] = 0$ for $j > i$, $\pi_j X_F \cong \pi_j X_F[i]$ for $j \leq i$), and for any prime number ℓ , $(X_F[i])_{(\ell)}$ for its localization at ℓ , which has the property that $\pi_j (X_F[i])_{(\ell)} \cong (K_j F)_{(\ell)} \cong K_j(F; \mathbb{Z}_{(\ell)})$ for $j \leq i$. If $\ell > \frac{i+1}{2}$, all k -invariants of $(X_F[i])_{(\ell)}$ are trivial and $(X_F[i])_{(\ell)}$ is a wedge of Eilenberg-MacLane spectra:

$$(X_F[i])_{(\ell)} \simeq \bigvee_{j=2}^i \Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)}))$$

(for any abelian group G , $H(G)$ denotes the Eilenberg-MacLane spectrum having all homotopy groups trivial except for G in dimension 0). Then, it is easy to compute

$$H_i(X_F; \mathbb{Z}_{(\ell)}) \cong H_i(X_F[i]_{(\ell)}; \mathbb{Z}) \cong \bigoplus_{j=2}^i H_i(\Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)})); \mathbb{Z}).$$

But it follows from [C], Théorème 2 or [A3], Proposition 1.3 that $H_i(\Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)})); \mathbb{Z})$ is trivial if $j < i < j + 2\ell - 2$. Consequently, the condition $i < 2\ell - 1$ produces the desired assertion since $H_i(X_F; \mathbb{Z}_{(\ell)}) \cong H_i(\Sigma^i H(K_i(F; \mathbb{Z}_{(\ell)}), i); \mathbb{Z}) \cong K_i(F; \mathbb{Z}_{(\ell)})$.

Remark 2.2. From the theorem, it is true that $K_i F$ and $H_i(X_F; \mathbb{Z})$ have isomorphic subgroups of ℓ -torsion divisible elements if $\ell > \frac{i+1}{2}$. We shall prove in another paper that for any bounded below spectrum, the cokernel of the Hurewicz homomorphism is a group of finite exponent. Consequently, *all* divisible elements in $H_i(X_F; \mathbb{Z})$ belong to the image of the Hurewicz homomorphism $\tilde{h}_i : K_i F \rightarrow H_i(X_F; \mathbb{Z})$ (but, perhaps, they are images of elements which are not divisible in $K_i F$).

Remark 2.3. If we look at integers $i \geq 1$, we may also consider the Hurewicz homomorphism $K_i(F; \mathbb{Z}_{(\ell)}) \rightarrow H_i(Y_F; \mathbb{Z}_{(\ell)})$, where Y_F denotes a 0-connected Ω -spectrum whose 0-th space is $BGL(F)^+$: then, the conclusion of Theorem 2.1 holds for primes $\ell > \frac{i}{2} + 1$.

It is also useful to consider the Hurewicz homomorphism on the space level

$$h_i : K_i(F; \mathbb{Z}_{(\ell)}) \longrightarrow H_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) \cong H_i(SL(F); \mathbb{Z}_{(\ell)})$$

for $i \geq 2$, and the commutative diagram

$$\begin{array}{ccc}
K_i(F; \mathbb{Z}_{(\ell)}) \cong \pi_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) & \xrightarrow{h_i} & H_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) \cong H_i(SL(F); \mathbb{Z}_{(\ell)}) \\
\downarrow \cong & & \downarrow \sigma \\
K_i(F; \mathbb{Z}_{(\ell)}) \cong \pi_i(X_F; \mathbb{Z}_{(\ell)}) & \xrightarrow{\tilde{h}_i} & H_i(X_F; \mathbb{Z}_{(\ell)}),
\end{array}$$

where σ denotes the iterated homology suspension. Thus, Theorem 2.1 has the following immediate consequence (see also [A2], Section 2).

Corollary 2.4. *If i is an integer ≥ 2 and ℓ a prime number $> \frac{i+1}{2}$, then the Hurewicz homomorphism $h_i : K_i(F; \mathbb{Z}_{(\ell)}) \rightarrow H_i(SL(F); \mathbb{Z}_{(\ell)})$ is a split injection.*

Since we know that $D(i) = 0$ for odd i 's, let us consider $i = 2n$ and obtain the following splitting result.

Corollary 2.5. *If n is a positive integer and ℓ a prime number $> n$, then the Hurewicz homomorphism $h_{2n} : D(2n)_\ell \rightarrow \overline{D}(2n)_\ell$ is a split injection.*

Of course, if F is totally real, $i = 2n$ an even integer with n odd and ℓ a prime $> n$, then Banaszak's formula for the order of $D(2n)_\ell$ asserts that $\overline{D}(2n)_\ell$ is *non-trivial* for suitable n and ℓ . If $F = \mathbb{Q}$ for instance, $D(2n)_\ell$ is non-trivial if ℓ is an irregular prime and n an odd integer such that ℓ divides the numerator of $\frac{B_{n+1}}{n+1}$. Actually, it turns out that, in general, $\overline{D}(i)_\ell$ is bigger than $D(i)_\ell$ ($\ell > \frac{i+1}{2}$).

Theorem 2.6. *If F is a totally real number field, there exist positive integers i and prime numbers $\ell > \frac{i+1}{2}$ such that the group $H_i(SL(F); \mathbb{Z})$ contains non-trivial ℓ -torsion divisible elements which do not belong to the image of the Hurewicz homomorphism $h_i : K_i F \rightarrow H_i(SL(F); \mathbb{Z})$. In particular, $H_i(SL(F); \mathbb{Z})$ may contain non-trivial divisible elements even if i is odd or if $i = 2n$ with n even.*

Proof. If i is a positive integer and ℓ a prime $> \frac{i+1}{2}$, then all k -invariants of the localized i -th Postnikov section $(BSL(F)^+[i])_{(\ell)}$ of $BSL(F)^+$ are trivial since this is the case for

the spectrum $(X_F[i])_{(\ell)}$ (see the proof of Theorem 2.1). Therefore, $(BSL(F)^+[i])_{(\ell)}$ is a product of Eilenberg-MacLane spaces:

$$(BSL(F)^+[i])_{(\ell)} \simeq \prod_{j=2}^i K(K_j(F; \mathbb{Z}_{(\ell)}), j).$$

This homotopy equivalence and the Künneth formula provide a calculation of

$$H_i(SL(F); \mathbb{Z}_{(\ell)}) \cong H_i((BSL(F)^+[i])_{(\ell)}; \mathbb{Z}) \cong H_i\left(\prod_{j=2}^i K(K_j(F; \mathbb{Z}_{(\ell)}), j); \mathbb{Z}\right).$$

This homology group has not only $H_i(K(K_i(F; \mathbb{Z}_{(\ell)}), i); \mathbb{Z}) \cong K_i(F; \mathbb{Z}_{(\ell)})$ as direct summand, but also mixed terms, for instance of the form

$$K_{2m}(F; \mathbb{Z}_{(\ell)}) \otimes \left(K_{j_1}(F; \mathbb{Z}_{(\ell)}) \otimes K_{j_2}(F; \mathbb{Z}_{(\ell)}) \otimes \cdots \otimes K_{j_s}(F; \mathbb{Z}_{(\ell)}) \right),$$

where $2m + j_1 + j_2 + \cdots + j_s = i$; however, the right hand side of this tensor product includes a free $\mathbb{Z}_{(\ell)}$ -module if j_1, j_2, \dots, j_s are $\equiv 1 \pmod{4}$ and ≥ 5 (see [Bo]). If this occurs for m odd, then all elements of $D(2m)_\ell$ are divisible in the above mixed term. Consequently, $\overline{D}(i)_\ell$ contains not only $D(i)_\ell$, but also $D(2m)_\ell$ for suitable choices of $m \leq \frac{i-5}{2}$. This may happen even if i is odd or if $i = 2n$ with n even.

Example 2.7. Take $F = \mathbb{Q}$ and $\ell = 691$. It is known that $D(22)_{691}$ is non-trivial (see [B1], Section VIII.3) and that $K_j\mathbb{Q}/\text{torsion}$ is infinite cyclic if $j \equiv 1 \pmod{4}$ and ≥ 5 . The argument introduced in the previous proof exhibits, for instance, non-trivial elements in $\overline{D}(27)_{691}$, in $\overline{D}(36)_{691}/D(36)_{691}$, and in $\overline{D}(66)_{691}/D(66)_{691}$.

It is easy to deduce from Theorem 2.1 that the divisible elements detected by Theorem 2.6 vanish under σ .

Corollary 2.8. *If i is an integer ≥ 2 and ℓ a prime number $> \frac{i+1}{2}$, then the iterated homology suspension $\sigma : H_i(SL(F); \mathbb{Z}_{(\ell)}) \rightarrow H_i(X_F; \mathbb{Z}_{(\ell)})$ satisfies $\sigma(\overline{D}(i)_\ell/D(i)_\ell) = 0$.*

Remark 2.9. As we mentioned in the introduction, all divisible elements in $K_i F$ belong to the image of the homomorphism $r_* : K_i O \rightarrow K_i F$ induced by the inclusion $r : O \hookrightarrow F$. If i is a positive integer and ℓ a prime $> \frac{i+1}{2}$, it follows obviously from Theorem 2.1 that the ℓ -torsion divisible elements in $H_i(X_F; \mathbb{Z})$ are also elements of the image of the induced homomorphism $r_* : H_i(X_O; \mathbb{Z}) \rightarrow H_i(X_F; \mathbb{Z})$. We do not know the answer of the following *question*: is $\overline{D}(i)_\ell$ contained in the image of $r_* : H_i(SL(O); \mathbb{Z}) \rightarrow H_i(SL(F); \mathbb{Z})$?

3. A vanishing theorem

The study of the Serre spectral sequence of the fibration

$$\prod_m BQP(O/m) \longrightarrow BQP(O) \longrightarrow BQP(F)$$

(introduced in Section 1) shows that $H_i(SL(F); \mathbb{Z})$ contains in general a lot of ℓ -torsion elements for all primes ℓ . The goal of this section is to prove that for certain choices of the integer i and the prime ℓ , the group $H_i(SL(F); \mathbb{Z})$ has no non-trivial ℓ -torsion divisible elements.

Theorem 3.1. *If N is a positive integer and ℓ a prime number $> N$ with the property that $D(2n)_\ell = 0$ for all positive $n \leq N$, then $\overline{D}(i)_\ell = 0$ for all positive $i \leq 2N$.*

Proof. As in the proof of Theorem 2.6, the assumption $\ell > N$ provides a homotopy equivalence

$$(BSL(F)^+[2N])_{(\ell)} \simeq \prod_{j=2}^{2N} K(K_j(F; \mathbb{Z}_{(\ell)}), j).$$

According to [B2], Section II.1, Corollary 1, the vanishing of $D(2n)_\ell$ implies the splitting

$$K_{2n}(F; \mathbb{Z}_{(\ell)}) \cong K_{2n}(O; \mathbb{Z}_{(\ell)}) \oplus \left(\bigoplus_m K_{2n-1}(O/m; \mathbb{Z}_{(\ell)}) \right).$$

Therefore, $K_{2n}(F; \mathbb{Z}_{(\ell)})$ is a direct sum of finitely generated $\mathbb{Z}_{(\ell)}$ -modules and the same is true for $H_k(K(K_{2n}(F; \mathbb{Z}_{(\ell)}), 2n); \mathbb{Z})$, for all $k \geq 1$ ($2 \leq 2n \leq 2N$). On the other hand, $K_j(F; \mathbb{Z}_{(\ell)})$ is finitely generated if j is odd because of the localization exact sequence. We may finally conclude by the Künneth formula that, for $i \leq 2N$,

$$H_i(SL(F); \mathbb{Z}_{(\ell)}) \cong H_i((BSL(F)^+[2N])_{(\ell)}; \mathbb{Z}) \cong H_i\left(\prod_{j=2}^{2N} K(K_j(F; \mathbb{Z}_{(\ell)}), j); \mathbb{Z}\right)$$

is again a direct sum of finitely generated $\mathbb{Z}_{(\ell)}$ -modules, and hence has no non-trivial ℓ -torsion divisible elements, since the ℓ -torsion subgroup of any finitely generated $\mathbb{Z}_{(\ell)}$ -module is finite. In other words, we get $\overline{D}(i)_\ell = 0$ for $i \leq 2N$.

Remark 3.2. It is shown in [BG] that, for $F = \mathbb{Q}$, the Kummer-Vandiver conjecture [W, p.159] holds if and only if $D(2n)_\ell = 0$ for n even, ℓ odd. It is known (loc.cit.) that this

conjecture holds for $\ell < 125'000$. Thus, the formula in the introduction for the order of $D(2n)_\ell$, for n odd, makes it easy to check the hypothesis of Theorem 3.1 for $\ell < 125'000$ and $F = \mathbb{Q}$.

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