Linear group homology properties of the inclusion of a ring of integers into a number field

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1. Introduction and statement of the main results

Let F be a number field, O the ring of algebraic integers in F, and let θ denote the inclusion $O \hookrightarrow F$. The localization exact sequence in algebraic K-theory splits into short exact sequences

$$0 \longrightarrow K_nO \xrightarrow{\theta_\sharp} K_nF \longrightarrow \bigoplus_m K_{n-1}(O/m) \longrightarrow 0$$

for all positive integers n, where $\theta_\sharp$ is the homomorphism induced by θ in K-theory and where m runs over the set of maximal ideals of O (see Section 5 of [Q1], Theorem 8 of [Q2] and Théorème 1 of [S2]); in particular, $\theta_\sharp$ is always injective. On the other hand, G. Banaszak investigated the subgroup of divisible elements in K_nF and explained the important role of these elements in relation with the Lichtenbaum-Quillen conjecture and étale K-theory (see [B1], [B2], [BG], [BZ]). If n is odd, K_nF is a finitely generated abelian group and has therefore no non-trivial divisible elements. If n is even, K_nF is a large torsion group but all its divisible elements belong to the image of $\theta_\sharp$ because $\bigoplus_m K_{n-1}(O/m)$ is a direct sum of finite cyclic groups and hence contains no non-trivial divisible elements.

In this note, we consider similar questions about the homomorphism

$$\theta_* : H_n(SL(O);\mathbb{Z}) \longrightarrow H_n(SL(F);\mathbb{Z})$$

induced by θ on the integral homology of the infinite special linear group ($n \geq 0$). Our main objective is to answer the following two questions.

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Question 1.1. For which dimensions n is the homomorphism θ_* injective?

Question 1.2. For which dimensions n does the image of θ_* contain all divisible elements of $H_n(SL(F);\mathbb{Z})$?

Both questions concern the structure of the groups $H_n(SL(F);\mathbb{Z})$. Remember that $H_n(SL(O);\mathbb{Z})$ is a finitely generated abelian group for all $n \geq 0$, while $H_n(SL(F);\mathbb{Z})$ is not finitely generated. However, it was shown in Section 2 of [A1] that $H_n(SL(F);\mathbb{Z})$ is the direct sum of a free abelian group of finite (and known) rank and a torsion group. Some information about this torsion group is given in [AZ], where we prove that for any $n \geq 0$, it contains finitely many non-trivial divisible homology classes. Since $SL(F) = \varinjlim_S SL(O_S)$ for any exhaustion of the set of maximal ideals in O by finite subsets S (here O_S denotes the ring where the elements of the ideals in S are inverted), the image of the induced homomorphism $H_n(SL(O_S);\mathbb{Z}) \to H_n(SL(F);\mathbb{Z})$ for a certain finite set S of maximal ideals contains the whole subgroup of classes divisible in $H_n(SL(F);\mathbb{Z})$.

Now, let us formulate the main results of the paper.

Definition 1.3. For a prime p , let d_p denote the smallest positive integer n for which K_nF contains non-trivial p-torsion divisible elements (observe that d_p is even and that it depends on F); if there are no non-trivial p-torsion divisible elements in K_nF for all $n \geq 1$, we say that $d_p = \infty$. For instance if $F = \mathbb{Q}$ and p an odd prime $< 125'000$, then according to Theorem 3 of [B2] and Corollary 4 of [BG], $d_p = \infty$ if and only if p is regular, and if p is irregular, d_p is the smallest even integer $2i$, with i odd, such that p divides the numerator of $\dfrac{B_{i+1}}{i+1}$, where B_{i+1} is the $(i+1)$-st Bernoulli number.

Theorem 1.4. *For any prime number p , the homomorphism*

$$\theta_* : H_n(SL(O);\mathbb{Z}_{(p)}) \longrightarrow H_n(SL(F);\mathbb{Z}_{(p)})$$

is injective for all integers n such that $2 \leq n \leq \min(2p-2, d_p+1)$.

Theorem 1.5. *Let p be a prime number and n an integer satisfying $2 \leq n \leq \min(2p-2, 2d_p-1)$. Then all p-torsion divisible homology classes in $H_n(SL(F);\mathbb{Z})$ belong to the image of the homomorphism*

$$\theta_* : H_n(SL(O);\mathbb{Z}) \to H_n(SL(F);\mathbb{Z}) .$$

Sections 2 and 3 are devoted to the proof of Theorems 1.4. and 1.5 respectively and to the discussion of related problems.

2. The injectivity of θ_*

The purpose of this section is to investigate Question 1.1. Write again θ for the map $BSL(O)^+ \to BSL(F)^+$, induced by the inclusion $O \hookrightarrow F$, between the spaces obtained by performing the plus construction on the classifying spaces of the respective special linear groups (remember that these spaces have the same homology as the associated special linear groups). Our argument is based on the comparison of these spaces with the corresponding products of Eilenberg-MacLane spaces.

Lemma 2.1. *Let p be an odd prime number, n an integer ≥ 2, and consider the product map*

$$\eta = \prod_{j=2}^{n} \eta_j : \prod_{j=2}^{n} K(K_j O, j) \longrightarrow \prod_{j=2}^{n} K(K_j F, j)\,,$$

where $\eta_j : K(K_j O, j) \to K(K_j F, j)$ is induced by the map θ (for $2 \leq j \leq n$). If $d_p = \infty$, the induced homomorphism

$$\eta_* : H_n(\prod_{j=2}^{n} K(K_j O, j); \mathbb{Z}_{(p)}) \longrightarrow H_n(\prod_{j=2}^{n} K(K_j F, j); \mathbb{Z}_{(p)})$$

is split injective for any $n \geq 2$; if $d_p < \infty$, it is split injective for $2 \leq n \leq d_p - 1$ or $n = d_p + 1$, and injective for $n = d_p$.

Proof. For all positive integers j, the inclusion $\theta : O \hookrightarrow F$ induces an injection $K_j O \to K_j F$ as mentioned in the introduction. Moreover, Banaszak proved in [B2], Corollary 1 that $K_j(O; \mathbb{Z}_{(p)}) \to K_j(F; \mathbb{Z}_{(p)})$ is a split injection for $j \leq d_p - 1$ (or for all j's if $d_p = \infty$) if p is odd. If $d_p = \infty$, η_* is obviously split injective for all $n \geq 2$. If $d_p < \infty$, the assertion is trivial for $n = 2$, and for $3 \leq n \leq d_p + 1$, we conclude by Künneth formula that

$$\eta_* : H_n(\prod_{j=2}^{n} K(K_j O, j); \mathbb{Z}_{(p)}) \cong K_n(O; \mathbb{Z}_{(p)}) \oplus H_n(\prod_{j=2}^{n-2} K(K_j O, j); \mathbb{Z}_{(p)}) \longrightarrow$$

$$K_n(F; \mathbb{Z}_{(p)}) \oplus H_n(\prod_{j=2}^{n-2} K(K_j F, j); \mathbb{Z}_{(p)}) \cong H_n(\prod_{j=2}^{n} K(K_j F, j); \mathbb{Z}_{(p)})$$

is an injection on the first factor (which splits if n is even $\leq d_p - 1$ and which is an isomorphism if n is odd) and a split injection on the second factor since $n-2 \leq d_p - 1$.

Definition 2.2. Let $M_1 := 1$, and for an integer $h \geq 2$ let M_h be the product of all primes $p \leq \frac{h}{2} + 1$.

Definition 2.3. Let $L_1 := 1$, and for an integer $k \geq 2$ let L_k denote the product of all primes p for which there exists a sequence of non-negative integers $(a_1, a_2, a_3, \ldots)$ satisfying:
(a) $a_1 \equiv 0 \pmod{2p-2}$, $a_i \equiv 0$ or $1 \pmod{2p-2}$ for $i \geq 2$,
(b) $a_i \geq p a_{i+1}$ for $i \geq 1$,
(c) $\sum_{i=1}^{\infty} a_i = k$.

Notice that L_k divides M_h if $k \leq h$. These integers occur in the computation of the stable homology groups of Eilenberg-MacLane spaces (see [C], Théorème 2):

Proposition 2.4. *For any abelian group G and any pair of integers i and j with $j < i < 2j$, one has $L_{i-j}\, H_i(K(G,j);\mathbb{Z}) = 0$. Consequently, if $n \geq 1$, the integer M_n fulfills $M_n\, H_i(K(G,j);\mathbb{Z}) = 0$ for all i and j with $j < i < 2j$ and $i - j \leq n$.*

Definition 2.5. Let $R_j := \prod_{k=1}^{j} L_k$ for $j \geq 2$. For example, $R_2 = 2$, $R_3 = 4$, $R_4 = 24$, $R_5 = 144$, $R_6 = 288, \ldots$ Then, define $\overline{R}_i := \prod_{j=2}^{i} R_j$ for $i \geq 2$. It turns out that a prime number p divides $\overline{R}_i$ if and only if $p \leq \frac{i}{2} + 1$.

If X is a CW-complex and n a positive integer, let us write $X[n]$ for the n-th Postnikov section of X (i.e., $X[n]$ is a CW-complex with $\pi_i X[n] = 0$ for $i > n$ and $\pi_i X \cong \pi_i X[n]$ for $i \leq n$).

Proposition 2.6. *(a) If X is an $(m-1)$-connected infinite loop space (with $m \geq 2$) and n an integer $\geq m+1$, then there exist maps*

$$X[n] \xrightarrow{\varphi} \prod_{j=m}^{n} K(\pi_j X, j) \xrightarrow{\psi} X[n]$$

such that the composition is homotopic to the $\overline{R}_{n-m+1}$-th power map.
(b) If $f : X \to Y$ is an infinite loop map between $(m-1)$-connected infinite loop spaces and n an integer $\geq m+1$, then there is commutative diagram

$$
\begin{array}{ccccc}
X[n] & \xrightarrow{\varphi} & \prod_{j=m}^{n} K(\pi_j X, j) & \xrightarrow{\psi} & X[n] \\
\downarrow f & & \downarrow \xi & & \downarrow f \\
Y[n] & \xrightarrow{\varphi} & \prod_{j=m}^{n} K(\pi_j Y, j) & \xrightarrow{\psi} & Y[n]\,,
\end{array}
$$

where $\xi_* : \pi_j X \to \pi_j Y$ is exactly the homomorphism induced by f for $m \leq j \leq n$.
(c) Moreover, if η denotes the product map $\prod_{j=m}^{n} \eta_j$, where η_j is the map $K(\pi_j X, j) \to K(\pi_j Y, j)$ induced by f (for $m \leq j \leq n$), and if we set $\zeta = \xi - \eta$, then, for any prime number p, the induced homomorphism

$$
\zeta_* : H_n(\prod_{j=m}^{n} K(\pi_j X, j); \mathbb{Z}_{(p)}) \longrightarrow H_n(\prod_{j=m}^{n} K(\pi_j Y, j); \mathbb{Z}_{(p)})
$$

is trivial assuming that $n \leq m + 2p - 3$.

Proof. The Postnikov k-invariants of X satisfy $R_{n-m+1} k^{n+1}(X) = 0$ in $H^{n+1}(X[n-1]; \pi_n X)$ for all $n \geq m+1$, since X is an infinite loop space (see for example [A3], Remark 1.6). Thus, Assertions (a) and (b) follow from the argument explained in Section 1 of [A3]. Because of the property of ξ_* stated in (b), the restriction of $\zeta = \xi - \eta$ to $K(\pi_j X, j)$ is actually an infinite loop map

$$
\zeta_j : K(\pi_j X, j) \longrightarrow Z_j := \prod_{i=j+1}^{n} K(\pi_i Y, i) \hookrightarrow \prod_{i=m}^{n} K(\pi_i Y, i)
$$

for any j such that $m \leq j \leq n-1$. Now, fix a prime p. It follows from [C] that for $j+1 \leq i \leq j+2p-3$, the homology groups with coefficients localized at p $H_i(K(\pi_j X, j); \mathbb{Z}_{(p)})$ contain only sums of products of homology classes represented by integral multiples of cycles of the form $\gamma_s(x)$ if j is even, respectively $x\gamma_s(y)$ if j is odd, where $x \in H_j(K(\pi_j X, j); \mathbb{Z}_{(p)})$ and $\deg(y) = j+1$ ($\gamma_s(-)$ denotes the s-th divided power); notice that $1 \leq s \leq p-1$ since $i \leq j + 2p - 3$. The ring homomorphism $(\zeta_j)_* : H_*(K(\pi_j X, j); \mathbb{Z}_{(p)}) \to H_*(Z_j; \mathbb{Z}_{(p)})$ maps any class x of degree j onto 0 because $H_j(Z_j; \mathbb{Z}_{(p)}) = 0$. If j is even, this implies that $s!(\zeta_j)_*(\gamma_s(x)) = (\zeta_j)_*(x^s) = ((\zeta_j)_*(x))^s = 0$, and consequently that $(\zeta_j)_*(\gamma_s(x)) = 0$ since p does not divide $s!$. Similarly if j is odd, it is obvious that $(\zeta_j)_*(x\gamma_s(y)) = 0$. The vanishing of the homomorphism

$$\zeta_* : H_n(\prod_{j=m}^{n} K(\pi_j X, j); \mathbb{Z}_{(p)}) \longrightarrow H_n(\prod_{j=m}^{n} K(\pi_j Y, j); \mathbb{Z}_{(p)})$$

follows now from Künneth formula.

Proof of Theorem 1.4. If $p = 2$, the statement of the theorem concerns only the dimension $n = 2$ where $\theta_\sharp : K_2 O \to K_2 F$ is injective. Thus, we may assume that p is an odd prime. According to the previous proposition for the map $\theta : BSL(O)^+ \to BSL(F)^+$ (with $m = 2$), we obtain a commutative diagram in homology localized at p .

$$\begin{array}{ccccc}
H_n(BSL(O)^+; \mathbb{Z}_{(p)}) & \xrightarrow{\varphi_*} & H_n(\prod_{j=2}^{n} K(K_j O, j); \mathbb{Z}_{(p)}) & \xrightarrow{\psi_*} & H_n(BSL(O)^+; \mathbb{Z}_{(p)}) \\
\downarrow \theta_* & & \downarrow \xi_* & & \downarrow \theta_* \\
H_n(BSL(F)^+; \mathbb{Z}_{(p)}) & \xrightarrow{\varphi_*} & H_n(\prod_{j=2}^{n} K(K_j F, j); \mathbb{Z}_{(p)}) & \xrightarrow{\psi_*} & H_n(BSL(F)^+; \mathbb{Z}_{(p)}) ,
\end{array}$$

such that both horizontal compositions are multiplication by the p-primary part $(\overline{R}_{n-1})_p$ of $\overline{R}_{n-1}$. Since $n \le 2p - 2$, it turns out by Proposition 2.6 (c) that ξ_* is exactly the homomorphism η_* introduced in Lemma 2.1; hence, Lemma 2.1 and the hypothesis $n \le d_p + 1$ (if $d_p < \infty$) imply that ξ_* is injective. Consequently, if x belongs to the kernel of $\theta_* : H_n(BSL(O)^+; \mathbb{Z}_{(p)}) \to H_n(BSL(F)^+; \mathbb{Z}_{(p)})$, then $\varphi_*(x) = 0$ and $(\overline{R}_{n-1})_p \, x = \psi_* \varphi_*(x) = 0$. The conclusion follows from the fact that p does not divide $\overline{R}_{n-1}$ because $n \le 2p - 2$.

Remark 2.7. The homomorphism $\theta_* : H_n(SL(O); \mathbb{Z}) \to H_n(SL(F); \mathbb{Z})$ is not injective for all n . Consider the case $O = \mathbb{Z}$, $F = \mathbb{Q}$, $n = 4(i-1)$ with i even, p a properly irregular prime with $p > i$ and such that p divides the numerator of $\frac{B_i}{i}$. There is a p-torsion element y in $H_{2(i-1)}(SL(\mathbb{Z}); \mathbb{Z})$ (see [S1], p. 290 and [A2], Section 2), and $\theta_*(y)$ is divisible in $H_{2(i-1)}(SL(\mathbb{Q}); \mathbb{Z})$ (by [B2], Theorem 3, and [AZ], Corollary 2.5). If w is a generator of the cyclic direct summand of $H_{2(i-1)}(SL(\mathbb{Z}); \mathbb{Z})$ containing y , then the Pontryagin product yw is non-trivial in $H_n(SL(\mathbb{Z}); \mathbb{Z})$, according to Corollary 2.3 of [A2], and belongs to the kernel of θ_* since $\theta_*(yw) = \theta_*(y)\, \theta_*(w) = 0$ because of the divisibility of $\theta_*(y)$. (We would like to thank C. Ausoni for this example.)

3. Divisible homology classes

Partial answers to Question 1.2 are given by the first two propositions while Theorem 1.5 is proved at the end of this section. If S is a set of maximal ideals of O, let $O_{(S)}$ denote the localization of O at S (i.e., the ring where the elements which are not in the ideals of S are inverted) and $\theta_{(S)}$ the inclusion $O_{(S)} \hookrightarrow F$.

Proposition 3.1. *Let n be an integer ≥ 2 and x any divisible homology class in $H_n(SL(F);\mathbb{Z})$. Then, for all finite sets S of maximal ideals of O, x belongs to the image of the homomorphism*

$$(\theta_{(S)})_* : H_n(SL(O_{(S)});\mathbb{Z}) \longrightarrow H_n(SL(F);\mathbb{Z})$$

induced by $\theta_{(S)}$. Moreover, there is a divisible element $y \in H_n(SL(O_{(S)});\mathbb{Z})$ such that $(\theta_{(S)})_*(y) = x$.

Proof. There is a fibration

$$\prod_{m\in S} BSL(O/m)^+ \longrightarrow BSL(O_{(S)})^+ \xrightarrow{\theta_{(S)}} BSL(F)^+$$

(see [Q2], Theorem 4 or [Q1], Section 7, Proposition 3.2). The associated Serre spectral sequence

$$E^2_{s,t} \cong H_s(BSL(F)^+; H_t(\prod_{m\in S} BSL(O/m)^+;\mathbb{Z}) \Longrightarrow H_{s+t}(BSL(O_{(S)})^+;\mathbb{Z})$$

has the property that $H_t(\prod_{m\in S} BSL(O/m)^+;\mathbb{Z})$ is finite whenever $t \geq 1$, because the set S is finite and the O/m's are finite fields. This shows that $E^r_{s,t}$ is a group of finite exponent, and consequently has no divisible elements except 0, for any $r \geq 2$, $s \geq 0$, $t \geq 1$. If x is divisible in $H_n(SL(F);\mathbb{Z}) \cong E^2_{n,0}$, then $d^r(x)$ must also be divisible in $E^r_{n-r,r-1}$: thus, $d^r(x) = 0$ for all $r \geq 2$ and $x \in E^\infty_{n,0}$, in other words, x belongs to the image of $(\theta_{(S)})_*$.
Another way to prove this assertion is to observe that all homotopy groups of the fibre of the above fibration are finite and to deduce, using Serre class theory, that the homomorphism $(\theta_{(S)})_*$ has finite kernel and finite cokernel. Since its cokernel is finite, x is actually divisible in the image of $(\theta_{(S)})_*$ and the argument of Lemma 11 of [B1] enables us to conclude that there exists a divisible element y in $H_n(SL(O_{(S)})^+;\mathbb{Z})$ such that $(\theta_{(S)})_*(y) = x$.

Proposition 3.2. *Let p be a prime and n an integer with $2 \leq n \leq 2p-2$. If a p-torsion divisible element x of $H_n(SL(F);\mathbb{Z})$ belongs to the image of the Hurewicz homomorphism $h_n : K_nF \to H_n(SL(F);\mathbb{Z})$, then*
(a) there is a p-torsion divisible element $z \in K_nF$ such that $h_n(z) = x$,
(b) x belongs to the image of $\theta_* : H_n(SL(O);\mathbb{Z}) \to H_n(SL(F);\mathbb{Z})$.

Proof. Assertion (a) is obvious since h_n is split injective on p-torsion for large primes p, i.e., if $p \geq \frac{n}{2}+1$ (see Corollary 2.4 of [AZ]). Assertion (b) follows from the fact, mentioned in the introduction, that any divisible element z of K_nF belongs to the image of $\theta_\sharp : K_nO \to K_nF$.

Lemma 3.3. *Let p be an odd prime such that $d_p < \infty$, n an integer satisfying $2 \leq n \leq 2d_p - 1$, and set $\mu = 1$ if $n < 4p-4$ and $\mu = p$ if $n \geq 4p-4$. If x is a p-torsion divisible element in $H_n(\prod_{j=2}^n K(K_jF,j);\mathbb{Z})$, then μx is the image of a p-torsion class under the homomorphism*

$$\eta_* : H_n(\prod_{j=2}^n K(K_jO,j);\mathbb{Z}) \longrightarrow H_n(\prod_{j=2}^n K(K_jF,j);\mathbb{Z})$$

induced by the map η introduced in the statement of Lemma 2.1.

Proof. Let us write

$$H_n(\prod_{j=2}^n K(K_jF,j);\mathbb{Z}_{(p)}) = H_n(Y \times \prod_{j=[\frac{n}{2}]+1}^n K(K_jF,j);\mathbb{Z}_{(p)})$$

where $Y = \prod_{j=2}^{[\frac{n}{2}]} K(K_jF,j)$ and $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$. If j is odd, $K_jF \cong K_jO$, and if j is even $\leq [\frac{n}{2}]$, $K_j(F;\mathbb{Z}_{(p)}) \cong K_j(O;\mathbb{Z}_{(p)}) \oplus (\bigoplus_m K_{j-1}(O/m;\mathbb{Z}_{(p)}))$ again by Corollary 1 of [B2], because $[\frac{n}{2}] \leq d_p - 1$. Therefore, for $2 \leq j \leq [\frac{n}{2}]$, $K_j(F;\mathbb{Z}_{(p)})$ is a direct sum of a free $\mathbb{Z}_{(p)}$-module of finite rank with an infinite direct sum of finite cyclic p-groups. On the other hand, since $\mu\, H_i(K(K_jF,j);\mathbb{Z}_{(p)}) = 0$ for $[\frac{n}{2}]+1 \leq j < i \leq n$ according to Proposition 2.4 (notice that μ is the p-primary part of $M_{[\frac{n}{2}]}$), we get

$$\mu\, H_i(\prod_{j=[\frac{n}{2}]+1}^n K(K_jF,j);\mathbb{Z}_{(p)}) \cong \mu\, K_i(F;\mathbb{Z}_{(p)})$$

for $[\frac{n}{2}]+1 \leq i \leq n$ and deduce from Künneth formula that

$$\mu\, H_n(\prod_{j=2}^{n} K(K_jF,j);\mathbb{Z}_{(p)}) \cong \mu\Big(\bigoplus_{i=[\frac{n}{2}]+1}^{n} K_i(F;\mathbb{Z}_{(p)}) \otimes H_{n-i}(Y;\mathbb{Z}_{(p)})\Big) \bigoplus \mu\Big(\bigoplus_{i=[\frac{n}{2}]+1}^{n-3} \operatorname{Tor}(K_i(F;\mathbb{Z}_{(p)}), H_{n-i-1}(Y;\mathbb{Z}_{(p)}))\Big)\,.$$

In that formula the groups $H_{n-i}(Y;\mathbb{Z}_{(p)})$, respectively $H_{n-i-1}(Y;\mathbb{Z}_{(p)})$, are direct sums of free $\mathbb{Z}_{(p)}$-modules of finite rank with infinite direct sums of finite cyclic p-groups. It then follows from the fact that $\otimes$ and Tor commute with direct sums that

$$\mu\, H_n(\prod_{j=2}^{n} K(K_jF,j);\mathbb{Z}_{(p)}) \cong \mu\bigg((\bigoplus_{i=[\frac{n}{2}]+1}^{n} K_i(F;\mathbb{Z}_{(p)}) \otimes A_i) \oplus G\bigg)\,,$$

where the groups A_i are free $\mathbb{Z}_{(p)}$-modules of finite rank and G is an infinite direct sum of groups of finite exponent; consequently, only the factors $K_i(F;\mathbb{Z}_{(p)}) \otimes A_i$ (for $[\frac{n}{2}]+1 \le i \le n$) may contain divisible elements. However, we explained in the introduction that any p-torsion divisible element of K_iF is the image of a p-torsion element of K_iO under the induced homomorphism $K_iO \to K_iF$ and we obtain the desired assertion.

Proof of Theorem 1.5. We may assume that $d_p < \infty$ since otherwise $H_n(SL(F);\mathbb{Z})$ contains no non-trivial divisible elements by Theorem 3.1 of [AZ]. If $p = 2$, n must be 2 and the theorem is a trivial consequence of the corresponding statement for the homomorphism $\theta_\sharp : K_2O \to K_2F$. If p is odd, look again at the diagram given by Proposition 2.6 for the map $\theta : BSL(O)^+ \to BSL(F)^+$ (with $m = 2$)

$$\begin{array}{ccccc}
H_n(BSL(O)^+;\mathbb{Z}) & \xrightarrow{\varphi_*} & H_n(\prod_{j=2}^{n} K(K_jO,j);\mathbb{Z}) & \xrightarrow{\psi_*} & H_n(BSL(O)^+;\mathbb{Z}) \\
\downarrow \theta_* & & \downarrow \xi_* & & \downarrow \theta_* \\
H_n(BSL(F)^+;\mathbb{Z}) & \xrightarrow{\varphi_*} & H_n(\prod_{j=2}^{n} K(K_jF,j);\mathbb{Z}) & \xrightarrow{\psi_*} & H_n(BSL(F)^+;\mathbb{Z})\,,
\end{array}$$

in which both horizontal compositions are multiplication by $\overline{R}_{n-1}$. If x is a p-torsion divisible homology class in $H_n(BSL(F)^+;\mathbb{Z})$, then $\varphi_*(x)$ is divisible in $H_n(\prod_{j=2}^{n} K(K_jF,j);\mathbb{Z})$, and Lemma 3.3 (in which $\mu = 1$ because of the condition $n \leq 2p-2$) implies that there is a p-torsion element $w \in H_n(\prod_{j=2}^{n} K(K_jO, j); \mathbb{Z})$ such that $\eta_*(w) = \varphi_*(x)$. It follows from Proposition 2.6 (c) that $\xi_*(w) = \eta_*(w) = \varphi_*(x)$. Therefore, $\overline{R}_{n-1}\, x = \psi_*(\phi_*(x))$ belongs to the image of θ_* . The proof is then complete because p does not divide $\overline{R}_{n-1}$ since $n \leq 2p-2$.

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