THE UCT, THE MILNOR SEQUENCE, AND A CANONICAL DECOMPOSITION OF THE KASPAROV GROUPS

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Abstract. Suppose that A is a C^*-algebra in the bootstrap category $\mathcal{N}$ with KK-filtration

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$$

and B is a C^*-algebra with a countable approximate unit. Then the graded Kasparov group $KK_*(A,B)$ is described both by the Universal Coefficient Theorem

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK_*(A,B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

and by the Milnor $\varprojlim^1$ sequence

$$0 \to \varprojlim{}^1 KK_*(A_i, B) \longrightarrow KK_*(A,B) \longrightarrow \varprojlim KK_*(A_i,B) \to 0.$$

It is demonstrated that these two descriptions are closely related and that $KK_*(A,B)$ decomposes unnaturally as the direct sum of the term

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

which stores index information, the term

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which is the $\mathbb{Z}$-adic completion of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$, and the term

$$\varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which houses the fine structure of $KK_*(A,B)$. (These groups depend only upon $K_*(A)$ and $K_*(B)$.) Further, the Milnor sequence itself splits unnaturally.

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1. Introduction

The graded Kasparov group $KK_*(A, B)$ associated to C^*-algebras A and B has been of use in a wide variety of applications in operator algebras and topology. It is best behaved when A is a separable nuclear C^*-algebra and when B has a countable approximate unit, and we assume these properties throughout.

Let $\mathcal{N}$ be the smallest full subcategory of separable nuclear C^*-algebras which contains all separable Type I C^*-algebras and which is closed under strong Morita equivalence, direct limits, extensions, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$. We may also require that if J is an ideal of A and if $J,\, A \in \mathcal{N}$ then so is A/J, and if A and A/J are in $\mathcal{N}$ then so is J. We refer to $\mathcal{N}$ as the bootstrap category.

The Universal Coefficient theorem may then be stated.

Theorem 1.1. *(Rosenberg and Schochet* [RS]*) If $A \in \mathcal{N}$ then the natural degree zero map*

$$\gamma : KK_*(A, B) \longrightarrow Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

induces a natural short exact sequence

$$\text{(UCT)} \qquad 0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A, B) \xrightarrow{\gamma} Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

which splits unnaturally. The map δ has degree one (that is, it takes odd degree elements to even degree elements and vice versa.)

The sequence (UCT) determines $KK_*(A, B)$ in terms of $K_*(A)$ and $K_*(B)$.

Next we recall the Milnor $\varprojlim^1$ sequence. This was first observed to hold for $KK_*(A, \mathbb{C})$ by L.G. Brown.[1]

Theorem 1.2. *(Milnor, cf.* [S3 Theorem 7.1] *for a proof of the general case.) Suppose that C^*-algebras $A = \varinjlim A_i$ and B are given. Then the natural degree zero maps*

$$\rho_i : KK_*(A, B) \longrightarrow KK_*(A_i, B)$$

induce a natural short exact sequence

$$0 \to \varprojlim{}^1 KK_*(A_i, B) \xrightarrow{\sigma} KK_*(A, B) \xrightarrow{\rho} \varprojlim KK_*(A_i, B) \to 0$$

with σ of degree one and $\rho = \varprojlim \rho_i$.

□

Next we record a most useful result of Roos.

Theorem 1.3. *(Roos* [R]*) Let $\{G_i\}$ be a direct sequence of abelian groups and let M be an abelian group. Then there is a natural short exact sequence*

[1] Direct and inverse limits are indexed by the natural numbers when i is used as the index and by an arbitrary directed set when α is used as the index.

$$0 \to \varprojlim{}^1 Hom_{\mathbb{Z}}(G_i, M) \to Ext^1_{\mathbb{Z}}(\varinjlim G_i, M) \to \varprojlim Ext^1_{\mathbb{Z}}(G_i, M) \to 0$$

and in addition

$$\varprojlim{}^1 Ext^1_{\mathbb{Z}}(G_i, M) \cong 0.$$

□

We wish to employ Theorem 1.2 for C^*-algebras $A \in \mathcal{N}$. Not every $A \in \mathcal{N}$ is the direct limit of commutative or even of Type I algebras. However for our purposes a much weaker sort of limit condition suffices. Here is the key definition. The unitalization of a C^*-algebra A is denoted A^+.

Definition 1.4. *A KK-filtration of a separable C^*-algebra A is an increasing sequence of commutative C^*-algebras*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$$

which satisfies the following conditions:

1. $A_i^+ \cong C(X_i)$ *for some finite CW-complex X_i.*
2. *Each map $K_*(A_i) \to K_*(A_{i+1})$ is an inclusion.*
3. $\varinjlim A_i$ *is KK-equivalent to A.*

It follows that each $K_*(A_i)$ is finitely generated and that

$$\varinjlim K_*(A_i) \cong K_*(A)$$

so that the sequence $\{K_*(A_i)\}$ is an increasing sequence of finitely generated subgroups with limit $K_*(A)$. Since the UCT is preserved under KK-equivalence, it follows that any KK-filtered C^*-algebra A satisfies the UCT for all B.

Any commutative unital separable C^*-algebra $C(X)$ may be written as the direct limit of a sequence $C(X_i)$ for finite complexes X_i by taking successive nerves of finite open covers of X. Since any $A \in \mathcal{N}$ is KK-equivalent to a commutative C^*-algebra by [RS, Cor. 7.5], this is close to saying that each $A \in \mathcal{N}$ has a KK-filtration. We shall prove the following Theorem in §5.

Theorem 1.5.

1. *Each $A \in \mathcal{N}$ has a KK-filtration.*
2. *If $\{A_i\}$ and $\{A_i'\}$ are KK-filtrations of A then there are isomorphisms*

(a)
$$\varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i'), K_*(B))$$

(b)
$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \cong \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i'), K_*(B))$$

(c)
$$\varprojlim KK_*(A_i, B) \cong \varprojlim KK_*(A_i', B)$$

which depend only upon the KK-equivalences specified, so that these groups depend only upon $K_*(A)$ and not upon the KK-filtrations.

(3) *If $\{A_i\}$ and $\{A'_i\}$ are KK-filtrations of A then there is an isomorphism of Milnor sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim^1 KK_*(A_i, B) & \longrightarrow & KK_*(A,B) & \longrightarrow & \varprojlim KK_*(A_i,B) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \varprojlim^1 KK_*(A'_i, B) & \longrightarrow & KK_*(A,B) & \longrightarrow & \varprojlim KK_*(A'_i,B) & \longrightarrow & 0 \end{array}$$

which depends only upon the choices of KK-equivalences

$$\varinjlim A_i \underset{KK}{\approx} A \underset{KK}{\approx} \varinjlim A'_i.$$

The following is our principal theorem. It gives a complete description of the relation between the Universal Coefficient Theorem and the Milnor sequence.

Theorem 1.6 Main Theorem. *Suppose that A has KK-filtration $\{A_i\}$. Then the following diagram*

(Main Diagram)

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & & & \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & & \\ & & \downarrow & & & & \downarrow \varprojlim \delta_i & & \\ 0 & \rightarrow & \varprojlim^1 KK_*(A_i,B) & \xrightarrow{\sigma} & KK_*(A,B) & \xrightarrow{\rho} & \varprojlim KK_*(A_i,B) & \rightarrow & 0 \\ & & \downarrow \psi & & \downarrow id & & \downarrow \tilde{\gamma} & & \\ 0 & \rightarrow & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\delta} & KK_*(A,B) & \xrightarrow{\gamma} & Hom_{\mathbb{Z}}(K_*(A), K_*(B)) & \rightarrow & 0 \\ & & \downarrow \varphi & & & & \downarrow & & \\ & & \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) & & & & 0 & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

is commutative, is natural with respect to A and B and has exact rows and columns. Each of the groups is independent of choice of KK-filtration and depends only upon $K_*(A)$ and $K_*(B)$. Further, each short exact sequence splits and these splittings are mutually coherent.[2]

The remainder of the paper is organized as follows. In Section 2 the Main Theorem is established. Section 3 is devoted to deducing some important consequences of the Main

[2]However, the splittings are not natural with respect to the pair (A, B).

Theorem including (3.1) the decomposition of $KK_*(A, B)$ into its three components. We also show how the recent work of Rørdam and Dadarlat-Loring is clarified by our results: their functors

$$KL_*(A, B)$$

and

$$Hom_\Lambda(\underline{K}_*(A), \underline{K}_*(B))$$

and associated exact sequences respectively turn out to be easily described in terms of the basic decomposition given by the Main Theorem and its corollaries. In Section 4 we turn the situation around and see how the proof of the Main Theorem helps us to understand the UCT itself. Section 5 is devoted to the existence and "uniqueness" of KK-filtrations.

Acknowledgments. This paper is an extension of joint work with Jerry Kaminker [KS] on $\varprojlim^1$ and its relation to KK-theory as well as an extension of joint work with Jonathan Rosenberg on the UCT [RS2, RS3]. It is a pleasure to acknowledge with gratitude both collaborations. In addition, we are grateful to Mikael Rørdam, to Marius Dadarlat and to Terry Loring for sending us early versions of their work and for freely sharing their ideas with us.

2. Proof of the Main Theorem

Proof of 1.6. A KK-equivalence $\varinjlim A_i \underset{KK}{\approx} A$ induces an isomorphism

$$KK_*(\varinjlim A_i, B) \cong KK_*(A, B)$$

which respects the UCT and the $\varprojlim^1$ sequences by Theorem 1.5. So for the purposes of this proof we may assume that $A = \varinjlim A_i$. Consider the Main Diagram. The two main rows are the Milnor $\varprojlim^1$ sequence and the UCT respectively. For each i there is a natural commutative square

$$\begin{array}{ccc} KK_*(A,B) & \xrightarrow{\rho_i} & KK_*(A_i,B) \\ \big\downarrow id & & \big\downarrow \gamma_i \\ KK_*(A,B) & \xrightarrow{\gamma_i\rho_i} & Hom_{\mathbb{Z}}(K_*(A_i),K_*(B)) \end{array}$$

and these induce a natural commutative diagram

$$\begin{array}{ccccc} KK_*(A,B) & \xrightarrow{\rho} & \varprojlim KK_*(A_i,B) & \xrightarrow{id} & \varprojlim KK_*(A_i,B) \\ \big\downarrow id & & \big\downarrow \varprojlim \gamma_i & & \big\downarrow \tilde{\gamma} \\ KK_*(A,B) & \xrightarrow{\varprojlim \gamma_i\rho_i} & \varprojlim Hom_{\mathbb{Z}}(K_*(A_i),K_*(B)) & \xrightarrow{\cong} & Hom_{\mathbb{Z}}(K_*(A),K_*(B)) \end{array}$$

where the bottom row is a factorization of γ. Thus $\gamma = \tilde{\gamma}\rho$. This implies that the squares in the Main Diagram commute.

We must identify the kernel and cokernel of the maps $\tilde{\gamma}$ and ψ in the Main Diagram. It is an immediate consequence of the Snake Lemma that $Ker(\psi) = 0$ and that $Cok(\tilde{\gamma}) = 0$, and we also know from that Lemma that

$$Ker(\tilde{\gamma}) \cong Cok(\varphi).$$

In order to identify $Ker(\tilde{\gamma})$ we argue as follows. The UCT sequence associated to the pair (A_i, B) has the form

$$(\diamondsuit_i) \qquad 0 \to Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \xrightarrow{\delta_i} KK_*(A_i, B) \xrightarrow{\gamma_i} Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to 0.$$

Taking the inverse limit of the short exact sequences $(\diamondsuit_i)$ over i yields a six term $\varprojlim$-$\varprojlim^1$ sequence. However,

$$\varprojlim{}^1 Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) = 0$$

by Theorem 1.3. After identifying

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \cong Hom_{\mathbb{Z}}(\varinjlim K_*(A_i), K_*(B)) \cong \varprojlim Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

there is a short exact sequence

$$0 \to \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \xrightarrow{\varprojlim \delta_i} \varprojlim KK_*(A_i, B) \xrightarrow{\tilde{\gamma}} Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

which is the right column of the Main Diagram. This establishes the exactness of the Main Diagram.[3] We see from Theorem 1.5 that each of the groups is independent of choice of KK-filtration and hence depends only upon $K_*(A)$ and $K_*(B)$.

It remains to show that each of the exact sequences in the Main Diagram splits. The UCT sequence map γ is split by the UCT; choose a splitting map Γ, so that $\gamma\Gamma = 1$. Given this choice there are canonical choices for each of the remaining splittings as follows:

(1) $\tilde{\gamma}$ is split by $\tilde{\Gamma} = \rho\Gamma$.

(2) ρ is split by $P = \Gamma\tilde{\gamma}$.

(3) σ is split by $\Sigma = 1 - P\rho$.

(4) ψ is split by $\Psi = \Sigma\delta$.

These verifications are quite routine. We note that the choice of Γ determines the remaining splittings, so that the obvious diagrams commute. The map Γ cannot be chosen to be natural, and hence the other splittings share in this defect. □

[3]This same argument implies that the natural map

$$\varprojlim{}^1 KK_*(A_i, B) \xrightarrow{\varprojlim^1 \gamma_i} \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

is an isomorphism. We take this isomorphism as an identification henceforth. Note that this group is the closure of zero in the Salinas [Sal] topology of $KK_*(A, B)$ and is hence of considerable interest. It is frequently (but not always) divisible and if non-zero then it is uncountable. It may have torsion. We study this "fine structure" systematically in [S5].

3. Some Consequences of the Main Theorem

Theorem 3.1. *Suppose that A has KK-filtration $\{A_i\}$. Then $KK_*(A, B)$ decomposes unnaturally as the direct sum of the terms*

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B)),$$

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)),$$

and

$$\varprojlim{}^1\, Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)).$$

Proof. The UCT splits unnaturally, and hence $KK_*(A, B)$ is the direct sum of two terms, namely

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

and

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)).$$

The term $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is the middle term in the Roos sequence which forms the left column of the Main Diagram. This short exact sequence splits, by Theorem 1.6, and hence $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ itself is the direct sum of two terms. They are

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

and

$$\varprojlim{}^1\, Hom_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which proves the Theorem. □

Corollary 3.2. *Suppose that A has KK-filtration $\{A_i\}$. Further, suppose that each $K_*(A_i)$ is free abelian. Then the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim{}^1\, KK_*(A_i, B) & \xrightarrow{\sigma} & KK_*(A, B) & \xrightarrow{\rho} & \varprojlim KK_*(A_i, B) & \longrightarrow & 0 \\ & & \Big\downarrow \psi & & \Big\downarrow id & & \Big\downarrow \tilde{\gamma} & & \\ 0 & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) & \xrightarrow{\delta} & KK_*(A, B) & \xrightarrow{\gamma} & Hom_{\mathbb{Z}}(K_*(A), K_*(B)) & \longrightarrow & 0 \end{array}$$

is commutative, is natural with respect to A and B, has exact rows, and the vertical maps are isomorphisms.

Proof. Each group

$$Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) = 0$$

since each $K_*(A_i)$ is free, so

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) = 0$$

and the result is immediate from the Main Theorem. □

Corollary 3.3. *Suppose that A has KK-filtration $\{A_i\}$. Then the Milnor $\varprojlim^1$ sequence*

$$0 \to \varprojlim{}^1 KK_*(A_i, B) \xrightarrow{\sigma} KK_*(A, B) \xrightarrow{\rho} \varprojlim KK_*(A_i, B) \to 0$$

is unnaturally split.

Proof. This is immediate from the Main Theorem. In fact the splitting is canonically related to the splitting of the UCT, as indicated in the proof of the Main Theorem. □

We note in passing that in the typical situations in algebraic topology where the Milnor sequence appears the $\varprojlim^1$ term is divisible and hence the analogue of Corollary 3.3 is immediate. However in the present context the $\varprojlim^1$ term may well *not* be divisible.[4] Nevertheless, in the presence of the UCT the sequence splits.[5]

Recall (cf. [FI]) that $PExt^1_{\mathbb{Z}}(G, M)$ is the subgroup of $Ext^1_{\mathbb{Z}}(G, M)$ consisting of pure extensions.

Corollary 3.4. *Suppose that A has KK-filtration $\{A_i\}$. Then*

(1) *There is a natural isomorphism*

$$PExt^1_{\mathbb{Z}}(K_*(A), K_*(B)) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)).$$

(2) *The natural surjection*

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

is the $\mathbb{Z}$-adic completion of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

Proof. If G is any group and $G = \varinjlim G_\alpha$ where the G_α are finitely generated then for any group M,

$$PExt^1_{\mathbb{Z}}(G, M) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(G_\alpha, M) \tag{3.5}$$

by a theorem of Jensen [J, p. 37]. Thus Part 1) is a result from pure algebra depending only upon the fact that the $K_*(A_i)$ are finitely generated. Combining Part 1) with Theorem 1.3 (Roos) yields a short exact sequence

(3.6)

$$0 \to PExt^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to 0.$$

On the other hand, the group $PExt^1_{\mathbb{Z}}(G, M)$ may also be described as

$$PExt^1_{\mathbb{Z}}(G, M) \cong \bigcap_{n \in \mathbb{N}} n Ext^1_{\mathbb{Z}}(G, M)$$

which is visibly the closure of $\{0\}$ in the $\mathbb{Z}$-adic topology. Hence the quotient of $Ext^1_{\mathbb{Z}}(G, M)$ by the subgroup $PExt^1_{\mathbb{Z}}(G, M)$ is always the $\mathbb{Z}$-adic completion. Combining this observation with the sequence (3.6) yields Part 2). □

[4] See [S5] for a general discussion of such matters.

[5] We know of no example in KK-theory where the Milnor sequence does not split. Perhaps it splits in general.

Corollary 3.7. *Suppose that A has KK-filtration $\{A_i\}$. Then the map*

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

is the completion of the group $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ in the topology induced by the KK-filtration and also in the $\mathbb{Z}$-adic topology.

□

M. Rørdam [R] defines a group $KL_*(A, B)$ to be the quotient of $KK_*(A, B)$ by the subgroup $PExt^1_{\mathbb{Z}}(K_*(A), K_*(B))$ (in a context where the UCT holds so that this is a subgroup of $KK_*(A, B)$).

Corollary 3.8. *Suppose that A has KK-filtration $\{A_i\}$. Then the functor $KL_*(A, B)$ is given by the isomorphism*

$$KL_*(A, B) \cong \varprojlim KK_*(A_i, B)$$

and there is a natural short exact sequence

$$0 \to \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to KL_*(A, B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

which splits unnaturally.

□

Following Dadarlat and Loring [DL] let

$$\underline{K}_j(A) \cong \oplus_n K_j(A; \mathbb{Z}/n)$$

where n ranges over all positive integers. Let

$$Hom_\Lambda(\underline{K}_*(A), \underline{K}_*(B))$$

denote all homomorphisms that respect the direct sum decomposition and the action of the Bockstein operations [S4]. Dadarlat and Loring use the Milnor sequence to establish [DL] a universal multi-coefficient exact sequence

$$(3.9) \qquad 0 \to PExt^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK_*(A, B) \to Hom_\Lambda(\underline{K}_*(A), \underline{K}_*(B)) \to 0$$

which we may identify.

Theorem 3.10. *Suppose that A has KK-filtration $\{A_i\}$. Then there is a natural isomorphism*

$$Hom_\Lambda(\underline{K}_*(A), \underline{K}_*(B)) \cong \varprojlim KK_*(A_i, B)$$

and the Dadarlat-Loring sequence (3.9) coincides with the Milnor sequence

$$0 \to \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \longrightarrow KK_*(A, B) \xrightarrow{\rho} \varprojlim KK_*(A_i, B) \to 0.$$

Further, the Dadarlat-Loring sequence unnaturally splits.

Proof. This is immediate by combining (3.9) with Corollary 3.4(1). □

Remark 3.11 It is tempting to think that the $\mathbb{Z}$-adic completion of $KK_*(A,B)$ is the group $\varprojlim KK_*(A_i,B)$. However the situation is more complex. By the UCT we know that there is an unnatural isomorphism

$$KK_*(A,B) \cong Hom_{\mathbb{Z}}(K_*(A),K_*(B)) \oplus Ext^1_{\mathbb{Z}}(K_*(A),K_*(B))$$

and since completions respect sums, there is an isomorphism

$$KK_*(A,B)^\frown \cong Hom_{\mathbb{Z}}(K_*(A),K_*(B))^\frown \oplus Ext^1_{\mathbb{Z}}(K_*(A),K_*(B))^\frown$$

where $G^\frown$ denotes the $\mathbb{Z}$-adic completion of G. We know that

$$Ext^1_{\mathbb{Z}}(K_*(A),K_*(B))^\frown \cong \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i),K_*(B)).$$

If the group

$$Hom_{\mathbb{Z}}(K_*(A),K_*(B))$$

were complete then exactness of the right column of the Main Diagram would imply that $\varprojlim KK_*(A_i,B)$ is complete and from this it would follow that this group is the completion of $KK_*(A,B)$. However, the group $Hom_{\mathbb{Z}}(K_*(A),K_*(B))$ is *not* $\mathbb{Z}$-adic complete in general. An abelian group is $\mathbb{Z}$-adic complete iff it is reduced and algebraically compact, by [FI, 39.1] so, for instance, $\mathbb{Z}$ is not $\mathbb{Z}$-adic complete. However we do have the following result.

Corollary 3.11. *Suppose that A has KK-filtration $\{A_i\}$. If $K_*(A)$ is a torsion group then the natural map*

$$KK_*(A,B) \longrightarrow \varprojlim KK_*(A_i,B)$$

is the $\mathbb{Z}$-adic completion of $KK_*(A,B)$.

Proof. The fact that $K_*(A)$ is a torsion group implies that the group

$$Hom_{\mathbb{Z}}(K_*(A),K_*(B))$$

is reduced and algebraically compact. Then the discussion of Remark 3.11 completes the proof. □

4. The UCT Revisited

The Main Diagram is overdetermined. We may use this fact to establish the following theorem. It is a variant of [RS2, Proposition 7.14] which in turn was adumbrated by [KS, p. 104]. We include a proof to emphasize the relation of the Main Diagram to these results.

Theorem 4.1. *Suppose given a direct sequence A_i of separable nuclear C^*-algebras with $A = \varinjlim A_i$ and a C^*-algebra B such that each (A_i, B) satisfies the UCT. Then (A, B) satisfies the UCT.*

We note quickly that if $A_i \in \mathcal{N}$ (which is the case typically) then this is not a new result; it follows from the [RS] proof of the UCT itself. However the proof in [RS] is by means of geometric realization. First one establishes the special case where $K_*(B)$ is injective and then one uses geometric realization to establish the general case. The proof to follow does not use geometric realization. It is much closer in spirit to the original proof of a special case of the UCT by L.G. Brown [B].

Proof. Combining Theorem 1.3 and Theorem 1.2 yields a commutative diagram with exact rows and columns of the form

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \varprojlim^1 KK_*(A_i,B) & \xrightarrow{\sigma} & KK_*(A,B) & \xrightarrow{\rho} & \varprojlim KK_*(A_i,B) & \to & 0 \\ & & \downarrow \psi & & \downarrow id & & & & \\ & & Ext^1_{\mathbb{Z}}(K_*(A),K_*(B)) & \xrightarrow{\delta} & KK_*(A,B) & & & & \\ & & \downarrow \varphi & & \downarrow & & & & \\ & & \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i),K_*(B)) & \xrightarrow{\pi} & 0 & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{4.2}$$

Apply the Snake Lemma to the first two columns of (4.2) and one obtains the long exact sequence

$$0 \to Ker(\sigma) \to Ker(\delta) \to Ker(\pi) \xrightarrow{\partial} Cok(\sigma) \to Cok(\delta) \to Cok(\pi) \to 0$$

which simplifies to the long exact sequence

$$0 \to Ker(\delta) \to \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i),K_*(B)) \xrightarrow{\partial} \varprojlim KK_*(A_i,B) \to Cok(\delta) \to 0$$

so that

$$Ker(\delta) \cong Ker(\partial) \quad \text{and} \quad Cok(\delta) \cong Cok(\partial).$$

Thus there is a long exact sequence

$$0 \to Ker(\partial) \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A,B) \to Cok(\partial) \to 0.$$

On the other hand, since each (A_i, B) satisfies the UCT, we may take the inverse limit of the UCT sequences

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \xrightarrow{\delta_i} KK_*(A_i,B) \xrightarrow{\gamma_i} Hom_{\mathbb{Z}}(K_*(A_i), K_*(B)) \to 0$$

to obtain the short exact[6] sequence

$$0 \to \varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) \xrightarrow{\varprojlim \delta_i} \varprojlim KK_*(A_i,B) \xrightarrow{\tilde{\gamma}} Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0.$$

It is easy to see that $\varprojlim \delta_i = \partial$ and hence

$$Ker(\partial) \cong Ker(\varprojlim \delta_i) = 0$$

and similarly

$$Cok(\partial) \cong Cok(\varprojlim \delta_i) \cong Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

which yields the UCT for the pair (A,B). $\square$

[6]since $\varprojlim^1 Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B)) = 0$ by Theorem 1.3

5. KK-filtrations

Given a $\mathbb{Z}/2$-graded abelian group G, we used geometric projective resolutions in [S2] to produce a commutative C^*-algebra A with

$$K_*(A) \cong G.$$

In later work with J. Rosenberg [RS2, RS3] we used more elaborate constructions to produce geometric injective resolutions. As one resolution is as good as another from the point of view of homological algebra, there was no reason to be concerned about the exact nature of these C^*-algebras.

The present situation is quite different. We wish to produce KK-filtrations, which means that not only must C^*-algebras A_i be produced with

$$K_*(A_i) \cong G_i$$

but each inclusion

$$G_i \hookrightarrow G_{i+1}$$

must be realized as an inclusion of C^*-algebras

$$A_i \hookrightarrow A_{i+1}.$$

This requires a bit more care but turns out not to be very difficult.

We recall some notation. For any C^*-algebra A, the *suspension* of A is defined by $SA = C_o(\mathbb{R}) \otimes A$. The natural n-fold covering map induces a map $n : S\mathbb{C} \to S\mathbb{C}$ and hence a mapping cone sequence

$$0 \to S^2\mathbb{C} \xrightarrow{\pi_n} Cn \to S\mathbb{C} \to 0$$

and the connecting homomorphism in K_* corresponds to multiplication by n.

Theorem 5.1. *There is a covariant functor $\mathcal{F}$ from the category of $\mathbb{Z}/2$-graded finitely generated abelian groups to the category of commutative C^*-algebras of the form $C_o(X)$ where X^+ (the one-point compactification of X) is a finite complex.*[7] *This functor respects finite direct sums, takes monomorphisms of abelian groups to inclusions of C^*-algebras, and has the property that for each $\mathbb{Z}/2$-graded finitely generated group G there is an isomorphism*

$$\Psi_G : K_*(\mathcal{F}G) \xrightarrow{\cong} G$$

and for each $\mathbb{Z}/2$-graded homomorphism $\alpha : G \to G'$ there is a commutative diagram

$$\begin{array}{ccc} K_*(\mathcal{F}G) & \xrightarrow{(\mathcal{F}\alpha)_*} & K_*(\mathcal{F}G') \\ \downarrow{\Psi_G} & & \downarrow{\Psi_{G'}} \\ G & \xrightarrow{\alpha} & G' \, . \end{array}$$

[7] In fact X^+ is the wedge of a finite number of compact connected smooth manifolds of dimension at most 3, since only 2-dimensional spheres, lens spaces, and their suspensions are required in the construction.

This proposition has a categorical interpretation. If we regard K_* as a functor

$$K_* : \mathcal{C}_{fg} \longrightarrow \mathcal{A}_{fg}$$

from the category $\mathcal{C}_{fg}$ of C^*-algebras with finitely generated K-theory to the category $\mathcal{A}_{fg}$ of $Z/2$-graded finitely generated abelian groups, then $\mathcal{F}$ is a functor $\mathcal{F} : \mathcal{A}_{fg} \to \mathcal{C}_{fg}$ and there is a natural equivalence

$$\Psi : K_*\mathcal{F} \xrightarrow{\cong} 1_{\mathcal{A}_{fg}}$$

so that $\mathcal{A}_{fg}$ is a retract of $\mathcal{C}_{fg}$.

We have chosen to build KK-filtrations as commutative C^*-algebras. Note, however, that commutativity is not at all essential for the present applications: any choices of $A_i \in \mathcal{N} \cap \mathcal{C}_{fg}$ would do. Similarly, it would suffice to have the various diagrams commute only up to homotopy, provided that the maps $A_i \to A_{i+1}$ are honest inclusions. See Remark 5.7.

Proof. As the functors $\mathcal{F}$ and K_* are finitely additive we may immediately restrict attention to the case where $G_1 = 0$, handling the G_1-term separately by taking suspensions. By unique decomposition of finitely generated abelian groups, we may reduce to the cases $G_0 = \mathbb{Z}$ and $\mathbb{Z}/p^j$. Define

$$\mathcal{F}\mathbb{Z} \,=\, S^2\mathbb{C}$$

so that $(S^2\mathbb{C})^+ \cong C(S^2)$ and define

$$\mathcal{F}(\mathbb{Z}/p^j) \,=\, Cp^j.$$

so that $(Cp^j)^+ \cong C(L_{p^j})$, a lens space. The map

$$\Psi_{\mathbb{Z}} : K_0(S^2\mathbb{C}) \longrightarrow \mathbb{Z}$$

is the isomorphism which sends the canonical Bott generator $(\eta - 1)$ to 1. The map

$$\Psi_{\mathbb{Z}/n} : K_0(Cn) \longrightarrow \mathbb{Z}/n$$

is defined as follows. Let

$$(\heartsuit) \qquad 0 \to S^2\mathbb{C} \xrightarrow{\pi_n} Cn \to S\mathbb{C} \to 0$$

be the mapping cone sequence associated to the degree n map of the circle. Then $(\pi_n)_*(1)$ is a generator of $K_0(Cn) \cong \mathbb{Z}/n$. Let

$$\Psi_{\mathbb{Z}/n}(\pi_n)_*(1) \,=\, [1]$$

where [1] denotes the image of 1 under the canonical projection $p_n : \mathbb{Z} \to \mathbb{Z}_n$. It is obvious from the definition that

$$\Psi_G : K_j(\mathcal{F}G) \xrightarrow{\cong} G_j.$$

It remains to define homomorphisms, and here too we may reduce down to a few cases.

Any homomorphism of finitely generated abelian groups is the direct sum of compositions of the following homomorphisms:

(1) The multiplication by k maps $M_k : \mathbb{Z} \to \mathbb{Z}$, defined for all $k \in \mathbb{Z}$;
(2) The canonical reduction map $p_n : \mathbb{Z} \to \mathbb{Z}/n$, defined for n a power of a prime;
(3) The multiplication by k maps $N_k : \mathbb{Z}/n \to \mathbb{Z}/n$, defined for $0 \leq k < n$ and n a prime power;
(4) The canonical map $w_m : \mathbb{Z}/n \to \mathbb{Z}/mn$, defined for both m and n powers of the same prime;
(5) The canonical map $x_n : \mathbb{Z}/mn \to \mathbb{Z}/n$, defined for both m and n powers of the same prime.

The first case is the multiplication by k map $M_k : \mathbb{Z} \to \mathbb{Z}$. Define $\mathcal{F}M_k$ to be the map

$$\mathcal{F}M_k : S^2\mathbb{C} \to S^2\mathbb{C}$$

which is the suspension of the degree k map $k : S\mathbb{C} \to S\mathbb{C}$. Then

$$\mathcal{F}M_{kk'} = \mathcal{F}M_k \mathcal{F}M_{k'}.$$

It is easy to see that $M_k \Psi_{\mathbb{Z}} = \Psi_{\mathbb{Z}}(\mathcal{F}M_k)_*$.

The second case is the reduction mod $n = p^j$ map $p_n : \mathbb{Z} \to \mathbb{Z}/n$. Define $\mathcal{F}p_n = \pi_n$, the natural inclusion in the mapping cone sequence ♡. We must show that the diagram

$$\begin{array}{ccc} K_0(S^2\mathbb{C}) & \xrightarrow{(\mathcal{F}p_n)_*} & K_0(Cn) \\ \downarrow{\scriptstyle \Psi_{\mathbb{Z}}} & & \downarrow{\scriptstyle \Psi_{\mathbb{Z}/n}} \\ \mathbb{Z} & \xrightarrow{p_n} & \mathbb{Z}/n \end{array}$$

commutes. It suffices to check on generators, and then we have

$$\Psi_{\mathbb{Z}/n}(\mathcal{F}p_n)_*(1) = \Psi_{\mathbb{Z}/n}((\pi_n)_*(1)) = [1] = p_n(1) = p_n\Psi_{\mathbb{Z}}(1)$$

as required.

To study the third case, fix $k < n$ and begin with the commuting diagram

$$\begin{array}{ccc} S\mathbb{C} & \xrightarrow{n} & S\mathbb{C} \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k} \\ S\mathbb{C} & \xrightarrow{n} & S\mathbb{C} \end{array}$$

which yields a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_n} & Cn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{F}M_k} & & \downarrow{\scriptstyle \mathcal{F}N_k} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_n} & Cn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \end{array}$$

which defines the map $\mathcal{F}N_k : Cn \to Cn$. It is a monomorphism by the Snake Lemma. We must show that the diagram

$$\begin{array}{ccc} K_0(Cn) & \xrightarrow{(\mathcal{F}N_k)_*} & K_0(Cn) \\ \downarrow{\scriptstyle \Psi_{\mathbb{Z}/n}} & & \downarrow{\scriptstyle \Psi_{\mathbb{Z}/n}} \\ \mathbb{Z}/n & \xrightarrow{N_k} & \mathbb{Z}/n \end{array}$$

commutes. This is a direct calculation:

$$\Psi_{\mathbb{Z}/n}(\mathcal{F}N_k)_*(1) = \Psi_{\mathbb{Z}/n}(\mathcal{F}N_k)_*(\mathcal{F}p_n)_*(1) = \Psi_{\mathbb{Z}/n}(\mathcal{F}p_n)_*(\mathcal{F}M_k)_*(1) =$$

$$= p_n\Psi_{\mathbb{Z}}(\mathcal{F}M_k)_*(1) = p_n N_k \Psi_{\mathbb{Z}}(1) = N_k p_n \Psi_{\mathbb{Z}}(1) = N_k \Psi_{\mathbb{Z}/n}(\mathcal{F}p_n)_*(1)$$

as required.

The fourth case is the natural map $w_n : \mathbb{Z}/n \to \mathbb{Z}/mn$ when both m and n are powers of p. Starting with the commutative diagram

$$\begin{array}{ccc} S\mathbb{C} & \xrightarrow{n} & S\mathbb{C} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle m} \\ S\mathbb{C} & \xrightarrow{mn} & S\mathbb{C} \end{array}$$

one obtains a commuting diagram of mapping cone sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_n} & Cn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{F}M_m} & & \downarrow{\scriptstyle \mathcal{F}w_m} & & \downarrow{\scriptstyle \mathcal{F}M_1} & & \\ 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_{mn}} & Cmn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \end{array}$$

which defines the map $\mathcal{F}w_m : Cn \to Cmn$. We must demonstrate that the diagram

$$\begin{array}{ccc} K_0(Cn) & \xrightarrow{(\mathcal{F}w_m)_*} & K_0(Cmn) \\ \downarrow{\scriptstyle \Psi_{\mathbb{Z}/n}} & & \downarrow{\scriptstyle \Psi_{\mathbb{Z}/mn}} \\ \mathbb{Z}/n & \xrightarrow{w_m} & \mathbb{Z}/mn \end{array}$$

commutes. Enlarge the diagram to obtain

$$\begin{array}{ccccc} K_0(S^2\mathbb{C}) & \xrightarrow{(\mathcal{F}p_n)_*} & K_0(Cn) & \xrightarrow{(\mathcal{F}w_m)_*} & K_0(Cmn) \\ \downarrow{\scriptstyle \Psi_{\mathbb{Z}}} & & \downarrow{\scriptstyle \Psi_{\mathbb{Z}/n}} & & \downarrow{\scriptstyle \Psi_{\mathbb{Z}/mn}} \\ \mathbb{Z} & \xrightarrow{p_n} & \mathbb{Z}/n & \xrightarrow{w_m} & \mathbb{Z}/mn \end{array}$$

The left square commutes by the second part of the argument, and the map $(\mathcal{F}p_n)_*$ is surjective, so it suffices to show that the outer rectangle commutes. This is direct, for

$$\Psi_{\mathbb{Z}/mn}(\mathcal{F}w_m)_*(\mathcal{F}p_n)_* = \Psi_{\mathbb{Z}/mn}(\mathcal{F}p_{mn})_*(\mathcal{F}M_n)_* =$$
$$p_{mn}\Psi_{\mathbb{Z}}(\mathcal{F}M_m)_* = p_{mn}M_m\Psi_{\mathbb{Z}} = w_m p_n \Psi_{\mathbb{Z}}$$

as required. This proves the fourth case.

The final case is the natural map $x_n : \mathbb{Z}/mn \to \mathbb{Z}/n$. From the commutative diagram

$$\begin{array}{ccc} S\mathbb{C} & \xrightarrow{mn} & S\mathbb{C} \\ \Big\downarrow m & & \Big\downarrow 1 \\ S\mathbb{C} & \xrightarrow{n} & S\mathbb{C} \end{array}$$

one obtains a commuting diagram of mapping cone sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_{mn}} & Cmn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \\ & & \Big\downarrow \mathcal{F}M_1 & & \Big\downarrow \mathcal{F}x_m & & \Big\downarrow m & & \\ 0 & \longrightarrow & S^2\mathbb{C} & \xrightarrow{\mathcal{F}p_n} & Cn & \longrightarrow & S\mathbb{C} & \longrightarrow & 0 \end{array}$$

which defines the map $\mathcal{F}x_m : Cmn \to Cn$ for both m and n powers of p. We must show that the square

$$\begin{array}{ccc} K_0(Cmn) & \xrightarrow{(\mathcal{F}x_m)_*} & K_0(Cn) \\ \Big\downarrow \Psi_{\mathbb{Z}/mn} & & \Big\downarrow \Psi_{\mathbb{Z}/n} \\ \mathbb{Z}/mn & \xrightarrow{x_m} & \mathbb{Z}/n \end{array}$$

commutes. Enlarge the square to the diagram

$$\begin{array}{ccccc} K_0(S^2\mathbb{C}) & \xrightarrow{(\mathcal{F}p_{mn})_*} & K_0(Cmn) & \xrightarrow{(\mathcal{F}x_m)_*} & K_0(Cn) \\ \Big\downarrow \Psi_{\mathbb{Z}} & & \Big\downarrow \Psi_{\mathbb{Z}/mn} & & \Big\downarrow \Psi_{\mathbb{Z}/n} \\ \mathbb{Z} & \xrightarrow{p_{mn}} & \mathbb{Z}/mn & \xrightarrow{x_m} & \mathbb{Z}/n. \end{array}$$

As before, the left square commutes, so it suffices to show that the outer rectangle commutes. This is clear by direct calculation:

$$\Psi_{\mathbb{Z}/n}(\mathcal{F}x_m)_*(\mathcal{F}p_{mn})_* = \Psi_{\mathbb{Z}/n}(\mathcal{F}M_i)_*(\mathcal{F}p_n)_* = \Psi_{\mathbb{Z}/n}(\mathcal{F}p_n)_* = p_n\Psi_{\mathbb{Z}} = p_{mn}x_m\Psi_{\mathbb{Z}}.$$

We note that by construction a monomorphism $G \to G'$ induces a monomorphism $\mathcal{F}G \to \mathcal{F}G'$. It is tedious but routine to show that with these definitions $\mathcal{F}$ is indeed a functor. $\square$

Note that the functor $\mathcal{F}$ is certainly not unique. For instance, define a new functor $\mathcal{J}$ as follows. Fix a positive integer k and let

$$\mathcal{J}(G) = S^{2k}(\mathcal{F}G).$$

It is easy to see that $\mathcal{J}$ would do just as well as $\mathcal{F}$. Presumably there are many other such functors. See also Remark 5.7.

Proposition 5.2. *Suppose that $\{G_i\}$ is an increasing sequence of $\mathbb{Z}/2$-graded finitely generated abelian groups. Then the sequence $\{\mathcal{F}G_i\}$ is an increasing sequence of separable commutative C^*-algebras, each diagram*

$$\begin{array}{ccc} K_*(\mathcal{F}G_i) & \longrightarrow & K_*(\mathcal{F}G_{i+1}) \\ \downarrow \cong & & \downarrow \cong \\ G_i & \longrightarrow & G_{i+1} \end{array}$$

commutes, and hence $K_*(\varinjlim \mathcal{F}G_i) \cong \varinjlim G_i$.

Proof. This is immediate from the constructions of Theorem 5.1. □

Proposition 5.3. *Suppose that $\alpha = \{\alpha_i\} : \{G_i\} \to \{G_i'\}$ is a morphism of increasing direct sequences of finitely generated abelian groups. Then there is an induced morphism*

$$\{\mathcal{F}\alpha_i\} : \{\mathcal{F}G_i\} \to \{\mathcal{F}G_i'\}$$

of increasing directed sequences of C^*-algebras, an induced map

$$\mathcal{F}\alpha : \varinjlim \mathcal{F}G_i \longrightarrow \varinjlim \mathcal{F}G_i'$$

and a commutative diagram

$$\begin{array}{ccccc} \varinjlim G_i & \xrightarrow{\cong} & \varinjlim K_*(\mathcal{F}G_i) & \xrightarrow{\cong} & K_*(\varinjlim \mathcal{F}G_i) \\ \downarrow \alpha = \varinjlim \alpha_i & & \downarrow \varinjlim \mathcal{F}\alpha_i & & \downarrow (\mathcal{F}\alpha)_* \\ \varinjlim G_i' & \xrightarrow{\cong} & \varinjlim K_*(\mathcal{F}G_i') & \xrightarrow{\cong} & K_*(\varinjlim \mathcal{F}G_i'). \end{array}$$

Proof. This is immediate from 5.1 and 5.2. □

Theorem 5.4. *Let A be some separable C^*-algebra in the bootstrap category $\mathcal{N}$. Write $K_*(A) = \varinjlim G_i$ where the G_i are an increasing sequence of $\mathbb{Z}/2$-graded finitely generated subgroups of $K_*(A)$. Then A is KK-equivalent to $\varinjlim \mathcal{F}G_i$ and each of the groups*

(a) $$\varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(\mathcal{F}G_i), K_*(B)) \cong \varprojlim{}^1 KK_*(\mathcal{F}G_i, B) \cong PExt^1_{\mathbb{Z}}(K_*(A), K_*(B),)$$

(b) $$\varprojlim Ext^1_{\mathbb{Z}}(K_*(\mathcal{F}G_i), K_*(B)),$$

and

(c) $$\varprojlim KK_*(\mathcal{F}G_i, B)$$

depend only upon $K_*(A)$ and upon $K_*(B)$.

Proof. First note that

$$K_*(A) \cong \varinjlim G_i \cong \varinjlim K_*(\mathcal{F}G_i) \cong K_*(\varinjlim \mathcal{F}G_i)$$

so that A and $\varinjlim \mathcal{F}G_i$ have the same K-theory groups. The fact that A is KK-equivalent to $\varinjlim \mathcal{F}G_i$ is then an immediate consequence of the UCT [RS Prop. 7.4].

Suppose that $K_*(A) \cong \varinjlim G_i \cong \varinjlim G'_i$ for different choices of subgroups. We may assume without loss of generality that $G_i \subseteq G'_i \subseteq G_{\nu(i)}$ for some strictly increasing function $\nu : \mathbb{N} \to \mathbb{N}$. Proposition 5.3 then yields induced morphisms

$$\{\mathcal{F}G_i\} \longrightarrow \{\mathcal{F}G'_i\} \longrightarrow \{\mathcal{F}G_{\nu(i)}\}$$

of increasing directed sequences of C^*-algebras, and this morphism induces the identity on K_* upon passage to limits. Thus the groups depend upon A and B alone.

For each group G_i we have a UCT sequence associated to $KK_*(\mathcal{F}G_i, B)$. Taking inverse limits over i yields the usual six term $\varprojlim$ - $\varprojlim^1$ sequence which degenerates to yield the isomorphisms

$$\varprojlim{}^1 KK_*(\mathcal{F}G_i, B) \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(\mathcal{F}G_i), K_*(B)) \cong$$

$$\cong PExt^1_{\mathbb{Z}}(K_*(\varinjlim \mathcal{F}G_i), K_*(B)) \cong PExt^1_{\mathbb{Z}}(K_*(A), K_*(B)) \tag{5.5}$$

and

(5.6)

$$0 \to \varprojlim Ext^1_{\mathbb{Z}}(K_*(\varinjlim \mathcal{F}G_i), K_*(B)) \to \varprojlim KK_*(\mathcal{F}G_i, B) \to Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

From (5.5) it is clear that the group (a) depends only upon $K_*(A)$ and $K_*(B)$. To analyze (5.6) we note that if

$$G = \varinjlim G_i = \varinjlim G'_i$$

for finitely generated subgroups G_i and G'_i then arguing as above we may assume that

$$G_i \subseteq G'_i \subseteq G'_{\nu(i)}$$

and from this it follows that

$$\varprojlim Ext^1_{\mathbb{Z}}(G_i, H) \cong \varprojlim Ext^1_{\mathbb{Z}}(G'_i, H)$$

for any H, so that (b) depends only upon $K_*(A)$ and $K_*(B)$. Using this fact, sequence (5.6) and the Five Lemma, it follows that

$$\varprojlim KK_*(\mathcal{F}G_i, B) \cong \varprojlim KK_*(\mathcal{F}G'_i, B)$$

and hence group (c) also depends only upon $K_*(A)$ and $K_*(B)$. □

Finally, we may prove Theorem 1.5.

Proof of Theorem 1.5. Given $A \in \mathcal{N}$, choose some increasing sequence of $\mathbb{Z}/2$-graded finitely generated abelian groups $\{G_i\}$ with

$$K_*(A) \cong \underrightarrow{lim}\, G_i.$$

Define $A_i = \mathcal{F}G_i$. Then each A_i is a commutative C^*-algebra of the correct form, and there are natural inclusions $A_i \hookrightarrow A_{i+1}$ corresponding to the inclusions $G_i \hookrightarrow G_{i+1}$. Thus there is an isomorphism

$$K_*(A) \cong K_*(\underrightarrow{lim}\, A_i).$$

Since both A and $\underrightarrow{lim}\, A_i$ are in the bootstrap category $\mathcal{N}$ and they have the same K-groups, they are KK-equivalent by [RS2 Cor. 7.5]. Thus $\{A_i\}$ is a KK-filtration of A. This proves Part (1) of Theorem 1.5. Part (2) is immediate from Theorem 5.4. □

Remark 5.7. Is it possible to build (presumably non-commutative) KK-filtrations such that each A_i is an *ideal* in A_{i+1}? This seems to be quite a difficult question. If A has a KK-filtration by ideals then by [S1] there is a spectral sequence of the form

$$E^1_{p,*} \cong K_*(A_p/A_{p-1}) \implies K_*(A)$$

so that in some sense the KK-filtration by ideals plays the role of a CW-decomposition of A from the point of view of KK-theory.

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