A SPECTRAL SEQUENCE FOR MOTIVIC COHOMOLOGY

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§0. Introduction

(0.1). The purpose of this paper is to construct a spectral sequence from the *motivic cohomology* of a field F to its algebraic K-theory:

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(\mathrm{Spec}(F), \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(\mathrm{Spec}(F)). \tag{0.1.1}$$

Here we *define* motivic cohomology via higher Chow groups [2], [5]

$$H^r_{\mathcal{M}}(X, \mathbb{Z}(s)) := CH^s(X, 2s - r).$$

Briefly, we write

$$\Delta^p := \mathrm{Spec}\,\big(F[t_0, \dots, t_p]/(\textstyle\sum t_i - 1)\big) \tag{0.1.2}$$

for the affine p-simplex. We have face and degeneracy maps defined in the usual way. Let $\mathcal{Z}^r(X, p)$ denote the free abelian group spanned by codimension r subvarieties of Δ^p_X meeting all faces properly. Let $\mathcal{Z}^r(X, \bullet)$ be the simplicial abelian group thus defined, with boundaries given by pullback of cycles along face maps. Finally, let

$$CH^r(X, n) := \pi_n\big(\mathcal{Z}^r(X, \bullet)\big) \cong H_n\big(\mathcal{Z}^r(X, \bullet)\big).$$

We admit to a certain presumption in defining motivic cohomology via higher Chow groups. At this point, one can only say that the higher Chow groups have some of the expected properties. Those who search for unicorns must beware being led astray by cows with one horn.

In §1 we use multi-relative K-theory with supports to define a graded complex of the form

$$\dots \to D \to D \to E \to D \to D \to E \to \dots$$

We show that, assuming a rather innocuous looking "moving lemma" called Theorem A, this complex is an exact couple, and the resulting spectral sequence has the form (0.1.1). §2 - §6 of the paper are devoted to proving theorem A. Finally, in §7, we prove that $t_{\geq 1}(\mathcal{Z}^2(\mathrm{Spec}(F), \bullet)[-4])$, the higher Chow complex shifted to the right 4 steps and then truncated so the resulting complex is supported in degrees 1 and 2, is quasi-isomorphic to the complex $\Gamma(2)$ introduced by Lichtenbaum [7]. It would follow from a variant of the Soulé Beilinson conjecture that truncation was unnecessary in this context. Indeed, one might formulate a "CEO" (cock-eyed

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optimist) conjecture that $\mathcal{Z}^r(X, \cdot)[-2r]$, sheafified for the étale topology on X, satisfies the six axioms listed in [17] and that the hypercohomology of this complex is linked to the value of the Zeta function of X at the point $s = r$ as suggested in [op. cit.].

This paper (we are embarrassed to admit) has taken six years to write. Along the way, any number of people have been helpful. We acknowledge many conversations over the years with S. Landsburg, as well as some very useful recent e-mail correspondence with M. Levine, J.-L Loday, and C. Weibel.

§1. The Exact Couple

(1.1) Introducing the players. We fix a field F, and we write $\Delta^p = \Delta^p_F$ for affine p-space over F as in (0.1.2). A subvariety $\sigma : t_{i_1} = \ldots = t_{i_q} = 0$ is called a (codimension q) face. A subvariety $V \subset \Delta^p$ is said to be in good position if $V \cap \sigma$ has codimension in $V \geq q$ for any codimension q face σ. Let

$$\mathcal{V}^n = \mathcal{V}^n(\Delta^p) \subset \Delta^p$$

denote the union of all codimension n subvarieties of Δ^p in good position. Let

$$\mathcal{W}^{n+1} \subset \mathcal{V}^n$$

denote the union of all subschemes of codimension $\geq n+1$ contained in $\mathcal{V}^n$. A number of arguments in this and subsequent sections involve general position so it will be convenient to assume the field F is infinite. The reduction to this case is straightforward using the fact that the K-groups have norms for finite extensions of fields and the fact that finite fields admit $\mathbb{Z}_\ell$-extensions for all ℓ. For example, the proofs of (1.2.2) and (2.4.1) involve showing that certain maps between K-groups are zero. If one knows that an element maps to zero after an ℓ-power extension, it follows that some power of ℓ kills the image of the element. Since ℓ was arbitrary, one concludes that the element maps to zero.

Given a scheme X, we write $K(X)$ for some space, functorial in X, whose homotopy groups calculate the Quillen K-theory $K_*(X)$. For $Y \subset X$ a closed subset, we write $K(X,Y) =$ homotopy fibre$(K(X) \to K(Y))$. This construction can be iterated. Given $Y_1, \ldots, Y_n \subset X$, we define the *multi-relative* K-groups

$$\begin{aligned} K(X; Y_1, \ldots, Y_n) := \text{homotopy fibre}\big(&K(X; Y_1, \ldots, Y_{n-1}) \\ &\to K(Y_n; Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n)\big). \end{aligned}$$

Of particular interest will be the case $X = \Delta^p$, $Y_i = \partial_i(\Delta^{p-1})$ where ∂_i denotes the inclusion of the codimension 1 face defined by $t_i = 0$. We write

$$K(\Delta^p, \partial) \text{ resp. } K(\Delta^p, \Sigma)$$

for the multi-relative K-theory relative to all (resp. all but the last) codimension 1 faces. Given $V \subset \Delta^p$ a closed subset, let $U = \Delta^p - V$. Define for $\Psi = \partial$ or Σ

$$K_V(\Delta^p, \Psi) := \text{homotopy fibre}\big(K(\Delta^p, \Psi) \to K(U \cap \Delta^p, U \cap \Psi)\big).$$

This is covariant in the sense that for $V \subset W$ we have

$$K_V(\Delta^p, \Psi) \to K_W(\Delta^p, \Psi).$$

Define

$$K_{\mathcal{V}^n}(\Delta^p, \Psi) := \varinjlim_{V \subset \mathcal{V}^n} K_V(\Delta^p, \Psi).$$

Finally, abusing notation, we write

$$K_{\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p,\Psi) := \varinjlim_{V\subset\mathcal{V}^n}\ \varinjlim_{W\subset\mathcal{V}^{n+1}} K_{V-W}(\Delta^p - W,\Psi).$$

(1.2) The contractible case. The groups $K_{\mathcal{V}^n}(\Delta^p,\Sigma)$ are relatively easy to understand.

Lemma (1.2.1). We have

$$K_{r,\mathcal{V}^n}(\Delta^p,\Sigma)\subset K_{r,\mathcal{V}^n}(\Delta^p;\partial_0,\ldots,\partial_{p-2})\subset\ldots\subset K_{r,\mathcal{V}^n}(\Delta^p)$$
$$K_{r,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p,\Sigma)\subset K_{r,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial_0,\ldots,\partial_{p-2})\subset\ldots\subset K_{r,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p).$$

proof. This is a formal consequence of the simplicial structure. For example, to show the first inclusion note we have fibrations

$$K_{\mathcal{V}^n}(\Delta^p,\Sigma)\to K_{\mathcal{V}^n}(\Delta^p;\partial_0,\ldots,\partial_{p-2})\to K_{\mathcal{V}^n}(\Delta^{p-1},\Sigma)$$
$$K_{\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p,\Sigma)\to K_{\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial_0,\ldots,\partial_{p-2})\to K_{\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^{p-1},\Sigma),$$

so it will suffice to show the inclusion

$$(\Delta^{p-1},\partial_0,\ldots,\partial_{p-2})\subset(\Delta^p,\partial_0,\ldots,\partial_{p-2})$$

admits a splitting, and that pulling back along this splitting carries cycles in good position to cycles in good position. On the level of rings, this follows because the surjection

$$F[t_0,\ldots,t_p]/(\sum t_i-1)\twoheadrightarrow F[u_0,\ldots,u_{p-1}]/(\sum u_i-1)$$

defined by $t_p\mapsto 0$ is split by mapping $u_i\mapsto t_i$ for $i<p-1$ and $u_{p-1}\mapsto t_{p-1}+t_p$. QED

Lemma (1.2.2). (i) The map $K_{r,\mathcal{V}^{n+1}}(\Delta^p,\Sigma)\to K_{r,\mathcal{V}^n}(\Delta^p,\Sigma)$ is zero for all r.
(ii) The map $K_{r,\mathcal{W}^{n+1}}(\Delta^p)\to K_{r,\mathcal{V}^n}(\Delta^p)$ is zero for all r.

proof. First, by (1.2.1) it suffices for the proof of (i) to verify the map

$$K_{r,\mathcal{V}^{n+1}}(\Delta^p)\to K_{r,\mathcal{V}^n}(\Delta^p)$$

This map factors through the map in (ii), so it suffices to prove (ii). Note that by the localization theorem, these K-theory with support groups coincide with $K'_r(\mathcal{W}^{n+1})\to K'_r(\mathcal{V}^n)$ (K-theory of coherent sheaves.) For the proof we may assume the field F is infinite. Indeed, if F is finite, for any prime ℓ there exists a $\mathbb{Z}_\ell$-extension F_ℓ of F. Assuming the map in question is zero over F_ℓ and using that these K-groups admit norms, one sees that the image of the map over F is ℓ-torsion. Since ℓ was arbitrary, this image is zero.

The proof of (ii) now is a sort of linear version of the Quillen trick [10], Th. 5.11. Given $W\subset\mathcal{W}^{n+1}$ of finite type, we can use Noether's lemma to show that a "general" surjective linear projection $L : \Delta^p \to \mathbb{A}^{p-1}$ is finite when restricted to W. Consider the fibre square

$$\begin{tikzcd}
X := W \times_{\mathbb{A}^{p-1}} \Delta^p \arrow[r, "\pi"] \arrow[d, "f"'] & \Delta^p \arrow[d, "L"] \\
W \arrow[r, "L|W"] & \mathbb{A}^{p-1}.
\end{tikzcd}$$

Since $\Delta^p \cong \mathbb{A}^{p-1} \times \mathbb{A}^1$ we get $X \cong W \times \mathbb{A}^1$. The map f admits a section σ such that $\pi \circ \sigma$ is the natural inclusion $W \hookrightarrow \Delta^p$. We have $\sigma_* : K'_r(W) \to K'_r(X)$ is the zero map by the structure of K'-theory for affine space. The map π is finite, hence proper, and $\pi(X) = L^{-1}L(W) \subset \mathcal{V}^n$ because L is general. It follows that $K'_r(W) \to K'_r(\mathcal{V}^n)$ is also the zero map. Passing to the limit over larger and larger such W completes the proof. QED

Recall that $\mathcal{Z}^n(\mathrm{Spec}(F), \bullet)$ is the simplicial abelian group built from cycles on simplices. Define

$$\mathcal{Z}^n(\mathrm{Spec}(F), \bullet)_{Norm} \subset \mathcal{Z}^n(\mathrm{Spec}(F), \bullet)$$

to be the elements on which all but (perhaps) the last face vanish. By standard homotopy theory, the last face map makes $\mathcal{Z}^n(\mathrm{Spec}(F), \bullet)_{Norm}$ a complex with homology isomorphic to the higher Chow groups.

Lemma (1.2.3).

$$K_{0,\mathcal{V}^n}(\Delta^p, \Sigma) \cong K_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^p, \Sigma) \cong \mathcal{Z}^n(\mathrm{Spec}(F), p)_{Norm}.$$

proof. We prove by induction on i for $-1 \le i \le p-1$

$$\begin{aligned} K_{0,\mathcal{V}^n}(\Delta^p; \partial_0, \dots, \partial_i) &\cong K_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^p; \partial_0, \dots, \partial_i) \\ &\cong \cap_{j=0}^{j=i} \ker \big(\partial_j : \mathcal{Z}^n(\mathrm{Spec}(F), p) \to \mathcal{Z}^n(\mathrm{Spec}(F), p-1)\big). \end{aligned}$$

For $i = -1$ we have

$$K_{0,\mathcal{W}^{n+1}}(\Delta^p) \to K_{0,\mathcal{V}^n}(\Delta^p) \to K_{0,\mathcal{V}^n - \mathcal{W}^{n+1}}(\Delta^p) \to 0.$$

The left hand map is 0 by (1.2.2)(ii), and since $\mathcal{V}^n - \mathcal{W}^{n+1}$ consists of generic points of components of $\mathcal{V}^n$, the right hand group is $\mathcal{Z}^n(\mathrm{Spec}(F), p)$.

Assume now $p-1 \ge i \ge 0$ and suppose the result established for fewer than i relative components. To simplify, write i in place of $\partial_0, \dots, \partial_i$ and $\mathcal{Z}^n(\mathrm{Spec}(F), p)_i = \cap_{j=0}^{j=i} \ker(\partial_j)$. We have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_{0,\mathcal{V}^n}(\Delta^p; i) \arrow[r] \arrow[d] & K_{0,\mathcal{V}^n}(\Delta^p; i-1) \arrow[r] \arrow[d, equal] & K_{0,\mathcal{V}^n}(\Delta^{p-1}; i-1) \arrow[d, equal] \\
0 \arrow[r] & K_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^p; i) \arrow[r] & K_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^p; i-1) \arrow[r] \arrow[d, equal] & K_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^{p-1}; i-1) \arrow[d, equal] \\
0 \arrow[r] & \mathcal{Z}^n(\mathrm{Spec}(F), p)_i \arrow[r] & \mathcal{Z}^n(\mathrm{Spec}(F), p)_{i-1} \arrow[r] & \mathcal{Z}^n(\mathrm{Spec}(F), p-1)_{i-1}
\end{tikzcd}$$

from which the assertion at step i follows. QED

(1.3) The candidate Our candidate for the exact couple comes from the diagram (defining i, j, and k.)

$$\begin{tikzcd}
K_{0,\mathcal{V}^{n+1}}(\Delta^p;\partial) \arrow[r] \arrow[d, "i"] & K_{0,\mathcal{V}^{n+1}}(\Delta^p;\Sigma) \arrow[d, "0"] & \\
K_{0,\mathcal{V}^n}(\Delta^p;\partial) \arrow[r, "j"] & K_{0,\mathcal{V}^n}(\Delta^p;\Sigma) \arrow[r, "k"] \arrow[d, "0"] & K_{0,\mathcal{V}^n}(\Delta^{p-1};\partial) \arrow[d, "i"] \\
 & K_{0,\mathcal{V}^{n-1}}(\Delta^p;\Sigma) \arrow[r] & K_{0,\mathcal{V}^{n-1}}(\Delta^{p-1};\partial)
\end{tikzcd} \tag{1.3.1}$$

The middle row is exact (k is pullback to the last face) so $\mathrm{Im}(j) = \ker(k)$. Also the two squares are commutative, so $j \circ i = i \circ k = 0$. We thus get a complex which we may write using Lemma (1.2.3)

$$\begin{aligned} \ldots \to K_{0,\mathcal{V}^{n+1}}(\Delta^p;\partial) \xrightarrow{i} K_{0,\mathcal{V}^n}(\Delta^p;\partial) \xrightarrow{j} \mathcal{Z}^n(\mathrm{Spec}(F),p)_{Norm} \\ \xrightarrow{k} K_{0,\mathcal{V}^n}(\Delta^{p-1};\partial) \xrightarrow{i} K_{0,\mathcal{V}^{n-1}}(\Delta^{p-1};\partial) \to \ldots \end{aligned} \tag{1.3.2}$$

Theorem (1.3.3). The complex (1.3.2) is exact.

Corollary (1.3.4). There is a spectral sequence

$$E_2^{p,q} = CH^{-q}(\mathrm{Spec}(F), -p-q) \Rightarrow K_{-p-q}(\mathrm{Spec}(F)).$$

proof of (1.3.4). Define

$$D_1^{n-p,-n} := K_{0,\mathcal{V}^n}(\Delta^p;\partial)\ ;\ E_1^{n-p,-n} := \mathcal{Z}^n(\mathrm{Spec}(F),p)_{Norm} \tag{1.3.4.1}$$

The exact sequence (1.3.2) now looks like

$$\ldots \to D \to D \to E \to D \to D \to E \to \ldots .$$

The formalism of exact couples [4] yields a spectral sequence with the desired E_2 term converging to $K_0(\Delta^{-p-q}, \partial)$. We have by standard K-theory that $K_*(\Delta^p) \cong K_*(\mathrm{Spec}(F))$, from which an easy induction argument yields

$$K_*(\Delta^p, \Sigma) = (0).$$

The long exact multi-relative sequence

$$\ldots \to K_r(\Delta^p, \partial) \to K_r(\Delta^p, \Sigma) \to K_r(\Delta^{p-1}, \partial) \to K_{r-1}(\Delta^p, \partial) \to \ldots$$

shows

$$K_p(\mathrm{Spec}(F)) \cong K_0(\Delta^p, \partial).$$

This completes the proof of the corollary. QED

Define ι and C to be the maps in the multi-relative sequence

$$\begin{aligned}&\dots K_{1,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\Sigma)\xrightarrow{C}K_{1,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^{p-1};\partial)\to\\&K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial)\xrightarrow{\iota}K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\Sigma)\to\dots.\end{aligned}$$

The main technical point in the proof of (1.3.3) is

Theorem (A). With notation as above, the map C is onto. Equivalently, the map ι is injective.

proof of (1.3.3) assuming Theorem A. Consider the diagram with exact rows and columns

$$\begin{tikzcd}
K_{1,\mathcal{V}^n}(\Delta^p;\Sigma) \arrow[r] \arrow[d] & K_{1,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\Sigma) \arrow[r] \arrow[d, "C"'] & K_{0,\mathcal{V}^{n+1}}(\Delta^p;\Sigma) \arrow[r, "0"] \arrow[d, "k"'] & K_{,\mathcal{V}^n}(\Delta^p;\Sigma) \arrow[d] \\
K_{1,\mathcal{V}^n}(\Delta^{p-1};\partial) \arrow[r, "B"] \arrow[d, "b"'] & K_{1,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^{p-1};\partial) \arrow[r, "\delta"] \arrow[d] & K_{0,\mathcal{V}^{n+1}}(\Delta^{p-1};\partial) \arrow[r, "i"] & K_{0,\mathcal{V}^n}(\Delta^{p-1};\partial) \\
K_{0,\mathcal{V}^n}(\Delta^p;\partial) \arrow[r, "D"] \arrow[d, "j"'] & K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial) & & \\
K_{0,\mathcal{V}^n}(\Delta^p;\Sigma) & & &
\end{tikzcd}$$

(Though there was not room to fit it on the page, note that $\ker(D)=\operatorname{Im}(i)$ for a different value of p.) With reference to the maps defined in this diagram, and letting n and p vary, we have:
sequence (1.3.2) exact $\Leftrightarrow \ker(j)\subset\operatorname{Im}(i)$ and $\ker(i)\subset\operatorname{Im}(k)\Leftrightarrow\operatorname{Im}(b)\subset\operatorname{Im}(i)$ and $\operatorname{Im}(\delta)\subset\operatorname{Im}(k)\Leftrightarrow\operatorname{Im}(b)\subset\ker(D)$ and $\operatorname{Im}(\delta)\subset\operatorname{Im}(k)\Leftrightarrow\operatorname{Im}(B)\subset\operatorname{Im}(C)$ and $\operatorname{Im}(\delta)\subset\operatorname{Im}(k)\Leftrightarrow C$ surjective. QED

Remark (1.3.5). For certain general position arguments in the sequel, it will be convenient to assume the field F is infinite. The proof of theorem A may be reduced to this case by the following standard argument. Let L/K be an extension of degree n. All the K-groups involved admit norm maps for finite field extensions. Assume theorem A holds for infinite fields, and let F be a finite field. Let $x\in\ker(\iota)$ be non-trivial, and let ℓ be a rational prime such that no power of ℓ kills x. Let Ω/F be a $\mathbf{Z}_\ell$-extension. By assumption, theorem A holds over the infinite field Ω, so x dies when pulled back to Ω. By a limit argument, x then dies over some finite extension L/F of degree a power of ℓ. Applying the norm, it follows that some power of ℓ kills x, a contradiction. QED

§2. Theorem A–Reduction to a moving lemma

(2.1) We will prove theorem A by showing the map

$$K_{0,\mathcal{Y}^n-\mathcal{Y}^{n+1}}(\Delta^p;\partial) \xrightarrow{\iota} K_{0,\mathcal{Y}^n-\mathcal{Y}^{n+1}}(\Delta^p;\Sigma) \tag{2.1.1}$$

is injective. In this section we will reduce the problem to a moving lemma. Given a collection of varieties, we will say we are in the *normal crossings* situation if all the varieties are smooth and all intersections are transverse.

Lemma (2.2). Let $Z, Y_1, \ldots, Y_n$ be subvarieties of X. Assume we are in the normal crossings situation then

$$K_Z(X; Y_1, \ldots, Y_n) \cong K(Z; Y_1 \cap Z, \ldots, Y_n \cap Z).$$

proof. Induction on n. For $n = 0$ this is a standard localization theorem. The general argument follows from the multi-relative sequence. QED

In the following proposition it is convenient to use ring-theoretic rather than scheme-theoretic notation. Our original argument for the proposition used a theorem of Ellis on multi-relative K_1. Unfortunately, Ellis' theorem is not correct. We are indebted to C. Weibel and J.-L. Loday for giving us counterexamples and to Weibel and M. Levine for help in finding a correct proof.

Proposition (2.3). Let R be a regular ring essentially of finite type over a field F, and let $x_1, \ldots, x_n \in R$ be such that $x = \prod x_i$ defines a reduced divisor with smooth components and normal crossings. Let $I = (x)$ and $I_j = (x_j)$. Then

$$K_0(R; I) \cong K_0(R; I_1, \ldots, I_n).$$

Lemma (2.3.1). With notation as above, R/I is K_1-regular, i.e. for indeterminants $T_1, \ldots, T_r$, we have

$$K_1(R/I) \cong K_1(R/I[T_1, \ldots, T_r]).$$

proof. Since the pair $R[T_1, \ldots, T_n]$, $IR[T_1, \ldots, T_n]$ satisfy the same hypotheses as R, I, it suffices to verify the lemma with $p = 1$, i.e. to show $NK_1(R/I) = (0)$. (Here NK denotes the "nil" groups [13].) By [op. cit.]

$$NK_*(R/I) \hookrightarrow \prod NK_*((R/I)_{\mathfrak{m}}),$$

where the product is taken over all localizations at maximal ideals $\mathfrak{m}$, so we may assume R local and $x_1, \ldots, x_n$ part of a system of parameters. By [12],

$$NK_*(R/I) \hookrightarrow NK_*(\hat{R}/I\hat{R}),$$

(completion) so we may assume R complete. By the structure theory of complete regular local rings over a field, we may assume $R \cong F[[x_1, \ldots, x_N]]$ with $N \geq n$.

We now argue by induction on n (compare [14], appendix). If $n = 1$, R/I is regular and $NK_*(R/I) = (0)$. Assume $n > 1$, and let $J = I_2 \cap \ldots \cap I_n$. Milnor's excision sequence [9] gives

$$\begin{aligned} NK_2(R/J) \oplus NK_2(R/I_1) \to NK_2(R/(I_1 + J)) \to NK_1(R/I) \\ \to NK_1(R/J) \oplus NK_1(R/I_1). \end{aligned}$$

From the structure of R, we see that the surjection $R/J \twoheadrightarrow R/(I_1 + J)$ is split, so

$$NK_1(R/I) \hookrightarrow NK_1(R/J) \oplus NK_1(R/I_1) = (0),$$

proving the lemma. QED

proof of (2.3). Weibel has introduced a K-theory $KH_*(R)$ which satisfies excision and is given by $KH_*(R) = \pi_* KH(R)$ for a certain non-connective spectrum $KH(R)$ [15]. There is a functorial map $K(R) \to KH(R)$ which is a homotopy equivalence for R regular. Let I and J be as in the proof of (2.3.1), and let $K(R; I_1, \ldots, I_n)$ be the spectrum calculating multi-relative Quillen K-theory. Write $\bar{I}_j$ for the image of I_j in R/I_1. Note that $J/I \cong \bar{I}_2 \cap \ldots \cap \bar{I}_n$. By induction, the middle and right hand vertical arrows in the diagram of fibrations below are defined and are homotopy equivalences

$$\begin{tikzcd}
KH(I) \arrow[r] & KH(J) \arrow[r] & KH(J/I) \\
K(R; I_1, \ldots, I_n) \arrow[u] \arrow[r] & K(R; I_2, \ldots, I_n) \arrow[u] \arrow[r] & K(R/I_1; \bar{I}_2, \ldots, \bar{I}_n). \arrow[u]
\end{tikzcd}$$

It follows that the left hand arrow is defined as well and is a homotopy equivalence.

If A is a K_n-regular ring, $KH_i(A) \cong K_i(A)$ for $i \leq n$ [op. cit., prop. 1.5]. Applying this to the sequence

$$KH_1(R) \to KH_1(R/I) \to KH_0(I) \to KH_0(R) \to KH_0(R/I)$$

and using (2.3.1), we get

$$K_0(R; I) \cong KH_0(I) \cong K_0(R; I_1, \ldots, I_n).$$

This proves (2.3). QED

Corollary (2.3.2). With hypotheses as above,

$$K_*(R; I_1, \ldots, I_n) \cong KH_*(R; I) \cong KH_*(I).$$

Corollary (2.3.3). With hypotheses as above, suppose further that $I \subset \mathrm{Rad}(R)$. Then $K_0(R; I_1, \ldots, I_n) = (0)$.

proof. In this case, $K_0(R; I) = (0)$ by [1], chap.IX, prop. (1.3).

(2.4). Consider the commutative diagram

$$\begin{tikzcd}
K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial) \arrow[r, "\iota"] \arrow[d, "a"'] & K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\Sigma) \arrow[d] \\
K_{0,\mathcal{V}^n-\mathcal{W}^{n+1}}(\Delta^p;\partial) \arrow[r] & K_{0,\mathcal{V}^n-\mathcal{W}^{n+1}}(\Delta^p;\Sigma).
\end{tikzcd}$$

All but the upper left hand group are known to be contained in $\mathcal{Z}^n(\mathrm{Spec}(F),p)$. To show ι is injective and prove theorem A, it will suffice to show the map labeled a is injective. Said another way, we will show the map

$$K_{0,\mathcal{W}^{n+1}-\mathcal{V}^{n+1}}(\Delta^p;\partial) \to K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial) \tag{2.4.1}$$

is zero. Intuitively, a class in bad position, i.e. supported on $\mathcal{W}^{n+1}$, can be "moved" to one in good position, i.e. on $\mathcal{V}^{n+1}$.

(2.5). To clarify the proof of theorem A for the reader, we finish this section with an outline of the argument. We consider diagrams of the form

(2.5.1)

$$\begin{tikzcd}
S(N) \arrow[r] & \ldots \arrow[r] & S(1) \arrow[r, equal] & \square^p \arrow[d, equal] & & & & \\
X(N) \arrow[u, "c_1"] \arrow[r] & \ldots \arrow[r] & X(1) \arrow[r] & X(0) \arrow[r] & \square^{p-2}\times\Delta^2 \arrow[r] & \ldots \arrow[r] & \square^1\times\Delta^{p-1} \arrow[r] & \Delta^p \\
\Delta^p \arrow[u, "c_2"] & & & & & & &
\end{tikzcd}$$

Here $\square^p = \mathrm{Spec}(F[x_1,\ldots,x_p])$ with *distinguished divisors* defined by $x_i = 0,1$. The $S(j)$ are iterated blowups of *faces*. (For details, see (3.1).) The $X(j)$ are unions of cubes $\square^p$ glued along divisors $x_i = 1$ (see (3.4)). Each cube in $X(j)$ coincides with the local structure at a *vertex* of $S(j)$. The vertical arrows c_1 and c_2 are correspondences constructed in (3.6) and (4.5.2), and the other arrows are regular morphisms. Each of the $S(j)$ and $X(j)$ come with a Cartier divisor which we frequently abbreviate ∂. We work with $KH_{0,supp}(Z;\partial)$ where Z denotes either $X(j)$, $S(j)$, or a variety of the form $\Delta^r\times\square^s$. Here "supp" refers to some given support. Note that by (2.3.2), KH in this context coincides with the multirelative Quillen K-theory $K_{0,supp}(Z;\partial)$ when Z is smooth and ∂ is a union of divisors with normal crossings. This is not the case, however, for $Z = X(j)$.

A class z in the image of (2.4.1) will be said to be supported on $\mathcal{W}^{n+1}$. Concerning such a class, we prove three basic results:

(i) Pulling z back to $X(N)$ and then via c_2 to Δ^p is the identity.

(ii) Pulling z back to $S(N)$ and then via c_1 to $X(N)$ coincides with pulling back directly to $X(N)$.

(iii) For a suitable choice of diagram (2.5.1) (i.e. a suitable tower of blowups of faces of $\square^p$), the pullback of z to $S(N)$ is zero.

§3. Towers of cubes

(3.1). Let S be a smooth algebraic scheme over our given field F. Let $D_1, \dots, D_n$ be smooth Cartier divisors on S. We assume we are in the normal crossings situation, i.e. that all intersections are transverse. We refer to the D_i as *distinguished divisors* and to intersections of the D_i as *faces*. Faces of dimension 0 (resp. 1) are called *vertices* (resp. *edges*). In our applications, vertices will consist of a single point.

We define $\mathcal{V}^{n+1}(S) \subset \mathcal{W}^{n+1}(S) \subset \mathcal{V}^n(S)$ just as in (1.1) by looking at subvarieties of the appropriate codimension meeting faces properly.

Let $\pi : S' \to S$ be the blowup of S along a face. We define distinguished divisors on S' to be the exceptional divisor together with the strict transforms of all distinguished divisors on S. Obviously, we can iterate this construction, defining $S'' \to S'$ by blowing up a face on S', etc. It is easy to check that

$$\pi^{-1}(\mathcal{V}^n(S)) \subset \mathcal{V}^n(S')\ ; \quad \pi^{-1}(\mathcal{W}^{n+1}(S)) \subset \mathcal{W}^{n+1}(S'). \tag{3.1.1}$$

(3.2). We take $S = S(1) = \mathrm{Spec}(F[x_1, \dots, x_p])$ with distinguished divisors defined by $x_i = 0, 1$. We consider a tower of blowups of faces

$$S(N) \to \dots \to S(1)$$

Let $v \in S(N)$ be a vertex. By induction on N we define an open immersion of some Zariski open neighborhood of $0 \in \square^p = \mathrm{Spec}(F[t_1, \dots, t_p])$ onto an open neighborhood of $v \in S(N)$,

$$\mu_v : \square^p \supset U \hookrightarrow S(N)\ ;\ \mu_v(0, \dots, 0) = v$$

so that the distinguished divisors through v pull back to the coordinate hyperplanes $t_i = 0$. When $N = 1$ define the pullback on functions by

$$\mu_v^*(x_i) = \begin{cases} t_i, & \text{if } x_i(v) = 0; \\ 1 - t_i, & \text{if } x_i(v) = 1. \end{cases}$$

This defines μ_v for $N = 1$. Suppose now $N > 1$ and let $\overline{v}$ be the image of v in S_{N-1}. Let $\overline{t}_1, \dots, \overline{t}_p$ be the coordinates at $\overline{v}$ defined inductively via $\mu_{\overline{v}}$. Over some neighborhood of $\overline{v}$, $S(N)$ is defined from $S(N-1)$ by blowing up $\{\overline{t}_i = 0 \mid i \in I\}$ for some (possibly empty) $I \subset \{1, \dots, p\}$. If $I = \emptyset$, we take $t_i = \overline{t}_i$. Otherwise there will be a unique $i \in I$ such that v does not lie on the strict transform in $S(N)$ of $\{\overline{t}_i = 0\}$. Define

$$t_j = \begin{cases} \overline{t}_j, & \text{if } j \notin I; \\ \overline{t}_j/\overline{t}_i, & \text{if } j \in I \text{ and } j \neq i; \\ \overline{t}_i, & \text{if } j = i. \end{cases}$$

The coordinate system at a vertex v defined in this way will be called the *distinguished coordinate system* at v.

Lemma (3.2.1). (i) The composition

$$\pi_v : \Box^p \overset{\mu_v}{\dashrightarrow} S(N) \to S$$

is everywhere regular.

(ii) The assignment

$$T \rightsquigarrow \coprod_{\substack{v \in T \\ v \text{ vertex}}} \Box^p$$

is functorial for blowups of faces on T.

proof. These assertions are straightforward. QED

Lemma (3.2.2). Every edge in $S(N)$ contains exactly two vertices.

proof. This is clear for $N = 0$. For $N \geq 1$, let $\rho : S(N) \to S(N-1)$ and let $\ell \subset S(N)$ be an edge. If $\rho(\ell)$ has dimension 1, then $\rho|\ell$ is an isomorphism, $\rho(\ell)$ is an edge of $S(N-1)$, and the vertices of ℓ and $\rho(\ell)$ coincide under ρ. If $\rho(\ell)$ is a point, it is a vertex of $S(N-1)$. The fibre $\rho^{-1}\rho(\ell)$ is then a projective space which is an intersection of distinguished divisors. The intersection of other distinguished divisors with $\rho^{-1}\rho(\ell)$ form the coordinate hyperplanes of the projective space. One is thus reduced to the situation where the ambient space is projective space, and the distinguished divisors are coordinate hyperplanes, where it is clear. QED

Definition (3.2.3). Let v, w be the two vertices on an edge ℓ, and let $t_1, \ldots, t_p$ (resp. $u_1, \ldots, u_p$) be the distinguished coordinate system at v (resp. at w). The *coordinate for* ℓ *at* v is the unique t_p which does not vanish on ℓ. Suppose t_r (resp. u_s) is the coordinate for ℓ at v (resp. at w). We define $g = g(v, w)$ in the symmetric group $\mathcal{S}_p$ such that $g(r) = s$ and, for $i \neq r$, the distinguished divisors defined locally by $t_i = 0$ and $u_{g(i)} = 0$ coincide.

(3.3). Our "little cubes" $\pi_v : \Box_v \to S$ are not yet really cubes. The sides $t_i = 0$ are defined, but we want to choose appropriate scale factors $c_{v,i}$ so the other sides are given by $t_i = c_{v,i}$. Choose once for all a point

$$c = (c_1, \ldots, c_p) \in S - \cup\{\text{distinguished divisors}\}.$$

Note that $\pi_v^{-1} : S - \cup\{\text{distinguished divisors}\} \to \Box_v$ is defined. Let

$$c_v = (c_{v,1}, \ldots, c_{v,p}) \in \Box_v$$

be the image of c.

Lemma (3.3.1). Let the situation and notation be as in (3.2.3). Define divisors $D_{vw} = \{t_r = c_{v,r}\} \subset \Box_v^p$ (resp. $D_{wv} = \{u_s = c_{w,s}\} \subset \Box_w^p$). Note D_{vw} (resp. D_{wv}) has coordinates t_α, $\alpha \neq r$ (resp. u_β, $\beta \neq s$). We have D_{vw} isomorphic to D_{wv} over S, the isomorphism carrying t_α to $u_{g(v,w)(\alpha)}$.

proof. To see that the two divisors are isomorphic over S, we argue by induction on N. When $N = 1$ the assertion is immediate. Suppose now that $N > 1$. Assume first that the image of the edge ℓ in $S(N-1)$, $\overline{\ell} \subset S(N-1)$ is an edge with vertices $\overline{v}$ and $\overline{w}$. We have $\square^p_v \to \square^p_{\overline{v}}$ (resp. $\square^p_w \to \square^p_{\overline{w}}$). The divisors D_{vw} and D_{wv} are the inverse images of the corresponding divisors $D_{\overline{vw}}$ and $D_{\overline{wv}}$ on $\square^p_{\overline{v}}$ and $\square^p_{\overline{w}}$. By induction, $D_{\overline{vw}} \cong D_{\overline{wv}}$, the isomorphism carrying (with obvious notation) coordinates $\overline{t}_\alpha$, $\alpha \neq r$ into coordinates $\overline{u}_\beta$, $\beta \neq s$, upto a permutation of indices. the origin in these coordinates is where the image of the edge $\overline{\ell}$ on $\square^p_{\overline{v}}$ (resp. on $\square^p_{\overline{w}}$) meets $D_{\overline{vw}}$ (resp. $D_{\overline{wv}}$). Further, D_{vw} and D_{wv} are coordinate patches on the blowup of $D_{\overline{vw}}$ and $D_{\overline{wv}}$ along the intersection of the center for the blowups on the $\square$'s with these divisors. The coordinate patches coincide, so $D_{vw} \cong D_{wv}$ as claimed.

It remains to consider the case when the image of the edge ℓ in $S(N-1)$ is a single vertex z. Let $y_1, \ldots, y_p$ be the coordinates at z. Let $\{y_i = 0 | i \in I\}$ define the center of the blowup $S(N) \to S(N-1)$. Then for some $r, s \in I$, the coordinates at v (resp. w) are y_i/y_r, $i \in I$, together with y_r and y_j, $j \notin I$ (resp. y_i/y_s, $i \in I$, together with y_s and y_j, $j \notin I$). We have

$$D_{vw} : y_s/y_r = c_{z,s}/c_{z,r} \text{ (resp. } D_{wv} : y_r/y_s = c_{z,r}/c_{z,s}).$$

The identification in question arises from the natural isomorphism

$$k[y_i/y_r, y_r, y_j]/(y_s/y_r - c_{z,s}/c_{z,r}) \cong k[y_i/y_s, y_s, y_j]/(y_r/y_s - c_{z,r}/c_{z,s})$$

which sends $y_r \mapsto (c_{z,r}/c_{z,s})y_s$. QED

Remark (3.3.2). Carrying along the constants $c_{\nu,i}$ defining the divisors D_ν is painful. We will tend to scale the coordinates at a vertex ν so $D_\nu : t_{\nu,r} = 1$.

(3.4). Let

$$X(0) = Y(0) = S(1) = \square^p = \mathrm{Spec}(F[x_1, \ldots, t_p])$$

with distinguished divisors defined by $x_i = 0, 1$. Define

$$Y(1) := \coprod \square^p_\nu \to Y(0)$$

where ν runs through the 2^p vertices of $S(1)$ and the maps on each $\square^p_\nu$ are the obvious isomorphisms mapping the origin to ν. Fix a general point $(c_1, \ldots, c_p)$ on Y(0) and define

$$Y(1) \to X(1) := \cup \square^p_\nu$$

where $\square^p_\nu$ and $\square^p_\mu$ are glued along the divisor $x_i = c_i$ if and only if ν and μ lie on a common edge $\ell \subset \square^p$ and x_i is the (necessarily unique) coordinate whose restriction to ℓ is non-constant.

Assume now we are given a tower $S(N) \to \ldots \to S(1)$ of blowups of faces. Define

$$Y(i) := \coprod \square^p_\nu$$

with ν running through vertices of $S(i)$. By (3.2.1), the association $S(i) \mapsto Y(i)$ is functorial, so we get a tower of disjoint cubes

$$Y(N) \to \ldots \to Y(1) \to Y(0).$$

By (3.3.1) if ν and μ lie on an edge in $S(i)$, there are well defined divisors $D_{\nu\omega} \subset \square^p_\nu$ and $D_{\omega\nu} \subset \square^p_\nu$ defined by setting appropriate distinguished coordinates equal to (appropriate) constants such that $D_{\nu\omega}$ and $D_{\omega\nu}$ can be identified over $Y(0)$. We define

$$Y(i) \to X(i) := \cup \square^p_\nu$$

by identifying cubes (associated to vertices on an edge) along these divisors. We get another tower

$$X(N) \to \ldots \to X(1) \to X(0).$$

The divisors $D_{\nu\mu}$ depend only on the point $c = (c_1, \ldots, c_p) \in \square^p$. As in (3.3.2), we will frequently scale the coordinates on $\square^p_\nu$ so the preimage of c is the point $(1, \ldots, 1)$. One can think of the coordinate hyperplanes $t_{\nu,i} = 0$ as *front faces* and the $D_{\nu\mu}$ defined after scaling by some $t_{\nu i} = 1$ as *back faces*. The cubes are glued along all back faces. Given $Z \subset \square^p_\nu$ we can choose c in general position so all intersections of back faces meet Z properly (or not at all). Note finally that the front faces glue to give Cartier divisors on $X(N)$.

(3.5) General position on $S(N)$ and $X(N)$. The basic result on general position is

Lemma (3.5.1). Given $Z \subset X(0) = \square^p$ closed such that no component of Z is contained in a face, we can find a tower $S(N) \to \ldots \to S(1)$ such that for $c \in \square^p$ general as above, the strict transform of Z in $Y(N)$ meets all faces properly. In other words, for ν a vertex in S(N), the strict transform of Z in $\square^p_\nu$ meets all intersections of faces (both front and back) properly.

proof. This is straightforward from [16] Th. (2.1.2) and general position for back faces. QED

Lemma (3.5.2). Let Z and $S(N)$ be as above, so the strict transform of Z on $Y(N)$ meets faces properly. Then the strict transform of Z on $S(N)$ also meets faces properly.

proof. Given $s \in S(N)$, let $\sigma \subset S(N)$ be the smallest face such that $s \in \sigma$. Let $\nu \in \sigma$ be a vertex. Then the rational map $S(N) \dashrightarrow \square_\nu$ is biregular on a neighborhood of s and carries germs of faces through s isomorphically to germs of faces on $\square_\nu$. If the strict transform of Z meets a face on $S(N)$ badly at s, the same will be true for the strict transform of Z on $\square_\nu$. QED

(3.6) Correspondences in K-theory. We need some mechanism to transfer classes in K-theory from $S(N)$ to $X(N)$. By [16], lemma (1.2.1), given ν a vertex of $S(N)$, we can find a diagram

$$\begin{tikzcd}
 & \Gamma_\nu \arrow[dl, "q_\nu"'] \arrow[dr, "p_\nu"] & \\
\square_\nu & & S(N)
\end{tikzcd}$$

such that q_ν is an iterated blowup of front faces. Indeed, we consider a diagram

$$\begin{tikzcd}
 & \Gamma_\nu(j) \arrow[r] \arrow[d] & \square_\nu \arrow[d] \\
S(j+1) \arrow[r] & S(j) \arrow[r] & S(1)
\end{tikzcd}$$

We construct $\Gamma_\nu(j)$ by induction, starting with $\Gamma_\nu(1) = \square_\nu$. By [16], lemma (1.2.1), (take $S(j) = S$, $S(j+1) = S'$, and $\Gamma_\nu(j) = V$ in the notation of [op. cit.]), there exists an iterated blowup of faces $\Gamma_\nu(j+1) \to \Gamma_\nu(j)$ fitting in a diagram

$$\begin{tikzcd}
\Gamma_\nu(j+1) \arrow[r] \arrow[d] & \Gamma_\nu(j) \arrow[d] \\
S(j+1) \arrow[r] & S(j)
\end{tikzcd}$$

Finally, we take $\Gamma_\nu = \Gamma_\nu(N)$.

Lemma (3.6.1). Let $Z \subset S(N)$ be a closed subvariety of codimension r meeting faces properly. Then $q_\nu p_\nu^{-1}(Z)$ coincides with the strict transform of Z.

proof. Let ω be the image of the vertex ν in $S(1)$. As explained in [16], there is an obvious torus action on $\square_\omega$, and $\square_\nu$ admits a torus action such that the map $\square_\nu \to \square_\omega$ is compatible with the actions. This torus action lifts to a toric structure on Γ_ν. There is also a compatible toric action on $S(N)$ which, however, does not stabilize all the faces. If $A \subset \Gamma_\nu$ is an orbit under this action, then so is $p_\nu(A)$, and the map $A \to p_\nu(A)$ has equidimensional fibres. Orbits in $S(N)$ are dense opens in (certain) faces, so if $Z \subset S(N)$ has codimension r and meets faces properly, $p_\nu^{-1}(Z) \cap A$ has codimension $\geq r$ in A. On the other hand, if we compute $p_\nu^{-1}(Z)$ by intersecting $\Gamma_\nu \times Z$ with the graph of p_ν on the smooth scheme $\Gamma_\nu \times S(N)$ we find that every irreducible component of $p_\nu^{-1}(Z)$ has codimension $\leq r$. Thus, no component of $p_\nu^{-1}(Z)$ can be contained in the complement of the open orbit of Γ_ν, proving the lemma. QED

We can choose the Γ_ν in a uniform way using [16], cor. (1.2.2) so that $q_\nu^{-1}(D_{\nu\mu}) \cong q_\mu^{-1}(D_{\mu\nu})$, and we can glue along these divisors to get a diagram

(3.6.2)
$$\begin{tikzcd}
 & \Gamma = \cup \Gamma_\nu \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X(N) \arrow[dr, "\pi_X"'] & & S(N) \arrow[dl, "\pi_S"] \\
 & \square^p = S(1). &
\end{tikzcd}$$

Recall we have already used (in the proof of (2.3)) the groups KH_* defined in [15]. These will be particularly useful in working with the K-theory of the non-normal schemes $X(N)$. We write $\partial \subset S(N)$ (resp. $\partial \subset X(N)$) for the Cartier divisor which is the sum of all the distinguished divisors (resp. the sum of all the front faces). The relative groups $KH_*(S(N), \partial)$ (resp. relative groups with support) coincide with the multi-relative K-groups (resp. multi-relative with support) (cf. the proof of (2.3)).

Lemma (3.6.2). Let $\mathcal{R}^n \subset \mathcal{V}^n(S(N))$ be the union of codimension n subvarieties of $S(N)$ whose strict transform on $X(N)$ meets faces properly.
(i) $\pi_S^{-1}(\mathcal{V}^n(\square^p)) \subset \mathcal{R}^n$.
(ii) There is a well-defined correspondence

$$q_*p^* : KH_{*,\mathcal{R}^n-\mathcal{R}^{n+1}}(S(N),\partial) \to KH_{*,\mathcal{V}^n-\mathcal{V}^{n+1}}(X(N),\partial).$$

(iii) We have

$$q_*p^*\pi_S^* = \pi_X^* : KH_{*,\mathcal{V}^n-\mathcal{V}^{n+1}}(\square^p,\partial) \to KH_{*,\mathcal{V}^n-\mathcal{V}^{n+1}}(X(N),\partial).$$

proof. Assertion (i) follows from (3.6.1). For (ii), we claim first that q is a local complete intersection morphism. This is straightforward for each q_ν, because iterated blowups with smooth centers are l.c.i. The $D_{\nu\mu}$, being in general position, are Tor-independent for the q_ν. It follows that for any set I of indices μ and any point $z \in q_\nu^{-1}(\cap_{\mu\in I} D_{\nu\mu})$ we can find a set of local defining equations $g_1,\ldots,g_r$ realizing $q_\nu^{-1}(\cap_{\mu\in I} D_{\nu\mu})$ as a complete intersection of codimension r in some affine space over the local ring at $q(z)$ on $\cap_{\mu\in I} D_{\nu\mu}$. Moreover, we can lift each g_j to a G_j defined on a neighborhood of z in Γ in such a way that the G_j give local defining equations for Γ in affine space over some neighborhood of $q(z)$.

We have that

$$\partial\Gamma := q^{-1}(\partial X(N)) = p^{-1}(\partial S(N))$$

is a Cartier divisor in Γ, so $\partial\Gamma \to \partial X(N)$ is also an l.c.i. morphism. The existence of a covariant map q_* on K-theory and on K-theory relative to ∂ is now straightforward. The assertion with supports follows by a homotopy fibre argument.

For the proof of (iii), we have $q_*p^*\pi_S^* = q_*q^*\pi_X^*$, so it will suffice to show

$$q_*q^* = \text{identity}$$

on K-theory relative to ∂. (Strictly speaking, it is necessary to show this identification is compatible with restriction to open sets, so we get the same equation on K-theory with supports. This will be clear.)

Consider the map of "doubles"

$$q \amalg q : \Gamma \amalg_{\partial\Gamma} \Gamma \to X(N) \amalg_{\partial X(N)} X(N).$$

Since KH_* satisfies excision, the relative K-theory will be a direct summand of the K-theory of the double, so it suffices to show

$$(q \amalg q)_*(q \amalg q)^* = \text{identity of } K\text{-theory}.$$

Writing $\mathcal{O}$ for the structure sheaf on the target, standard arguments in K-theory (the projection formula) reduce us to showing the natural map

$$\mathcal{O} \to \mathbf{R}(q \amalg q)_*(q \amalg q)^*\mathcal{O}$$

is a quasi-isomorphism. Let $\partial X(N) : f = 0$ on $X(N)$. Let $i_1, i_2 : \Gamma \hookrightarrow \Gamma \amalg_{\partial\Gamma} \Gamma$ be the inclusions of the factors. We have an exact sequence

$$0 \to i_{1*}\mathcal{O}_\Gamma \overset{\times f}{\to} \mathcal{O}_{\Gamma \amalg_{\partial\Gamma} \Gamma} \to i_{2*}\mathcal{O}_\Gamma \to 0.$$

The i_j are closed immersions and have no higher derived images, so an exact sequence argument reduces us to showing $\mathcal{O}_{X(N)} \to \mathbf{R}q_*\mathcal{O}_\Gamma$ is a quasi-isomorphism. We have decompositions $X(N) = \cup\Box_\nu$ and $\Gamma = \cup\Gamma_\nu$, so a similar argument reduces us to showing $\mathcal{O}_{\Gamma_\nu} \to \mathbf{R}q_{\nu*}\mathcal{O}_{\Gamma_\nu}$ is a quasi-isomorphism. But q_ν is a iterated blowup with smooth centers, and by induction one reduces to the case of a single blowup. This case is well known. QED

§4. Simplicial constructions

(4.1). We want to relate K-theory classes on one of the assemblages of cubes $X(N)$ to classes on Δ^p. We begin by recalling some standard constructions in topology. As usual, we write $\Delta^p = \operatorname{Spec}\big(F[t_0, \ldots, t_p]/(\sum t_i - 1)\big)$. By $\operatorname{Map}(\Delta^p, \Delta^q)$ we mean the set of increasing maps $0, \ldots, p \to 0, \ldots, q$. We identify $\operatorname{Map}(\Delta^p, \Delta^q)$ with a set of algebraic maps $\Delta^p \to \Delta^q$ in the obvious way. We define

$$\operatorname{Map}(\Delta^p, \Delta^{q_1} \times \ldots \times \Delta^{q_r}) := \Pi \operatorname{Map}(\Delta^p, \Delta^{q_i}).$$

For fixed $\Delta^{q_1} \times \ldots \times \Delta^{q_r}$ there is a natural simplicial set given in degree p by $\operatorname{Map}(\Delta^p, \Delta^{q_1} \times \ldots \times \Delta^{q_r})$. We write $A_{\bullet}(\Delta^{q_1} \times \ldots \times \Delta^{q_r})$ for the chain complex computing the homology of this simplicial set. Thus

$$A_{\bullet}(\Delta^{q_1} \times \ldots \times \Delta^{q_r}) := \mathbb{Z}[\operatorname{Map}(\Delta^p, \Delta^{q_1} \times \ldots \times \Delta^{q_r})].$$

There is a chain map, the Eilenberg-MacLane map (cf. [8], p. 133)

$$g : A_{\bullet}(\Delta^{q_1}) \otimes \ldots \otimes A_{\bullet}(\Delta^{q_r}) \to A_{\bullet}(\Delta^{q_1} \times \ldots \times \Delta^{q_r})$$

This map is functorial in the sense that a map $\Delta^s \to \Delta^{q_i}$ induces a commutative square

$$\begin{tikzcd}
A_{\bullet}(\Delta^{q_1}) \otimes \ldots \otimes A_{\bullet}(\Delta^{s}) \otimes \ldots \otimes A_{\bullet}(\Delta^{q_r}) \arrow[r, "g"] \arrow[d] & A_{\bullet}(\Delta^{q_1} \times \ldots \times \Delta^{s} \times \ldots \times \Delta^{q_r}) \arrow[d] \\
A_{\bullet}(\Delta^{q_1}) \otimes \ldots \otimes A_{\bullet}(\Delta^{q_r}) \arrow[r, "g"] & A_{\bullet}(\Delta^{q_1} \times \ldots \times \Delta^{q_r})
\end{tikzcd}$$

When $r = 1$, $g =$ identity.

Let $\operatorname{id}(q) \in A_q(\Delta^q)$ be the identity map. Let $\square^p = (\Delta^1)^p$. Write

$$F_p := g\big(\operatorname{id}(1) \otimes \ldots \otimes \operatorname{id}(1)\big) \in A_p(\square^p). \tag{4.1.1}$$

It will be convenient to treat such formal sums of maps as though they were themselves maps, so we will write

$$F_p : \Delta^p \to \square^p.$$

Let $\sigma \in \mathcal{S}^p$ be a permutation, and write $\sigma : \square^p \to \square^p$ for the corresponding permutation of coordinates. By [8], lemma 29.11, we have

$$\sigma \circ F_p = \operatorname{sgn}(\sigma) \cdot F_p. \tag{4.1.2}$$

Write

$$\partial = \sum (-1)^i \partial_i : \Delta^{p-1} \to \Delta^p.$$

Let δ^0 and $\delta^1 : \Delta^0 \to \Delta^1$ be the inclusions at 0 and 1 respectively. Let

$$\delta_i^j : \square^{p-1} \to \square^p$$

put j in the i-th place, $j = 0, 1;\ 1 \leq i \leq p$. Since g is a chain map, the diagram

$$\begin{tikzcd}
\big(A_\bullet(\Delta^1) \otimes \ldots \otimes A_\bullet(\Delta^1)\big)_p \arrow[r, "g"] \arrow[d, "d"'] & A_p(\square^p) \arrow[d, "\circ\partial"] \\
\big(A_\bullet(\Delta^1) \otimes \ldots \otimes A_\bullet(\Delta^1)\big)_{p-1} \arrow[r, "g"] & A_{p-1}(\square^p)
\end{tikzcd}$$

commutes. Also

$$d\big(\mathrm{id}(1) \otimes \ldots \otimes \mathrm{id}(1)\big) = \sum (-1)^{i+j} \mathrm{id}(1) \otimes \ldots \otimes \delta^j \circ \mathrm{id}(0) \otimes \ldots \otimes \mathrm{id}(1)$$

so

$$F_p \circ \partial = g \circ d\big(\mathrm{id}(1) \otimes \ldots \otimes \mathrm{id}(1)\big) = \sum (-1)^{i+j} \delta^j_{i*} F_{p-1}.$$

If we define

$$\delta := \sum (-1)^{i+j} \delta^j_i : \square^{p-1} \to \square^p, \tag{4.1.3}$$

the above formula becomes

$$F_p \circ \partial = \delta \circ F_{p-1} : \Delta^{p-1} \to \square^p. \tag{4.1.4}$$

(4.2). Let G be a simplicial abelian group. Write $C(G)$ for G viewed as a chain complex with boundary maps $d = \sum (-1)^i \partial_i$. Define $N(G) \subset G$ by

$$N(G)_n = \ker(\partial_0) \cap \ldots \cap \ker(\partial_{n-1}).$$

Note $N(G) \hookrightarrow C(G)$ as a subcomplex. Let $D(G) \subset C(G)$ be the subcomplex generated by degenerate elements of G.

Lemma (4.2.1). We have $C(G) = N(G) \oplus D(G)$. This decomposition if functorial in G.

proof. This is well known in topology (e.g. [8], Cor. 22.2.) QED

We apply this with $G = A_\bullet(\Delta^p)$. Define

$$N_p \in N(A_\bullet(\Delta^p))_p$$

to be the projection of $\mathrm{id}(p) \in C(A(\Delta^p))_p$. We view N_p as a formal sum of maps $\Delta^p \to \Delta^p$,

$$N_p : \Delta^p \to \Delta^p.$$

Lemma (4.2.2). (i) $N_p \circ \partial_j = 0; \quad 0 \leq j \leq p-1$.
(ii) $N_p \circ \partial_p = \partial \circ N_{p-1}$.

proof. Assertion (i) is clear. For (ii), write $n : A_\bullet \to N(A_\bullet)$ for the projection. We have

$$\mathrm{id}(p) \circ \partial = \sum (-1)^i \mathrm{id}(p) \circ \partial_i = \sum (-1)^i \partial_i \circ \mathrm{id}(p-1).$$

Since n is a functorial chain map,

$$N_p\circ\partial_p = N_p\circ\partial = n(\mathrm{id}(p))\circ\partial = n(\mathrm{id}(p)\circ\partial) = \sum(-1)^i n\big(\partial_i\circ\mathrm{id}(p-1)\big) =$$
$$= \sum(-1)^i\partial_i\circ n(\mathrm{id}(p-1)) = \partial\circ N_{p-1},$$

proving (ii). QED

(4.3). Recall we have constructed towers $X(N)\to\ldots\to X(0)$ where each $X(i)$ is a union of cubes $\square^p_\nu$ with given coordinate system $x_1,\ldots,x_p$ and vertex $\nu=(0,\ldots,0)$. For each vertex $\nu\in X(j)$ one has a (possibly empty) subset $I=I_\nu\subset\{1,\ldots,p\}$ such that the cubes $\square^p_\omega$ on $X(j+1)$ lying over $\square^p_\nu$ are open affines in the blowup of $\square^p_\nu$ along the subscheme defined by $x_i=0,\ i\in I$. Define

$$\Theta^p_I:\square^{p+1}=\square^p\times\square^1\to\square^p;\quad (u_1,\ldots,u_p;t)\mapsto(\Theta_{1,I},\ldots,\Theta_{p,I}) \tag{4.3.1}$$

where $\Theta_{i,I}=tu_i$ if $i\in I$ and $\Theta_{i,I}=u_i$ otherwise. The composition $\Theta_I\circ\delta:\square^p\to\square^p$ is given by

$$\Theta_I\circ\delta=\sum_{i=1}^p(-1)^{i+1}\Theta_I\circ\delta^1_i+(-1)^{p+2}\text{identity}+\mathcal{F}\quad\text{if } I\neq\emptyset$$
$$\Theta_I\circ\delta=\sum_{i=1}^p(-1)^{i+1}\Theta_I\circ\delta^1_i+\mathcal{F}\quad\text{if } I=\emptyset$$

where $\mathcal{F}$ consists of maps $\square^p\to\square^p$ with image in the front faces (i.e. faces where some $x_i=0$). Notice that for $i\in I$ there exists a $\omega\mapsto\nu$ such that writing $\iota_\omega:\square_\omega\hookrightarrow X(j+1)$ and $\pi_{\omega\nu}:\square_\omega\to\square_\nu$ we have

$$\Theta_I\circ\delta^1_i=\pi_{\omega\nu}\circ\iota_\omega.$$

For $i\notin I,\ i\leq p$ we have

$$\delta^1_I(\square^p)=\Theta_I^{-1}\{x_i=1\}.$$

Finally,

$$\Theta_I\circ\delta^1_{p+1}=\text{ identity }:\square^p\to\square^p.$$

We now assign signs $\epsilon(\nu)$ to vertices of $X(j)$ as follows. When $j=0$ so $X(j)=\square^p=\mathrm{Spec}(F[t_1,\ldots,t_p])$, $\epsilon(\nu):=(-1)^r$ with r being the number of i such that $t_i(\nu)=1$. In general, writing $\pi(j):X(j)\to X(0)$ we define $\epsilon(\nu):=\epsilon(\pi(j)(\nu))$. Define

$$G(j):=\sum_{\text{vertices of }X(j)}\epsilon(\nu)\iota_\nu:\square^p\to X(j) \tag{4.3.2}$$
$$H(j):=\sum_{\text{vertices of }X(j)}\epsilon(\nu)\iota_\nu\Theta_{I_\nu}:\square^{p+1}\to X(j)$$

Lemma (4.3.3).

$$H(j)\circ\delta\circ F_p = (-1)^p\big(G(j)\circ F_p - \pi(j+1,j)\circ G(j+1)\circ F_p\big) + \mathcal{E}$$

where $\mathcal{E}$ is a sum of maps $\Delta^p \to \partial X(j)$ and $\pi(j+1,j) : X(j+1) \to X(j)$.

proof. This is a repeat with slightly different notation of the proof of Prop. (3.4.2) in [16]. The letter n in [op. cit.] is here replaced by p, and Ψ^{n+1} becomes F_{p+1}. (The crucial point is that Ψ and F are both alternating with respect to permutation of coordinates.) The terms $H(j)\circ\delta_i^0\circ F_p$ map Δ^p to $\partial X(j)$ and can be subsumed in $\mathcal{E}$. The same is true for terms $H(j)\circ\delta_{p+1}^0\circ F_p$ at vertices where there is a nontrivial blowup. At vertices ν where no blowup occurs, $(-1)^{p+1}H(j)\circ\delta_{p+1}^0\circ F_p$ contributes $(-1)^{p+1}\iota_\nu\circ F_p$. Also $(-1)^{p+2}H(j)\circ\delta_{p+1}^1\circ F_p = (-1)^p G(j)\circ F_p$

Given a vertex $\nu \in X(j)$, and an integer i with $1 \le i \le p$, let $I = I_\nu \subset \{1,\dots,p\}$ be as above. Let $t_{\nu,1},\dots,t_{\nu,p}$ be the coordinates on $\square_\nu$. Suppose first that $i \notin I$. Let $\ell \subset S(j)$ be the edge defined near ν by $\{t_{\nu,r} = 0 \mid r \neq i\}$, and let μ be the other vertex on ℓ ([16], lemma (1.3.2)). Let $t_{\mu,1},\dots,t_{\mu,p}$ be the coordinates on $\square_\mu$, and let $t_{\mu,h}$ be the unique coordinate not vanishing on ℓ. It follows from [op. cit.], lemma (3.4.2.3) that

$$(-1)^i\epsilon(\nu)\Theta_{I_\nu}\circ\delta_i^1\circ F_p = (-1)^{h+1}\epsilon(\mu)\Theta_{I_\mu}\circ\delta_h^1\circ F_p,$$

so these terms cancel in $H(j)\circ\delta\circ F_p$.

Finally, if $i \in I_\nu$, there exists a unique vertex $\omega \in X(j+1)$ lying over ν such that the strict transform of $\{t_{\nu,i} = 0\}$ in $S(j+1)$ does not vanish at ω Let as above $\pi_{\omega\nu} : \square_\omega \to \square_\nu$. The argument at the end of the proof of Prop. (3.4.2) in [op. cit.] shows (note $\epsilon(\nu) = \epsilon(\omega)$)

$$(-1)^{i+1}\epsilon(\nu)\Theta_{I_\nu}\circ\delta_i^1\circ F_p = (-1)^{p+1}\epsilon(\omega)\pi_{\omega\nu}\circ\iota_\omega\circ F_p.$$

This completes the proof of the lemma. QED

(4.4). A final simplicial construction we will need is *subdivision.* This is done in some detail in [op. cit.], §3, so we will only sketch the construction and leave the reader with a picture. Consider a product of simplices $\prod\Delta := \Delta^{q_1}\times\ldots\times\Delta^{q_s}$ with $\sum q_i = p$. Depending on the choice of a general point $c = (c_1,\dots,c_s) \in \prod\Delta$ one defines a linear combination of maps

$$\text{subdivision}_{\Pi\Delta} : \Delta^p \to \Pi\Delta.$$

In the case when all $q_i = 1$, we have

$$\text{subdivision} = \pi(1,0)\circ G(1)\circ F_p : \Delta^p \to \square^p. \tag{4.4.1}$$

Write

$$\partial(\Pi\Delta) := \coprod_i \big(\Delta^{q_1}\times\ldots\times\Delta^{q_i-1}\times\ldots\times\Delta^{q_s}\big).$$

Let

$$\partial_{\Pi\Delta} : \partial(\Pi\Delta) \to \Pi\Delta$$

be the obvious sum with appropriate signs. This doesn't quite make sense since the various maps are not defined on the disjoint union. However, the formula

$$\text{subdivision}_{\Pi\Delta}\circ\partial = \partial_{\Pi\Delta}\circ\text{subdivision}_{\partial(\Pi\Delta)} \tag{4.4.2}$$

makes sense and is true.

(4.5) Another correspondence. The following construction will be used:

Construction (4.5.1). Let X be a variety, $Y \subset X$ a closed subvariety. Let W be a smooth variety, $D \subset W$ a normal crossings divisor. (We refer to intersections D_I of components D_i of D as faces.) Let $f_i : W \to X$ be morphisms, and write $f = \sum_{i=1}^r n_i f_i$ for a finite formal linear combination. Assume for any face $\iota : D_I \hookrightarrow W$ we have that $f\circ\iota : D_I \to Y$ in the sense that if $S \subset \{1, \ldots, r\}$ is the set of integers i such that $f_i(D_I) \notin Y$, one has formally $\sum_{i\in S} n_i f_i\circ\iota = 0$. Then there is an induced map well defined upto homotopy

$$f^* : KH(X;Y) \to KH(W;D).$$

proof. By excision, we can identify KH(D) with the homotopy limit (cf. [15], cor. 4.10)

$$\mathcal{KH}(D) := \prod KH(D_i) \rightrightarrows \prod KH(D_{ij}) \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} \prod KH(D_{ijk}) \cdots .$$

Let

$$\mathcal{S} := \prod \mathcal{S}(D_i) \rightrightarrows \prod \mathcal{S}(D_{ij}) \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} \cdots$$

denote the homotopy fibre of the map

$$\sigma : \prod_r \mathcal{KH}(D) \to \mathcal{KH}(D)$$

given by mapping $(x_1, \ldots, x_r) \mapsto \sum n_i x_i$ in the sense of the H-space structure. We will show the map

$$(f_1, \ldots, f_r) : KH(X;Y) \to \prod_r \mathcal{KH}(D)$$

lifts to a map with target $\mathcal{S}$. To this end, it suffices to show the map

$$KH(X;Y) \to \prod_r KH(W) \to \prod_r KH(D_I)$$

lifts to a map $\rho_I : KH(X;Y) \to \mathcal{S}(D_I)$ and that this lifting is functorial in the sense that for $D_J \subset D_I$ the composition $KH(X;Y) \to \mathcal{S}(D_I) \to \mathcal{S}(D_J)$ coincides with ρ_J. Let $\iota : D_I \hookrightarrow W$. Let $S \subset \{1, \ldots, r\}$ be the set of i such that $f_i\circ\iota(D_I) \not\subset Y$. The map

$$\Sigma\circ \prod_{i\in S}(f_i\circ\iota)^* : KH(X) \to \prod_{i\in S} KH(D_I) \to KH(D_I)$$

is canonically homotopic to a point because $\sum_{i\in S} n_i f_i|D_I$ cancels formally. Thus

$$\Sigma\circ \prod_{1\le i\le r}(f_i\circ\iota)^* \simeq \Sigma\circ\prod_{i\notin S}(f_i\circ\iota)^*.$$

But for $i\notin S$, the composition

$$KH(X;Y)\to KH(X)\xrightarrow{f_i} KH(W)\to KH(D_I)$$

is again canonically homotopic to a point because $f_i\circ\iota(D_I)\subset Y$. We get

$$\Sigma\circ\prod_{1\le i\le r}(f_i\circ\iota)^*\simeq \text{pt}.$$

This defines the desired lifting $\rho_I : KH(X;Y)\to \mathcal{S}(D_I)$. Finally, the diagram

$$\begin{tikzcd}
 & & KH(W;D) \arrow[d] \\
KH(X;Y) \arrow[r] \arrow[d] & \prod_r KH(W) \arrow[r] \arrow[d] & KH(W) \arrow[d] \\
\mathcal{S} \arrow[r] & \prod_r \mathcal{KH}(D) \arrow[r] & \mathcal{KH}(D)
\end{tikzcd}$$

with bottom row and right hand column fibrations defines the map

$$KH(X;Y)\to KH(W;D).$$

QED.

We consider several examples of this construction.

Example (4.5.2). Take $W=\Delta^p$ and $D=\partial(\Delta^p)$. Let $f=G(j)\circ F_p\circ N_p:\Delta^p\to X(j)$. It follows from lemma (3.4.2.3) in [op. cit.] (this argument was already used in the proof of lemma (4.3.3) above) that $G(j)\circ F_p\circ\partial:\Delta^{p-1}\to\partial(X(j))$, i.e. that the back faces cancel. Using (4.2.2), we get a map

$$(G(j)\circ F_p\circ N_p)^*: KH_*(X(j);\partial)\to K_*(\Delta^p;\partial).$$

(As usual, the group on the right is multi-relative K-theory. Since the faces are smooth, we could also have written KH_*.) We will see in (5.2)(ii) below that this map preserves supports,i.e given $V\subset\mathcal{V}^n(X(j))$ and $T\subset V\cap\mathcal{V}^{n+1}$, we can choose c general so one has

(4.5.2.1) $$KH_{*,V-T}(X(j);\partial)\to K_{*,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial).$$

Example (4.5.3). Take $W = \Delta^p$ and $D = \partial(\Delta^p)$. Let $f = H(j)\circ F_{p+1}\circ N_{p+1}\circ \partial_{p+1} : \Delta^p \to X(j)$. Since $N_{p+1}\circ\partial_{p+1}\circ\partial_i = 0$ for $0 \le i \le p$, we may apply the construction with $Y = \emptyset$. We get a commutative triangle

$$\begin{tikzcd}
KH_{*,V-T}(X(j),\partial) \arrow[rr] \arrow[dr, "f^*"'] & & KH_{*,V-T}(X(j)) \arrow[dl] \\
& KH_{*,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial) &
\end{tikzcd}$$

(Again, the fact that f^* preserves good position is checked in (5.2)(ii).) On the other hand, using (4.3.3), (4.1.4) and (4.2.2) we have

$$\begin{aligned}
(4.5.3.1)\qquad f &= H(j)\circ F_{p+1}\circ N_{p+1}\circ\partial_{p+1} = H(j)\circ\delta\circ F_p\circ N_p = \\
&= (-1)^p\big(G(j)\circ F_p\circ N_p - \pi(j+1,j)\circ G(j+1)\circ F_p\circ N_p\big) + \mathcal{E}
\end{aligned}$$

where $\mathcal{E}$ consists of maps $\Delta^p \to \partial X(j)$.

Example (4.5.4). Take $W = \Delta^p$ and $D = \partial(\Delta^p)$. Fix $r \le p$ and let $\text{subdivision}_{\Box\times\Delta} : \Delta^{p+1} \to \Box^r \times \Delta^{p+1-r}$ be subdivision (4.4). Define

$$f = \text{subdivision}_{\Box\times\Delta}\circ N_{p+1}\circ\partial_{p+1}.$$

Again given $V \subset \mathcal{V}^n(\Box\times\Delta)$ and $T \subset V \cap \mathcal{V}^{n+1}$ of finite type, we may choose $c \in \Box\times\Delta$ general and get a factorization

$$f^* : KH_{0,V-T}(\Box\times\Delta;\partial(\Box\times\Delta)) \to KH_{0,V-T}(\Box\times\Delta) \to KH_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p;\partial)$$

.

Example (4.5.5). By [16], Prop.(3.2.1), there exists $H_p : \Delta^{p+1} \to \Delta^p$ such that

$$\text{identity} \pm \text{subdivision}_{\Delta^p} = H_p\circ\partial - \partial\circ H_{p-1}.$$

Composing with N_p yields

$$H_p\circ N_{p+1}\circ\partial_{p+1} = N_p \pm \text{subdivision}_{\Delta^p}\circ N_p + \mathcal{E},$$

where $\mathcal{E} : \Delta^p \to \partial\Delta^p$. As in the previous two examples, pullback via the map on the left factors through absolute (i.e. non-relative) K-theory. Similarly, because $N \simeq \text{id}$, there exists $h_p : \Delta^{p+1} \to \Delta^p$ such that

$$h_p\circ\partial = N_p \pm \text{identity} + \mathcal{E}'$$

and N_p and identity differ by a map factoring through absolute K-theory.

§5. From simplices to cubes and back

(5.1). Define

$$\eta_p : \Delta^{p-1}\times\Delta^1 \to \Delta^p,$$
$$\eta_p((u_0,\dots,u_{p-1}),(v_0,v_1)) = (u_0v_0,\dots,u_{p-1}v_0,v_1);$$
$$\psi_p : \Delta^p\times\Delta^1 \to \Delta^p,$$
$$\psi_p((y_0,\dots,y_p),(x_0,x_1)) = (x_0y_0,\dots,x_0y_{p-1},x_1+x_0y_p).$$

The following identities are straightforward

$$\psi_p\circ\partial_{0,\Delta^1} = \rho \underset{\text{def}}{=} \text{constant map to point } (0,\dots,0,1) : \Delta^p\to\Delta^p$$
$$\psi_p\circ\partial_{1,\Delta^1} = \text{identity} : \Delta^p\to\Delta^p$$
$$\psi_p\circ\partial_{i,\Delta^p} = \partial_{i,\Delta^p}\circ\psi_{p-1} : \Delta^{p-1}\times\Delta^1\to\Delta^p,\ 0\le i\le p-1$$
$$\psi_p\circ\partial_{p,\Delta^p} = \eta_p : \Delta^{p-1}\times\Delta^1\to\Delta^p.$$

From these, we deduce the identity in the free abelian group of (not necessarily linear) maps between products of simplices:

(5.1.1) $$\partial_{\Delta^p}\circ\psi_{p-1} - \psi_p\circ\partial_{\Delta^p\times\Delta^1} = (-1)^{p+1}(\text{identity} - \rho - \eta_p - \partial_p\circ\psi_{p-1}).$$

(Here $\partial_{\Delta^p\times\Delta^1}$ is the sum of maps $\Delta^{p-1}\times\Delta^1\subset\Delta^p\times\Delta^1$ and $\Delta^p\subset\Delta^p\times\Delta^1$ with the usual signs.)

Consider the diagram

(5.1.2) $$\begin{tikzcd}
\Delta^2\times(\Delta^1)^{p-1} \arrow[dr, "a_2"] & \Delta^3\times(\Delta^1)^{p-2} \arrow[dr, "\cdots"] & & \Delta^p\times\Delta^1 \arrow[dr, "a_p"] & \\
\square=(\Delta^1)^p \arrow[u, phantom, "\cup"] \arrow[r, "b_2"'] & \Delta^2\times(\Delta^1)^{p-2} \arrow[u, phantom, "\cup"] \arrow[r] & \cdots \arrow[r] & \Delta^{p-1}\times\Delta^1 \arrow[u, phantom, "\cup"] \arrow[r, "b_p"'] & \Delta^p
\end{tikzcd}$$

where $a_i = \psi_i\times(\mathrm{id})^{p-i}$ and $b_i = \eta_i\times(\mathrm{id})^{p-i}$.

We will augment our tower of cubes and consider

$$X(N)\to\ldots\to X(1)\xrightarrow{\pi(1,0)} X(0)=\square^p \xrightarrow{b_2} \Delta^2\times(\Delta^1)^{p-2}\to\ldots\to\Delta^{p-1}\times\Delta^1\xrightarrow{b_p}\Delta^p.$$

Let $\Omega : X(N)\to\Delta^p$ be the composed map.

Proposition (5.2). Let $V\subset\mathcal{V}^n(\Delta^p)$ be of finite type. Then for a general choice of $c=(c_1,\dots,c_p)\in\square^p$ we have
(i) $\Omega^{-1}(V)\subset\mathcal{V}^n(X(N))$.
(ii) Let $f:\Delta^p\to X(N)$ be some component of the formal linear combination of

maps $G(N)\circ F_p\circ N_p : \Delta^p \to X(N)$ defined in §4. Then $f^{-1}\Omega^{-1}(V) \subset \mathcal{V}^n(\Delta^p)$.
(iii) Let $W \subset \mathcal{W}^{n+1}$ and $T \subset W \cap \mathcal{V}^{n+1}$ be of finite type. Then the induced map

$$(\Omega\circ G(N)\circ F_p\circ N_p)^* : K_{0,W-T}(\Delta^p, \partial) \to K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p, \partial)$$

coincides with the natural map defined by the inclusions $W \subset \mathcal{V}^n$ and $T \subset \mathcal{V}^{n+1}$.

proof. We begin with a general lemma.

Lemma (5.2.1). Let X and Y be smooth varieties. Let $D_i \subset X$ (resp. $E_j \subset Y$ be smooth Cartier divisors, and assume we are in the normal crossings situation. For $\sigma \subset X$ a face, let $\partial\sigma \subset \sigma$ be the union of all subfaces. Let $f : X \to Y$ be a morphism. Assume for all faces $\sigma \subset X$, there exists a face $\tau \subset Y$ such that the restriction of f induces a dominant map with equidimensional fibres $\sigma - \partial\sigma \to \tau - \partial\tau$. Then $f^{-1}(\mathcal{V}^n(Y)) \subset \mathcal{V}^n(X)$.

proof of lemma. Straightforward. QED

Turning to the proof of (5.2), it is easy to see the map η_p (and hence also the maps b_i) satisfies the hypotheses of (5.2.1). Thus $b_2^{-1} \ldots b_p^{-1}(V) \subset \mathcal{V}^n(X(0))$. Recall we wrote $Y(j) \to X(j)$ for the disjoint union of the cubes. Again by (5.2.1) it is straightforward to check that the inverse image of $\mathcal{V}^n(\square^p)$ in $Y(N)$ meets the front faces properly. We can insure that the inverse image of any given subset of finite type in $\mathcal{V}^n(\square^p)$ meets the back faces properly by choosing the point c general. This proves (i).

Note for (ii) that if say g was a component of $F_p : \Delta^p \to \square^p$ and $T \subset \square^p$ met faces properly, one could not assert that $f^{-1}(T) \subset \Delta^p$ met faces properly. (E.g. take $T = (1/2, 1/2) \in \square^2$.) However, pulling back to $X(1)$, one cube is replaced by 2^p cubes with back faces in general position. More precisely, let $\square_\nu \hookrightarrow X(j)$ be a cube with $j \geq 1$. Let g be a component of $F_p\circ N_p$, and let σ be a face of Δ^p. Note the image $g(\sigma) \subset \square_\nu$ depends on the point $c \in X(0)$. In fact, the fixed set of $g(\sigma)$ as c varies is a front face of $\square_\nu$. Thus, if $T \subset \square_\nu$ has codimension n and meets front faces properly, then for a general choice of c, $g^{-1}(T) \subset \mathcal{V}^n(\Delta^p)$. The proof of (ii) is now straightforward.

For (iii) we first consider the case $N = 1$, i.e. we look at

$$(b_p\circ \ldots \circ b_2\circ \pi(1,0)\circ G(1)\circ F_p\circ N_p)^* : K_{0,W-T}(\Delta^p, \partial) \to K_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p, \partial).$$

For $z \in K_{0,W-T}(\Delta^p, \partial)$ we find from (5.1.1) (ignoring maps with image in $\partial(\Delta^p)$) that $(a_p\circ \partial_{\Delta^p\times\square^1})^*(z) = \pm z \pm b_p^*(z)$. This can be iterated in an obvious way using the diagram (5.1.2) to get

$$\sum_{r=2}^{r=p} \pm\big(b_p\circ \ldots b_{r+1}\circ a_r\circ \partial_{\Delta^r\times\square^{p-r+1}}\big)^*(z) = \pm z \pm \big(b_p\circ \ldots \circ b_2\big)^*(z).$$

We combine this formula with (4.4.1) and example (4.5.4) to get

$$\sum_{r=2}^{r=p} \pm\big(b_p\circ \ldots b_{r+1}\circ a_r\circ \text{subdivision}_{\Delta^r\times\square^{p+1-r}}\circ N_{p+1}\circ \partial_{p+1}\big)^*(z) =$$
$$= \pm\big(\text{subdivision}_{\Delta^p}\circ N_p\big)^*(z) \pm \big(b_p\circ \ldots \circ b_2\circ \text{subdivision}_{\square^p}\circ N_p\big)^*(z).$$

By (4.5.4), the left hand side factors through the absolute (i.e. non-relative) K_0 of the various $\Delta^r \times \square^{p+1-r}$. Since z has support on $\mathcal{W}^{n+1}(\Delta^p)$, an argument like (1.2.2)(ii) shows the left side vanishes, so

$$\big(\text{subdivision}_{\Delta^p} \circ N_p\big)^*(z) = \pm\big(b_p \circ \ldots \circ b_2 \circ \text{subdivision}_{\square^p} \circ N_p\big)^*(z). \tag{5.2.2}$$

The same argument, using (4.5.5) shows

$$\big(\text{subdivision}_{\Delta^p} \circ N_p\big)^*(z) = z. \tag{5.2.3}$$

Finally, for $N > 1$, the same sort of argument using (4.5.3) and (4.5.3.1) gives

$$\big(G(j) \circ F_p \circ N_p\big)^*(w) = \big(\pi(j+1, j) \circ G(j+1) \circ F_p \circ N_p\big)^*(w), \tag{5.2.4}$$

with w the pullback of z to $X(j)$. Combining (5.2.2)-(5.2.4), summing over j, and using (4.4.1) proves (iii). QED

Remark (5.3). $G(N) \circ F_p \circ N_p$ is c_2 from (2.5.1).

§6. The Key Point

Proposition (6.1). Let $W \subset \mathcal{W}^{n+1}(\Delta^p)$ and $T \subset W \cap \mathcal{V}^{n+1}$ be of finite type. Let $z \in KH_{0,W-T}(\square^p;\partial)$. Then there exists a tower $\pi : S(N) \to \ldots \to S(1) = \square^p$ such that $\pi^*(z) = 0$ in $KH_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(S(N);\partial)$.

Lemma (6.1.1). Let S be a smooth variety with normal crossings divisor ∂ (e.g. $S = S(j)$ in some tower over $\square^p$). Let $A \subset \mathcal{W}^{n+1}(S)$ be closed of pure codimension $n+1$. Assume no component of A lies in ∂. Let $\pi : S' \to S$ be an iterated blowup of faces such that the strict transform $A' \subset \mathcal{V}^{n+1}(S')$. Let $\pi^{-1}(A) = A' \cup E$. Let $\tau \subset S$ be the union of those faces σ such that the intersection $A \cap \sigma$ is not proper, i.e. has some component of codimension in $S < n+1+\mathrm{cod}(\sigma)$. Then $\pi(E) \subset \tau \cap A$.

proof. On $S - \tau$, A meets faces properly, so the strict transform coincides with the total transform. Thus, $E \subset \pi^{-1}(\tau)$, proving the lemma. QED

proof of (6.1). We may assume that $W \subset \partial \cap \mathcal{V}^n$. Indeed, the map

$$KH_{0,\mathcal{W}^{n+1}-\mathcal{V}^{n+1}}(S) \to KH_{0,\mathcal{V}^n-\mathcal{V}^{n+1}-\partial\cap\mathcal{V}^n}(S)$$

is seen to be 0 by an argument like (1.2.2), so the image of z in the group with supports in $\mathcal{V}^n - \mathcal{V}^{n+1}$ lifts to some w with supports in $\partial \cap \mathcal{V}^n$. We may replace z by w. We also fix projective embeddings of S and S' simply so we can talk about hypersurface sections.

Suppose, we have an iterated blowup of faces $\pi : S' \to S$ and we are given $E \subset \partial \cap \mathcal{V}^n(S')$. Let $R = \pi(E)$ and assume $E = \pi^{-1}(R)$. Assume $\pi^*(z) \in KH_{0,\mathcal{V}^n-\mathcal{V}^{n+1}}(S';\partial)$ lifts to a class z' with support on $E - \mathcal{V}^{n+1} \cap E$. We argue by induction on $\dim(R)$ beginning with $S' = S$, $R = E = W$.) Let $V \subset S$ be a complete intersection of large degree and codimension n such that $R \subset V$ and V is general containing R. Since $E = \pi^{-1}(R) \subset \mathcal{W}^{n+1}(S')$ it is easy to check that $R \subset \mathcal{W}^{n+1}(S)$, so by general position $V \subset \mathcal{V}^n(S)$. Let $V' = \pi^{-1}(V)$. We have $E \subset V' \subset \mathcal{V}^n(S')$. By general position again, V' meets faces transversally outside E and also transversally over generic points of R. In other words, if $\sigma' \subset S'$ is a face, then writing $(\sigma' \cap V')_{\mathrm{sing}}$ for the singular set, we have $\pi((\sigma' \cap V')_{\mathrm{sing}}) \subset R$ and the left side has dimension strictly smaller than the right. Let $B \subset \partial \cap V'$ be a hypersurface section of $\partial \cap V'$ of large degree with $B \supset (\sigma' \cap V')_{\mathrm{sing}}$ for all faces σ'. We assume B general with that property. Let $C \subset \partial \cap V'$ be a hypersurface section such that $C \subset \mathcal{V}^{n+1}$ and the class z' lifts to $E - C \cap E$.

Let $\mathcal{D} \subset V'$ be the union of all Cartier divisors $D \subset V'$ such that $D \cap \partial \subset (B \cup C)$. The pair $(V' - \mathcal{D}, \partial \cap V' - B - C)$ has normal crossings and is Jacobson. By (2.3.2) and (2.3.3) we have

$$KH_{0,V'-\mathcal{D}}(S';\partial) \cong K_0(V' - \mathcal{D}; \partial \cap V' - B - C) = (0).$$

This means we can find a Cartier divisor $D \subset \mathcal{D}$ a closed subset $F \subset D$, and a class y in $KH_{0,D-F}(S';\partial)$ such that the image of y with supports in $\mathcal{V}^n(S') - \mathcal{V}^{n+1}(S')$ coincides with the image of z'.

Let $\rho : S'' \to S'$ be an iterated blowup of faces such that the strict transform D'' of D is in good position, i.e. $D'' \subset \mathcal{V}^{n+1}(S'')$. Let $\rho^{-1}(D) = D'' \cup E''$. By

(6.1.1) $\rho(E'')$ is contained in the union of bad (i.e. improper) faces $D \cap \sigma'$. But $D \cap \sigma' \subset (B \cup C \cap \sigma')$. Since $C \subset \partial \cap \mathcal{V}^{n+1}$, we must have $B \cap \sigma'$ bad. By the general position condition for B, this means $(\sigma' \cap V')_{\text{sing}}$ contains some component of $\sigma' \cap V'$. Since V' meets faces transversally over generic points of R, we must have $\dim(\pi \circ \rho(E'')) < \dim(R)$. Since our class y pulls back to a class with support on $D'' \cup E''$, and since $D'' \subset \mathcal{V}^{n+1}(S'')$, we replace E by $(\pi \circ \rho)^{-1}(\pi \circ \rho)E''$ and R by $\pi \circ \rho(E'')$ and conclude by induction on $\dim(R)$. QED

(6.2). We can now complete the proof of theorem A. Let $\pi : S(N) \to \square^p$ be as in (6.1). Let $\mathcal{R}^n \subset \mathcal{V}^n(S(N))$ be the union of codimension n subvarieties of $S(N)$ whose strict transform on $X(N)$ meets faces properly (cf. (3.6.2)). Let $M^i \subset \mathcal{V}^i(S(N))$ be of finite type such that $\pi^*(z)$ dies in

$$KH_{0,(\mathcal{R}^n \cup M^n)-(\mathcal{R}^{n+1} \cup M^{n+1})}(S(N); \partial).$$

Blowing up more, replacing $S(N)$ by some $S(N_1)$ and taking the base point c general, we can arrange that the inverse image of the M^i (which coincides with the strict transform since these sets are in good position) in $S(N_1)$ has strict transform on $X(N_1)$ meeting faces properly. Replacing N by N_1 we see that $\pi^*(z)$ vanishes in $KH_{0,\mathcal{R}^n - \mathcal{R}^{n+1}}(S(N); \partial)$. It follows from (3.6.2)(iii) that

$$0 = \pi_X^*(z) \in KH_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(X(N); \partial).$$

Take

$$z_0 \in \text{Image}(KH_{0,\mathcal{W}^{n+1} - \mathcal{V}^{n+1}}(\Delta^p; \partial) \to KH_{0,\mathcal{V}^n - \mathcal{V}^{n+1}}(\Delta^p; \partial)$$

and (With notation as in (5.1.2)) $z = b_2^* \circ \ldots \circ b_p^*(z_0)$. We conclude from (5.2)(iii) that $z_0 = 0$. This shows the map (2.4.1) is zero and completes the proof of theorem A.

§7. The Chow Complex and $\Gamma(2)$

(7.1). The purpose of this section is to compare the complex $\Gamma(2, \mathrm{Spec}(F))$, introduced in [7], with $\mathcal{Z}^2(\mathrm{Spec}(F), \bullet)$. We view the latter as a complex in negative degrees which we shift four steps to the right and truncate to get a complex

$$t_{\geq 1}(\mathcal{Z}^2(\mathrm{Spec}(F), \bullet)[-4])$$

satisfying

$$H^i(t_{\geq 1}(\mathcal{Z}^2(\mathrm{Spec}(F), \bullet)[-4])) = \begin{cases} CH^2(F, 2) & i = 2 \\ CH^2(F, 3) & i = 1 \\ 0 & i \neq 0, 1 \end{cases} .$$

Theorem (7.2). There is a canonical isomorphism in the derived category

$$\Gamma(2, \mathrm{Spec}(F)) \approx t_{\geq 1}(\mathcal{Z}^2(\mathrm{Spec}(F), \bullet)[-4]).$$

In particular,

$$CH^2(F, 3) \cong K_3(F)_{\text{indecomposable}}.$$

proof. Let $\mathcal{P} \subset \Delta^1$ be the set of all F-rational points distinct from 0 and 1. By definition ([7], 1.5), $\Gamma(2, \mathrm{Spec}(F))$ is the complex, placed in degrees 1 and 2

$$K_2(\Delta^1 - \mathcal{P}, \partial) \to \bigoplus_{P \in \mathcal{P}} K_1(P). \tag{7.2.1}$$

Let $\mathcal{Q} \subset \Delta^1_F$ be the set of all closed points. We consider the complex in degrees 1 and 2

$$K_2(\Delta^1 - \mathcal{Q}, \partial) \to \bigoplus_{Q \in \mathcal{Q}} K_1(Q). \tag{7.2.2}$$

We assert that the obvious map of complexes from (7.2.1) to (7.2.2) is a quasi-isomorphism. To see this, one considers the diagram

(7.2.3)

$$\begin{tikzcd}
{} \arrow[r, "\psi"] & K_3(F) \arrow[r, "\alpha"] \arrow[d, equal] & K_2(\Delta^1 - \mathcal{P}, \partial) \arrow[r] \arrow[d] & \bigoplus_{P \in \mathcal{P}} K_1(P) \arrow[r, "f"] \arrow[d] & K_2(F) \arrow[r] \arrow[d, equal] & 0 \\
{} \arrow[r, "\phi"] & K_3(F) \arrow[r, "\beta"] & K_2(\Delta^1_F - \mathcal{Q}, \partial) \arrow[r] & \bigoplus_{Q \in \mathcal{Q}} K_1(Q) \arrow[r, "g"] & K_2(F) \arrow[r] & 0
\end{tikzcd}$$

The two rows arise from localization and the identification $K_i(\Delta^1_F, \partial) \cong K_{i+1}(F)$.. To show they are exact, it suffices to show f surjective. This is clear since $K_2(F)$ is generated by symbols, and for $x \in F^\times$, $f(x|_P) = \{x, \frac{P}{P-1}\}$. The image of ψ is known ([7], 2.6) to coincide with $K_3(F)_{ind} := \mathrm{Coker}(K_3^{\mathrm{Milnor}}(F) \to K_3(F))$. Using this together with the compatibility of the bottom sequence in (7.2.3) with the norm map in a finite extension L/K, one shows easily that $\mathrm{Image}(\phi) = K_3(F)_{ind}$ also. It now follows that (7.2.1) and (7.2.2) are quasi-isomorphic.

Next consider the diagram

(7.2.4)

$$\begin{tikzcd}
\mathcal{C} : & K_2(\Delta^2_F - \mathcal{V}^1, \partial) \arrow[r, "\psi"] \arrow[d, "\beta"'] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \partial) \arrow[r] \arrow[d, "\alpha"'] & K_{0,\mathcal{V}^2}(\Delta^2, \partial) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_2(\Delta^2_F - \mathcal{V}^1, \Sigma) \arrow[r] \arrow[d, "\delta"'] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \Sigma) \arrow[r] \arrow[d, "\gamma"'] & K_{0,\mathcal{V}^2}(\Delta^2, \Sigma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2(\Delta^1_F - \mathcal{V}^1, \partial) \arrow[r] & K_{1,\mathcal{V}^1}(\Delta^1, \partial) \arrow[r] \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

where C is the top row, viewed as a complex in degrees $[0, 2]$. Note that $K_{i,\mathcal{V}^n}(\Delta^n, \Sigma) = K_{i,\mathcal{V}^n}(\Delta^n, \partial) = \bigoplus_{P \in \Delta^n - \partial \text{ closed}} K_i(P)$. This explains the equality sign in the last column and makes it straightforward to check surjectivity of γ. (Surjectivity of γ also follows from theorem A in §1.) The columns are parts of multi-relative exact sequences and are exact.

Lemma (7.2.5). For any n and any p, the complex

$$0 \to K_p(\Delta^n - \mathcal{V}^1, \Sigma) \to K_{p-1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^n, \Sigma) \to \ldots \to K_{0,\mathcal{V}^p-\mathcal{V}^{p+1}}(\Delta^n, \Sigma) \to 0$$

is exact. (If $p > n$ the complex ends with $K_{p-n,\mathcal{V}^n}(\Delta^n, \Sigma)$.) In particular, the middle row of (7.2.4) is exact.

proof of lemma. As a consequence of (1.2.2)(i) we have exact sequences

$$0 \to K_{r,\mathcal{V}^s}(\Delta^n, \Sigma) \to K_{r,\mathcal{V}^s-\mathcal{V}^{s+1}}(\Delta^n, \Sigma) \to K_{r-1,\mathcal{V}^{s+1}}(\Delta^n, \Sigma) \to 0.$$

The lemma follows by linking these sequences for various r and s together. (For the case $p \leq n$ one needs (1.2.3) to get surjectivity on the right.) QED

Lemma (7.2.6). The map δ in (7.2.4) is surjective.

proof. We remind the reader that the notations $K_*(\Delta^n, \Sigma)$ and $K_*(\Delta^n, \partial)$ introduced in §1 refer to multi-relative K-theory. In what follows we shall need to work with both ordinary relative and multi-relative groups. Let A be a ring, and let $I_1, \ldots, I_n \subset A$ be ideals. Let $J = \cap I_\ell$. Write $K_*(A, J)$ for the relative K-theory and $K_*(A; I_1, \ldots, I_n)$ for the multirelative groups. The following two facts are straightforward from the definitions.

(7.2.6.1) There exists a functorial map $K_*(A, J) \to K_*(A; I_1, \ldots, I_n)$.

(7.2.6.2) If $I_j + (\cap_{i \neq j} I_i) = A$ for every j, then $K_*(A, J) \cong K_*(A; I_1, \ldots, I_n)$.

In the sequel, the ring A will be the ring of functions on some localization of Δ^n, and $I_0, \ldots, I_n$ will be the ideals defining the codimension 1 faces. We will frequently use geometric notation for A. For example, by (7.2.6.1) and (7.2.6.2) we have a commutative diagram

(7.2.6.3)
$$\begin{tikzcd}
K_2(\Delta^2 - \mathcal{V}^1, I_0 \cap I_1) \arrow[r] \arrow[d] & K_2(\Delta^2 - \mathcal{V}^1; I_0, I_1) \arrow[d, "\delta"'] \\
K_2(\Delta^1 - \mathcal{V}^1, I_0 \cap I_1) \arrow[r, "\cong"] & K_2(\Delta^1 - \mathcal{V}^1; I_0, I_1).
\end{tikzcd}$$

The scheme $\Delta^n - \mathcal{V}^1$ is the prime ideal spectrum of the semi-localization R_n of the ring of functions on Δ^n at the vertices. We have $I_0 \cap \ldots \cap I_n \subset I_0 \cap \ldots \cap I_{n-1} \subset \mathrm{Rad}(R_n)$, from which it follows that $K_2(\Delta^n - \mathcal{V}^1, \mathcal{I})$ (where $\mathcal{I}$ denotes either of the above ideals) is generated by "pointy bracket symbols" [18] $< a, b >$ with $a \in R_n$ and $b \in \mathcal{I}$ or vice versa. In particular, the left hand vertical arrow in (7.2.6.3) is surjective, so δ is onto as well, proving the lemma. QED

Lemma (7.2.7). The map α in (7.2.4) is injective.

proof. By the remark just below (7.2.4), it will suffice to show the map

(7.2.7.1)
$$K_{2,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \Sigma) \to \bigoplus_{P \in \Delta^1 - \{0,1\} \text{ closed}} K_2(P)$$

is onto. Define $\pi : \Delta^2 \to \Delta^1$ by $\pi(x_0, x_1, x_2) = (x_0, x_1 + x_2)$. Given $P \in \Delta^1 - \{0, 1\}$ a closed point, let R_P denote the localization of the line $\pi^{-1}(P)$ at the two points where it meets $\partial(\Delta^2)$. Let $\mathcal{I} \subset R_P$ be the ideal of $\pi^{-1}(P) \cap \{x_1 = 0\}$. Note the map in (7.2.7.1) is restriction to the locus $\{x_2 = 0\}$. We have a commutative diagram

$$\begin{tikzcd}
K_2(R_P, \mathcal{I}) \arrow[r] \arrow[d] & K_{2,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \Sigma) \arrow[d] \\
K_2(P) \arrow[r] & \bigoplus_{Q \in \Delta^1 - \{0,1\} \text{ closed}} K_2(Q)
\end{tikzcd}$$

For $x \in R_P$ and $y \in \mathcal{I}$ such that $1 + xy \in R_P^\times$, the symbol $< x, y >$ is defined in $K_2(R_P, \mathcal{I})$. It maps to the Steinberg symbol $\{x(P), 1 + x(P)y(P)\} \in K_2(P)$. It is now easy to check (using the fact that $K_2(P)$ is generated by Steinberg symbols) surjectivity for the left hand vertical arrow and hence, since P was arbitrary, for the right hand arrow as well. QED

Returning to the proof of (7.2), we now can rewrite (7.2.4) as a diagram with exact columns and exact middle row:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Image}(\psi) \arrow[r] \arrow[d] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\partial) \arrow[r] \arrow[d] & K_{0,\mathcal{V}^2}(\Delta^2,\partial) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2(\Delta^2-\mathcal{V}^1,\Sigma) \arrow[r] \arrow[d] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\Sigma) \arrow[r] \arrow[d] & K_{0,\mathcal{V}^2}(\Delta^2,\Sigma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2(\Delta^1-\mathcal{V}^1,\partial) \arrow[r] \arrow[d] & K_{1,\mathcal{V}^1}(\Delta^1,\partial) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{7.2.8}$$

Combining (7.2.3) and (7.2.8), we deduce a quasi-isomorphism in the derived category

$$\Gamma(2,F)\overset{\approx}{\to} t_{\geq 1}\mathcal{C}. \tag{7.2.9}$$

where $\mathcal{C}$ is defined in (7.2.4).

Lemma (7.2.10). Let I_i be the ideal defining the i-th face of Δ^2, and let $K_i=\cap_{j\neq i}I_j$. Let $J=I_0\cap I_1\cap I_2$. Then

$$K_{*,\mathcal{V}^1}(\Delta^2,J)\cong K_{*,\mathcal{V}^1}(\Delta^2;I_i,K_i)\cong K_{*,\mathcal{V}^1}(\Delta^2;I_0,I_1,I_2)=K_{*,\mathcal{V}^1}(\Delta^2,\partial).$$

proof. Let X be a scheme, Y, Z, W closed subschemes of X with $Z\cap W=\emptyset$. Let $K(X)$ be some space which is functorial in X whose homotopy groups calculate K-theory. We have a diagram

$$\begin{tikzcd}
K_Z(X,Y) \arrow[r] \arrow[d] & K(X,Y) \arrow[r] \arrow[d] & K(X-Z,Y-Y\cap Z) \arrow[d] \\
K_Z(X) \arrow[r] \arrow[d] & K(X) \arrow[r] \arrow[d] & K(X-Z) \arrow[d] \\
K_{Z\cap Y}(Y) \arrow[r] & K(Y) \arrow[r] & K(Y-Z\cap Y),
\end{tikzcd}$$

where the middle and right columns are homotopy fibrations (defining $K(*,*)$) as are all three rows (defining $K_*(*)$). It follows that the right hand column is a fibration as well. Assume now Y (resp. X) is regular in some open set containing $Y\cap Z$ (resp. Z). Then by the existence of K'-theory and localization we have $K_Z(X)\simeq K_Z(X-W)$ and $K_{Z\cap Y}(Y)\simeq K_{Z\cap Y}(Y-Y\cap W)$. It follows that $K_Z(X,Y)\simeq K_Z(X-W,Y-Y\cap W)$. More generally, if $Y_1,\dots,Y_n$ are closed subschemes of X which are regular and meet transversally in some neighborhood of Z, an obvious induction in n gives

$$K_Z(X; Y_1, \dots, Y_n) \simeq K_Z(X - W; Y_1 - Y_1 \cap W, \dots, Y_n - Y_n \cap W).$$

We apply this with $Z \subset \mathcal{V}^1 \subset \Delta^2 = X$ and Y_0, Y_1, Y_2 faces. We take Z of finite type and W a principal divisor containing the vertices of Δ^2 and not meeting Z. From (7.2.6.2) applied with A the ring of functions on $\Delta^2 - W$, we get $K_{*,Z}(\Delta^2, J) \cong K_{*,Z}(\Delta^2; I_i, K_i) \cong K_{*,Z}(\Delta^2, \partial)$. Lemma (7.2.10) follows by passing to the limit over larger and larger Z. QED

Returning to the proof of (7.2), we consider the following diagram of complexes, where $\mathcal{D}$ is the complex on top, placed in degrees $[0, 2]$. (Here f and g are the evident restrictions and $\gamma := \beta \circ f$.)
(7.2.11)

$$\begin{tikzcd}
\mathcal{D} : & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3, \partial) \oplus K_2(\Delta^3 - \mathcal{V}^1, \Sigma) \arrow[r] \arrow[d, "{(0,g)}"'] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3, \Sigma) \arrow[r, "\gamma"] \arrow[d, "f"'] & K_{0,\mathcal{V}^2}(\Delta^2, \partial) \arrow[d, equal] \\
\mathcal{C} : & K_2(\Delta^2_F - \mathcal{V}^1, \partial) \arrow[r, "\psi"] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \partial) \arrow[r, "\beta"] & K_{0,\mathcal{V}^2}(\Delta^2, \partial)
\end{tikzcd}$$

Lemma (7.2.12). The diagram (7.2.11) induces a quasi-isomorphism

$$t_{\geq 1}\mathcal{D} \overset{\approx}{\to} t_{\geq 1}\mathcal{C}.$$

proof. f is surjective (theorem A from §1) with $\ker(f) = K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3, \partial) \subset \mathcal{D}^0$. The lemma will follow once we show that $\mathrm{Image}(\psi \circ g) \supset \mathrm{Image}(\psi)$. Writing ψ as a composition, $K_2(\Delta^2_F - \mathcal{V}^1, \partial) \xrightarrow{\bar{\psi}} K_{1,\mathcal{V}^1}(\Delta^2, \partial) \to K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2, \partial)$, it will suffice to show

$$\mathrm{Image}(\bar{\psi} \circ g) \supset \mathrm{Image}(\bar{\psi}). \tag{7.2.12.1}$$

Let $T = F[x_0, x_1, x_2]/(\sum x_i - 1,\ x_0x_1x_2)$ be the coordinate ring of the triangle. Consider the two diagrams

$$\begin{tikzcd}
 & & K_2(T)/K_2(F) \arrow[d, "\cong"', "v"] \\
K_2(\Delta^2 - \mathcal{V}^1, J) \arrow[r] \arrow[d] & K_{1,\mathcal{V}^1}(\Delta^2, J) \arrow[r, "u"] \arrow[d, "\cong"', "s"] & K_1(\Delta^2, J) \arrow[d] \\
K_2(\Delta^2 - \mathcal{V}^1; I_i, K_i) \arrow[r, "\partial_i"] \arrow[d] & K_{1,\mathcal{V}^1}(\Delta^2; I_i, K_i) \arrow[r] \arrow[d, "\cong"'] & K_1(\Delta^2; I_i, K_i) \arrow[d] \\
K_2(\Delta^2 - \mathcal{V}^1, \partial) \arrow[r, "\bar{\psi}"] & K_{1,\mathcal{V}^1}(\Delta^2, \partial) \arrow[r] & K_1(\Delta^2, \partial) \arrow[d, "\cong"'] \\
 & & K_3(F)
\end{tikzcd}$$

$$\begin{tikzcd}
K_2(\Delta^3 - \mathcal{V}^1; I_i, K_i) \arrow[r, "\text{onto}"] \arrow[d] & K_2(\Delta^2 - \mathcal{V}^1; I_i, K_i) \arrow[d] \\
K_2(\Delta^3 - \mathcal{V}^1, \partial) \arrow[r, "g"] & K_2(\Delta^2 - \mathcal{V}^1, \partial)
\end{tikzcd}$$

The rows in the top diagram are exact. Surjectivity for the top arrow in the second diagram follows from work of Loday and Guin-Waléry [18, Th. 3]. Let A be a ring with ideals J_1 and J_2. Given $z_i \in J_i$ such that $1 + z_1 z_2 \in A^\times$, they define an element $<< z_1, z_2 >> \in K_2(A; J_1, J_2)$. (They use the notation $< z_1, z_2 >$.) When $J_1 \cap J_2 \subset \mathrm{Rad}(A)$, these symbols generate $K_2(A; J_1, J_2)$, from which surjectivity in our situation is clear.

Let $z : K_2(T)/K_2(F) \to K_3(F)$ be the composition of the vertical arrows on the right of the upper diagram. To prove (7.2.12.1), it will suffice (as a few moments thought shows) to verify that $v(\ker(z))$ is contained in the subgroup of $K_1(\Delta^2, J)$ generated by the images of $u \circ s^{-1} \circ \partial_i$ for $i = 0, 1, 2$. Dayton and Roberts [19] define elements $< x_i, a x_j x_k^2 > \in \ker(z)$ for $a \in F$ and $\{i, j, k\} = \{0, 1, 2\}$. They show the mapping

$$(a_1, a_2, a_3) \mapsto < x_0, a_1 x_1 x_2^2 > + < x_0, a_2 x_2 x_1^2 > + < x_1, a_3 x_2 x_0^2 >$$

defines an isomorphism $F^{\oplus 3} \cong \ker(z)$. Lemma (7.2.12) will now follow once we verify:

$$v(< x_i, a x_j x_k^2 >) = u s^{-1} \partial_i (<< x_i, a x_j x_k^2 >>). \tag{7.2.12.2}$$

Let X be a smooth variety, and let $E, T \subset X$ be closed with $E \cap T = \emptyset$. Let $X \mapsto K(X)$ be a functor to spaces such that the homotopy of $K(X)$ is $K_*(X)$. Consider the diagram

$$\begin{tikzcd}
K'(E) \arrow[r] \arrow[d] & K(X, T) \arrow[d] \\
K(X) \arrow[r, equal] \arrow[d] & K(X) \arrow[d] \\
K(X - E) \arrow[r] & K(T).
\end{tikzcd}$$

The bottom square is commutative, and the columns are taken to be fibrations, defining the spaces at the top. We apply this with $X = \Delta^2,\ T : J = (x_0 x_1 x_2) = 0,\ E : 1 + a x_i x_j x_k^2 = 0$. We get

$$\begin{tikzcd}
<< x_i, a x_j x_k^2 >> \in K_2(\Delta^2 - E; I_i, K_i) \arrow[r] \arrow[d] & K_2(\Delta^2 - E) \arrow[r] \arrow[d] & K_2(T) \arrow[d] \\
K_{1,E}(\Delta^2; I_i, K_i) = K_1'(E) & = K_{1,E}(\Delta^2, J) \arrow[r] & K_1(\Delta^2, J)
\end{tikzcd}$$

which implies (7.2.12.2). This proves lemma (7.2.12). QED

Back yet again to the proof of (7.2). Consider yet another diagram:
(7.2.13)

$$\begin{tikzcd}
\mathcal{D}: K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\partial)\oplus K_2(\Delta^3-\mathcal{V}^1,\Sigma) \arrow[r] \arrow[d, "{(1,0)}"'] & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\Sigma) \arrow[r, "\gamma"] \arrow[d, "\zeta"'] & K_{0,\mathcal{V}^2}(\Delta^2,\partial) \arrow[d, equal] \\
\mathcal{E}: K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\partial) \arrow[r, "\phi"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\Sigma) \arrow[r] & K_{0,\mathcal{V}^2}(\Delta^2,\Sigma)
\end{tikzcd}$$

Lemma (7.2.14). Diagram (7.2.13) defines a quasi-isomorphism $t_{\geq 1}\mathcal{D}\to t_{\geq 1}\mathcal{E}$.

proof. The map ζ is surjective by lemma (7.2.5). The same lemma implies exactness of

$$K_2(\Delta^3-\mathcal{V}^1,\Sigma)\to K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\Sigma)\to K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\Sigma).$$

Lemma (7.2.14) follows immediately. QED

This is the last diagram. We promise:

$$\begin{tikzcd}
K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\partial) \arrow[r, "\bar{\phi}"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\partial) \arrow[r, "i"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\Sigma) \\
K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^4,\Sigma) \arrow[r, "\kappa"] \arrow[u, "\varphi"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^4,\Sigma) \arrow[u, "\pi"] &
\end{tikzcd}$$

The map φ is surjective by theorem A in §1, and κ is surjective by (7.2.5), so $i\circ\pi$ and $i\circ\bar{\phi}$ have the same image. We now have (OK. So we lied):

$$\begin{tikzcd}
\mathcal{E}: & K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^3,\partial) \arrow[r, "i\circ\bar{\phi}"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\Sigma) \arrow[r] \arrow[d, equal] & K_{0,\mathcal{V}^2}(\Delta^2,\Sigma) \arrow[d, equal] \\
\mathcal{F}: & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^4,\Sigma) \arrow[r, "i\circ\pi"] & K_{0,\mathcal{V}^2-\mathcal{V}^3}(\Delta^3,\Sigma) \arrow[r] & K_{0,\mathcal{V}^2}(\Delta^2,\Sigma),
\end{tikzcd}$$

and $t_{\geq 1}\mathcal{E}\to t_{\geq 1}\mathcal{F}$ is a quasi-isomorphism. It follows from (1.2.3) that $t_{\geq 1}\mathcal{F}=t_{\geq 1}(\mathcal{Z}^2(F,\bullet)[-4])$. We get a string of quasi-isomorphisms

$$\Gamma(2,F)\xrightarrow{(7.2.9)}t_{\geq 1}\mathcal{C}\xrightarrow{(7.2.12)}t_{\geq 1}\mathcal{D}\xrightarrow{(7.2.14)}t_{\geq 1}\mathcal{E}\to t_{\geq 1}\mathcal{F}=t_{\geq 1}(\mathcal{Z}^2(F,\bullet)[-4]).$$

This completes the proof of (7.2). QED

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