

# Roitman's theorem for singular complex projective surfaces

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## Abstract

Let  $X$  be a complex projective surface with arbitrary singularities. We construct a generalized Abel–Jacobi map  $A_0(X) \rightarrow J^2(X)$  and show that it is an isomorphism on *torsion* subgroups. Here  $A_0(X)$  is the appropriate Chow group of smooth 0-cycles of degree 0 on  $X$ , and  $J^2(X)$  is the intermediate Jacobian associated with the mixed Hodge structure on  $H^3(X)$ . Our result generalizes a theorem of Roitman for smooth surfaces: if  $X$  is smooth then the torsion in the usual Chow group  $A_0(X)$  is isomorphic to the torsion in the usual Albanese variety  $J^2(X) \cong \text{Alb}(X)$  by the classical Abel–Jacobi map.

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\* Members of GNSAGA of CNR, partially supported by ECC Science Plan

\*\* Partially supported by NSF grants

# Introduction

If  $X$  is a smooth projective surface over the complex numbers  $\mathbf{C}$ , the classical Abel–Jacobi map goes from the Chow group  $A_0(X)$  of cycles of degree 0 to the (group underlying the) Albanese Variety  $Alb(X)$ . Roitman’s Theorem [35] states that this map induces an isomorphism on torsion subgroups. (See [9] for a nice compendium).

The goal of this paper is to remove the word “smooth” from Roitman’s theorem. For this we shall modify the definition of  $A_0(X)$ , replace  $Alb(X)$  with Griffiths’ intermediate Jacobian  $J^2(X)$ , and construct a generalization of the Abel–Jacobi map.

**Main Theorem.** *Let  $X$  be a reduced projective surface over  $\mathbf{C}$ . Then there is a natural map from  $A_0(X)$  to  $J^2(X)$  inducing an isomorphism on torsion:*

$$A_0(X)_{tors} \cong J^2(X)_{tors}.$$

*In particular, the torsion subgroup is a finite direct sum of copies of  $\mathbf{Q}/\mathbf{Z}$ .*

If  $X$  is a normal surface, this theorem is a reformulation of a theorem of Collino and Levine [8] [24], because (as we will show in Corollary 4.3),  $J^2(X)$  is isomorphic to the Albanese of any desingularization of  $X$ .

Gillet studied the Abel–Jacobi map in [18] when  $X$  is a singular surface with “ordinary multiple curves” (*e.g.* a seminormal surface with smooth normalization  $\tilde{X}$ ). He proved in [18, Theorem B] that if  $\tilde{X}$  satisfied some extra hypotheses ( $p_g = 0$ , etc.) then the Abel–Jacobi map is surjective with finite kernel. Thus we deduce:

**Corollary.** *Let  $X$  be a surface with ordinary multiple curves such that  $H^2(X, \mathcal{O}_X) = 0$ . Assume that Bloch’s conjecture holds for the normalization  $\tilde{X}$  of  $X$ . Then the Abel–Jacobi map is an isomorphism.*

$$A_0(X) \cong J^2(X)$$

We now describe the ingredients in our main theorem. If  $X$  is a proper surface over  $\mathbf{C}$ , the intermediate Jacobian  $J^2(X)$  is defined to be

$$J^2(X) \stackrel{\text{def}}{=} \frac{H^3(X, \mathbf{C})}{F^2H^3 + H^3(X, \mathbf{Z}(2))}.$$

Here  $F^2H^3$  refers to the Hodge filtration of [11] and the coefficients  $\mathbf{Z}(2)$  refer to the embedding of  $\mathbf{Z}$  in  $\mathbf{C}$  sending 1 to  $(2\pi i)^2$ .

When  $X$  is a smooth surface, it is well known that  $J^2(X)$  is isomorphic to the Albanese Variety  $Alb(X)$ . Now suppose that  $X$  is a singular surface. We will show in Corollary 4.5 that  $J^2(X)$  is a complex torus, and that if  $X'$  is a resolution of singularities for  $X$  then  $J^2(X)$  is an extension of  $Alb(X')$  by a torus. That is, the map  $J^2(X) \rightarrow Alb(X')$  forms a 1–motive in the sense of Deligne [11]; we call it the *Albanese 1–motive* of  $X$ . Given this, the torsion subgroup of  $J^2(X)$  is a finite direct sum of copies of  $\mathbf{Q}/\mathbf{Z}$ .

The modified version of  $A_0(X)$  is defined as a subgroup of the Levine–Weibel Chow group  $CH_0(X)$  of zero-cycles on  $X$  [26]. By definition,  $CH_0(X)$  is the abelian group generated by the smooth closed points on  $X$ , modulo the subgroup generated by all terms  $D = \sum n_i P_i$  (with  $P_i$  smooth on  $X$ ) such that  $D = (f)$  for some rational function  $f$  on some curve  $C$ , the curve being locally defined by a single equation on the surface  $X$ .

If  $X$  is a surface with  $c$  proper components, there is a natural surjection  $CH_0(X) \rightarrow \mathbf{Z}^c$ , called the degree map. By definition,  $A_0(X)$  is the kernel of the degree map.

In order to prove our Main Theorem, we need to reinterpret  $CH_0(X)$  in terms of algebraic  $K$ -theory. Let  $SK_0(X)$  denote the subgroup of  $K_0(X)$  generated by the classes of smooth points on the surface  $X$ . Then  $CH_0(X)$  is isomorphic to  $SK_0(X)$ , by the map sending a smooth point to its class in  $K_0(X)$ . This is the Riemann–Roch Theorem if  $X$  is smooth. It was proven for affine surfaces in [26, Theorem 2.3]. For arbitrary quasi-projective surfaces it is due to Levine [23], who proved that both groups are isomorphic to  $H^2(X, \mathcal{K}_2)$  (cf. [32], [7]). The isomorphism

$$CH_0(X) \cong H^2(X, \mathcal{K}_2) \cong SK_0(X)$$

is often called “Bloch’s formula” for surfaces.

We have laid this paper out as follows. In §1 we present some basic facts about Deligne cohomology of a proper but singular scheme. The corresponding Deligne Chern classes which will be used in later sections is introduced in §2. In §3 we construct and compare the Mayer-Vietoris sequences for  $K$ -theory and Deligne cohomology that we shall need. In §4 we compute  $J^2(X)$  for any proper surface  $X$ . Our computation shows that  $J^2(X)$  is part of a 1-motive  $Alb(X)$  which we call the *Albanese 1-motive of  $X$* . In §5 we describe the structure of  $SK_1$  of any curve over any algebraically closed field. In §6 and §7 we establish some technical results about  $\mathcal{K}_2$ -cohomology, ending with the exact sequence of Theorem 7.7 for a normal surface  $X$  over any field containing  $\frac{1}{n}$ .

$$0 \rightarrow H^1(X, \mathcal{K}_2)/n \rightarrow NH_{\acute{e}t}^3(X) \rightarrow {}_nCH_0(X) \rightarrow 0.$$

Finally we prove the Main Theorem in §8.

## Notation

All schemes we consider will be separated and of finite type over a field  $k$ . We call such a scheme a *curve* if it is 1-dimensional, and a *surface* if it is 2-dimensional. When  $X$  is an algebraic scheme over  $\mathbf{C}$ , we will write  $H^*(X, \mathbf{Z})$  and  $H^*(X, \mathbf{C})$  for the singular cohomology of the associated analytic space  $X_{an}$  as well as for the mixed Hodge structure on it, given by Deligne [11]. The weight filtration on  $H^*(X, \mathbf{Z})$  will be written as  $W_i H^*$ , and the Hodge filtration on  $H^*(X, \mathbf{C})$  will be written as  $F^i H^*$ .

The notation  $\mathbf{Z}(r)$  denotes the subgroup  $(2\pi i)^r \mathbf{Z}$  of  $\mathbf{C}$ . Unless we wish to call attention to the relation with  $H^*(X, \mathbf{C})$ , we will write  $H^*(X, \mathbf{Z})$  instead of  $H^*(X, \mathbf{Z}(r))$ .

The notation  $\mathbf{Z}(r)_{\mathcal{D}}$  denotes the Deligne complex on a smooth scheme  $X$  over  $\mathbf{C}$  (see §1). We will use the Deligne complex to define the Deligne cohomology of proper schemes; in the affine case the definition of Deligne-Beilinson cohomology is different (one needs to consider logarithmic poles), and we remain silent about this. Similarly, the Zariski sheaves  $\mathcal{H}_{\mathcal{D}}^*(r)$  (defined as the higher direct images of  $\mathbf{Z}(r)_{\mathcal{D}}$ ) are used only for proper schemes, as a technical device. (See §1, (2.4), 6.5 and 8.2.)

The Zariski sheaf  $\mathcal{K}_q$  on  $X$  is obtained by sheafifying the Quillen higher  $K$ -theory functor  $U \mapsto K_q(U)$ . The  $\mathcal{K}$ -cohomology groups  $H^p(X, \mathcal{K}_q)$  are just the Zariski cohomology of these sheaves. As indicated in the introduction, when  $X$  is a surface the most important  $\mathcal{K}$ -cohomology group is  $H^2(X, \mathcal{K}_2) \cong CH_0(X)$ .

Similarly, we shall write  $\mathcal{K}_q(\mathbf{Z}/n)$  and  $\mathcal{H}^q(\mu_n^{\otimes i})$  for the Zariski sheaves associated to the presheaves sending  $U$  to  $K_q(U; \mathbf{Z}/n)$  and  $H_{\acute{e}t}^q(U, \mu_n^{\otimes i})$ , respectively. In general, we will always use calligraphic letters for Zariski sheaves.

Finally, we will use some standard notation. Let  $H$  be an abelian group or sheaf of abelian groups. Then  $H_{tors}$  will denote its torsion subgroup. For each integer  $n$  we will write  $H/n$  for  $H/nH$ , and  ${}_nH$  for the subgroup  $\{x \in H : nx = 0\}$  of  $H$ .

## 1 Deligne cohomology groups

For  $X$  smooth (possibly affine) over  $\mathbf{C}$  we let  $\mathbf{Z}(r)_{\mathcal{D}}$  denote the ‘‘Deligne complex’’

$$0 \rightarrow \mathbf{Z}(r) \rightarrow \mathcal{O}_{X_{an}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X_{an}}^{r-1} \rightarrow 0$$

of sheaves on the complex analytic manifold  $X_{an}$ , where  $\mathbf{Z}(r) \stackrel{\text{def}}{=} (2\pi i)^r \mathbf{Z}$  is in degree 0. The *analytic* Deligne cohomology groups of the smooth scheme  $X$  are defined to be

$$H_{\mathcal{D}}^q(X, \mathbf{Z}(r)) = H^q(X, \mathbf{Z}(r)_{\mathcal{D}}) \stackrel{\text{def}}{=} \mathbf{H}_{an}^q(X, \mathbf{Z}(r)_{\mathcal{D}}).$$

We then have exact sequences of complexes of sheaves on  $X_{an}$ :

$$0 \rightarrow \Omega_{X_{an}}^{\leq r}[-1] \rightarrow \mathbf{Z}(r)_{\mathcal{D}} \xrightarrow{\varepsilon_X} \mathbf{Z} \rightarrow 0. \quad (1)$$

We can also define the Deligne cohomology groups of a smooth simplicial scheme  $X_{\bullet}$  by considering  $\mathbf{Z}(r)_{\mathcal{D}}$  as a complex of analytic sheaves on  $X_{\bullet}$ . This yields an exact sequence of complexes parallel to (1) by [11, 5.1.9.(II)].

Now let  $X$  be a singular scheme. A *smooth proper hypercovering*  $X_{\bullet} \rightarrow X$  of  $X$  (cf. [11, 6.2.5–6.2.8]) is a simplicial scheme  $X_{\bullet}$  with smooth components  $X_i$ , each proper over  $X$ , together with a morphism to  $X$  satisfying ‘‘universal cohomological descent.’’ We define the Deligne cohomology of  $X$  to be:

$$H_{\mathcal{D}}^q(X, \mathbf{Z}(r)) \stackrel{\text{def}}{=} \mathbf{H}_{an}^q(X_{\bullet}, \mathbf{Z}(r)_{\mathcal{D}}).$$

This definition is independent of the choice of smooth proper hypercovering by [SGA4, Exposé Vbis, 5.1.7 and 5.2.4]. There is a canonical descent isomorphism  $H^*(X, \mathbf{Z}) \cong$

$H^*(X_\bullet, \mathbf{Z})$ , so the map  $\varepsilon$  in (1) induces a natural map  $\varepsilon_X: H_{\mathcal{D}}^*(X, \mathbf{Z}(r)) \rightarrow H_{an}^*(X, \mathbf{Z})$ . It is well-known (see [1, 1.6.4]) that  $\varepsilon_X$  preserves products.

For  $X$  proper with arbitrary singularities we have a standard long exact sequence

$$\cdots \xrightarrow{\varepsilon} H^q(X, \mathbf{Z}) \rightarrow H^q(X, \mathbf{C})/F^r \rightarrow H_{\mathcal{D}}^{q+1}(X, \mathbf{Z}(r)) \xrightarrow{\varepsilon} H^{q+1}(X, \mathbf{Z}) \rightarrow \cdots \quad (2)$$

induced by (1) and  $\mathbf{Z} \cong \mathbf{Z}(r) \subset \mathbf{C}$ , as well as

$$\cdots \rightarrow F^r H^q(X, \mathbf{C}) \rightarrow H^q(X, \mathbf{C}/\mathbf{Z}(r)) \rightarrow H_{\mathcal{D}}^{q+1}(X, \mathbf{Z}(r)) \rightarrow F^r H^{q+1}(X, \mathbf{C}) \rightarrow \cdots \quad (3)$$

If  $X$  is a proper surface then from (2) we have an exact sequence

$$0 \rightarrow J^2(X) \rightarrow H_{\mathcal{D}}^4(X, \mathbf{Z}(2)) \xrightarrow{\varepsilon} H^4(X, \mathbf{Z}(2)) \rightarrow 0. \quad (4)$$

Any map  $i: Y \rightarrow X$  lifts to a morphism  $i: Y_\bullet \rightarrow X_\bullet$  between hypercoverings; see [SGA4, Exposé *Vbis*, 5.1.7 and 5.2.4] or [11, 6.2.8]. The *relative* Deligne cohomology of this map is defined in the notation of [11, 6.3.3] to be:

$$H_{\mathcal{D}}^q(X \bmod Y, \mathbf{Z}(r)) \stackrel{\text{def}}{=} \mathbf{H}_{an}^q(X_\bullet \bmod Y_\bullet, \mathbf{Z}(r)_{\mathcal{D}} \bmod \mathbf{Z}(r)_{\mathcal{D}})$$

By [11, 6.3.2.2] we have a functorial long exact sequence

$$\cdots \rightarrow H_{\mathcal{D}}^q(X \bmod Y, \mathbf{Z}(r)) \rightarrow H_{\mathcal{D}}^q(X, \mathbf{Z}(r)) \rightarrow H_{\mathcal{D}}^q(Y, \mathbf{Z}(r)) \rightarrow \cdots \quad (5)$$

and of course there are relative versions of (2) and (3) which depend functorially on the pair  $(X, Y)$ , such as

$$\cdots \xrightarrow{\varepsilon} H^q(X \bmod Y, \mathbf{Z}) \rightarrow H^q(X \bmod Y)/F^r \rightarrow H_{\mathcal{D}}^{q+1}(X \bmod Y, \mathbf{Z}(r)) \xrightarrow{\varepsilon} \cdots \quad (6)$$

## Low degree Deligne cohomology

We will need the following calculation of  $H_{\mathcal{D}}^q(X, \mathbf{Z}(2))$  for  $q \leq 2$ . Given a proper scheme  $X$  over  $\mathbf{C}$ , we fix a smooth proper hypercovering  $X_\bullet \xrightarrow{\pi} X$ . By abuse of notation, we shall write  $\mathcal{H}_{\mathcal{D}}^q(2)$  for the complexes  $R^q \omega_* \mathbf{Z}(2)_{\mathcal{D}}$  of Zariski sheaves on either  $X_\bullet$  or  $X$ ,  $\omega$  denoting either  $\omega_\bullet: X_{an} \rightarrow X_{zar}$  or  $\omega = \pi \omega_\bullet: X_{an} \rightarrow X_{zar}$ .

**Proposition 1.1** *For  $X$  proper and connected over  $\mathbf{C}$  we have:*

- (i)  $H_{\mathcal{D}}^0(X, \mathbf{Z}(2)) = 0$ ;
- (ii)  $H_{\mathcal{D}}^1(X, \mathbf{Z}(2)) \cong \mathbf{C}/\mathbf{Z}(2) \cong \mathbf{C}^*$ ;
- (iii)  $H_{\mathcal{D}}^2(X, \mathbf{Z}(2))_{tors} \cong H^1(X, \mathbf{Q}/\mathbf{Z})$ .

Moreover, if  $X$  is irreducible then we have

$$(iv) H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) = H_{zar}^0(X, \mathcal{H}_{\mathcal{D}}^2(2)) = H_{zar}^0(X_\bullet, \mathcal{H}_{\mathcal{D}}^2(2))$$

and there are edge homomorphisms:

$$(v) \quad H_{zar}^1(X, \mathcal{H}_{\mathcal{D}}^2(2)) \hookrightarrow H_{zar}^1(X., \mathcal{H}_{\mathcal{D}}^2(2)) \hookrightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2)) \quad (\text{these are injections});$$

$$(vi) \quad H_{zar}^2(X, \mathcal{H}_{\mathcal{D}}^2(2)) \longrightarrow H_{zar}^2(X., \mathcal{H}_{\mathcal{D}}^2(2)) \rightarrow H_{\mathcal{D}}^4(X, \mathbf{Z}(2)).$$

**Proof** It is well-known that  $H_{an}^1(X, \mathbf{Z})$  is torsion-free. Hence (i) and (ii) follow immediately from (2). (Cf. the proof of Lemma 2.17 in [18].) Part (iii) follows from this and (3), since  $H^1(X, \mathbf{C}/\mathbf{Z}(2))_{tors} \cong H^1(X, \mathbf{Q}/\mathbf{Z})$ . In order to prove parts (iv), (v) and (vi) we use the Leray spectral sequences for  $\omega$  and  $\omega_*$ :

$$\begin{aligned} E_2^{p,q} &= H_{zar}^p(X, \mathcal{H}_{\mathcal{D}}^q(i)) \Rightarrow H_{\mathcal{D}}^{p+q}(X, \mathbf{Z}(i)) \\ E_2^{p,q} &= H_{zar}^p(X., \mathcal{H}_{\mathcal{D}}^q(i)) \Rightarrow H_{\mathcal{D}}^{p+q}(X, \mathbf{Z}(i)) \end{aligned} \quad (7)$$

with  $i = 2$  (cf. [18, (2.13)]). For this, we need to compute  $\mathcal{H}_{\mathcal{D}}^0(2)$  and  $\mathcal{H}_{\mathcal{D}}^1(2)$ .

When  $U$  is smooth we may identify the analytic sheaf  $\mathcal{O}_U/\mathbf{Z}(2)$  with  $\mathcal{O}_U^*$  and obtain a quasi-isomorphism between  $\mathbf{Z}(2)_{\mathcal{D}}$  and the complex  $0 \rightarrow \mathcal{O}_U^* \xrightarrow{dlog} \Omega_U^1$ . It follows that there is a distinguished triangle of complexes of analytic sheaves on  $X$ .

$$\mathbf{C}^*[-1] \rightarrow \mathbf{Z}(2)_{\mathcal{D}} \rightarrow \Omega_X^1/dlog(\mathcal{O}_X^*)[-2] \rightarrow \mathbf{C}^*.$$

Applying  $\omega_*$  and  $R^1\omega_*$  immediately yields  $\mathcal{H}_{\mathcal{D}}^0(2) = 0$  and  $\mathcal{H}_{\mathcal{D}}^1(2) = \omega_*\mathbf{C}^* = \mathbf{C}^*$  on both  $X_{zar}$  and  $X_{zar}$ . Therefore in either spectral sequence (7) the row  $q = 0$  vanishes and in row  $q = 1$  we have  $H_{zar}^p(X, \mathbf{C}^*) = H_{zar}^p(X., \mathbf{C}^*)$ . The exact sequences of low degree terms in (7) become:

$$\begin{aligned} 0 \rightarrow H_{zar}^1(X, \mathbf{C}^*) \rightarrow H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) \rightarrow H^0(X, \mathcal{H}_{\mathcal{D}}^2) \xrightarrow{d_2} H_{zar}^2(X, \mathbf{C}^*) \rightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2)) \\ 0 \rightarrow H_{zar}^1(X, \mathbf{C}^*) \rightarrow H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) \rightarrow H^0(X., \mathcal{H}_{\mathcal{D}}^2) \xrightarrow{d_2} H_{zar}^2(X, \mathbf{C}^*) \rightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2)). \end{aligned}$$

The map between these sequences identifies them, and  $H^0(X, \mathcal{H}_{\mathcal{D}}^2) \cong H^0(X., \mathcal{H}_{\mathcal{D}}^2)$  by the 5-lemma. If  $X$  is irreducible then  $H_{zar}^p(X, \mathbf{C}^*) = 0$  for  $p \neq 0$ . Hence  $H_{\mathcal{D}}^2(X, \mathbf{Z}(2))$  is isomorphic to  $H^0(X, \mathcal{H}_{\mathcal{D}}^2)$ . Parts (v) and (vi) follows similarly.  $\bullet$

For each  $n$  there is a distinguished triangle of complexes of analytic sheaves on  $X$ :

$$\mathbf{Z}/n[-1] \xrightarrow{\delta} \mathbf{Z}(2)_{\mathcal{D}} \xrightarrow{n} \mathbf{Z}(2)_{\mathcal{D}} \xrightarrow{\bar{\epsilon}} \mathbf{Z}/n \xrightarrow{\delta[1]} \mathbf{Z}(2)_{\mathcal{D}}[1]. \quad (8)$$

The comparison theorem between the analytic and étale sites, together with universal cohomological descent, yields  $H^q(X., \mathbf{Z}/n) \cong H^q(X, \mathbf{Z}/n) \cong H_{ét}^q(X, \mathbf{Z}/n)$ . Fixing an  $n^{\text{th}}$  root of unity allows us to identify  $\mu_n$ ,  $\mu_n^{\otimes 2}$  and  $\mathbf{Z}/n$  on  $X_{ét}$ . If  $X$  is proper, the cohomology of the triangle (8) yields ‘‘Kummer sequences’’

$$0 \rightarrow H_{\mathcal{D}}^q(X, \mathbf{Z}(2))/n \xrightarrow{\bar{\epsilon}} H_{ét}^q(X, \mu_n^{\otimes 2}) \xrightarrow{\delta} H_{\mathcal{D}}^{q+1}(X, \mathbf{Z}(2))_{n-tors} \rightarrow 0. \quad (9)$$

By 1.1 this identifies  $\mu_n \cong H_{\acute{e}t}^0(X, \mathbf{Z}/n)$  with the  $n$ -torsion in  $\mathbf{C}^* \cong H_{\mathcal{D}}^1(X, \mathbf{Z}(2))$ , and identifies  $H_{\acute{e}t}^1(X, \mu_n^{\otimes 2}) \cong H_{\acute{e}t}^1(X, \mathbf{Z}/n)$  with the  $n$ -torsion in  $H_{\mathcal{D}}^2(X, \mathbf{Z}(2))$ .

Now consider the morphism  $\omega: X_{\bullet, an} \rightarrow X_{zar}$ . Applying the higher direct image  $R^q\omega_*$  to (8) yields an exact sequence of Zariski sheaves:

$$\xrightarrow{\bar{\varepsilon}} \mathcal{H}^{q-1}(\mu_n^{\otimes 2}) \xrightarrow{\delta} \mathcal{H}_{\mathcal{D}}^q(2) \xrightarrow{n} \mathcal{H}_{\mathcal{D}}^q(2) \xrightarrow{\bar{\varepsilon}} \mathcal{H}^q(\mu_n^{\otimes 2}) \xrightarrow{\delta}. \quad (10)$$

In particular,  $\delta$  identifies  $\mathcal{H}^1(\mu_n^{\otimes 2})$  with the  $n$ -torsion subsheaf of  $\mathcal{H}_{\mathcal{D}}^2(2)$ .

The map  $H_{\acute{e}t}^{*-1}(X, \mu_n^{\otimes 2}) \xrightarrow{\delta} H_{\mathcal{D}}^*(X, \mathbf{Z}(2))$  is also the abutment of a morphism of Leray spectral sequences. At the  $E_2$ -level it is  $H_{zar}^p(X, \mathcal{H}^{q-1}(\mu_n^{\otimes 2})) \xrightarrow{\delta} H_{zar}^p(X, \mathcal{H}_{\mathcal{D}}^q(2))$ . If  $X$  is proper and irreducible then the bottom row of both spectral sequences degenerates (e.g.  $H_{zar}^p(X, \mathbf{Z}/n) = 0$  for  $p \neq 0$ ) and we obtain the following result.

**Corollary 1.2** *If  $X$  is proper and irreducible, there is a commutative diagram whose rows are the exact sequences of low degree terms of Leray spectral sequences:*

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(X, \mathcal{H}^1(\mu_n^{\otimes 2})) & \rightarrow & H_{\acute{e}t}^2(X, \mu_n^{\otimes 2}) & \rightarrow & H^0(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \xrightarrow{d_2} H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})) \\ & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ 0 & \rightarrow & H^1(X, \mathcal{H}_{\mathcal{D}}^2(2)) & \rightarrow & H_{\mathcal{D}}^3(X, \mathbf{Z}(2)) & \rightarrow & H^0(X, \mathcal{H}_{\mathcal{D}}^3(2)) \xrightarrow{d_2} H^2(X, \mathcal{H}_{\mathcal{D}}^2(2)). \end{array}$$

## 2 Chern classes in Deligne cohomology

For each scheme  $X$  of finite type over  $\mathbf{C}$  the exponential map  $\mathcal{O}_{X_{an}} \rightarrow \mathcal{O}_{X_{an}}^*$  induces a quasi-isomorphism between  $\mathbf{Z}(1)_{\mathcal{D}} = (\mathbf{Z} \rightarrow \mathcal{O}_{X_{an}})$  and  $\mathcal{O}_{X_{an}}^*[-1]$ . This quasi-isomorphism also holds over a simplicial scheme  $X_{\bullet}$  by naturality, so  $\mathbf{H}^q(X_{\bullet}, \mathbf{Z}(1)_{\mathcal{D}}) \cong H^{q-1}(X_{\bullet}, \mathcal{O}_{X_{\bullet}}^*)$ . This gives a natural map from  $H_{an}^1(X, \mathcal{O}_X^*)$  to  $\mathbf{H}^2(X_{\bullet}, \mathbf{Z}(1)_{\mathcal{D}}) \cong H_{an}^1(X_{\bullet}, \mathcal{O}_{X_{\bullet}}^*)$  for every smooth proper hypercovering  $X_{\bullet} \rightarrow X$ . Composing with the determinant map  $K_0(X) \rightarrow \text{Pic}(X)$  and the natural map  $\text{Pic}(X) \rightarrow H_{an}^1(X, \mathcal{O}_X^*)$  yields a map  $c_1: K_0(X) \rightarrow \mathbf{H}^2(X_{\bullet}, \mathbf{Z}(1)_{\mathcal{D}})$ .

Now the splitting principle holds for Deligne cohomology by [18, 5.2]. (Warning: if  $X$  is not proper this differs slightly from the splitting principle proven in [1, 1.7.2]!) Thus the map  $c_1$  extends to Chern classes  $c_i: K_0(X) \rightarrow \mathbf{H}^{2i}(X_{\bullet}, \mathbf{Z}(i)_{\mathcal{D}})$  for vector bundles. When  $X$  is proper, these are the Deligne-Beilinson Chern classes

$$c_i: K_0(X) \rightarrow H_{\mathcal{D}}^{2i}(X, \mathbf{Z}(i)).$$

Recall from (1) that there is a map  $\varepsilon_X: \mathbf{H}^{2i}(X_{\bullet}, \mathbf{Z}(i)_{\mathcal{D}}) \rightarrow H_{an}^{2i}(X, \mathbf{Z})$ , and that it is product-preserving.

**Lemma 2.1** (cf. Beilinson [1, 1.7]) *The composition of  $c_i$  with the map  $\varepsilon_X$  is the classical Chern class of the associated topological vector bundle [30]:*

$$c_i^{an}: K_0(X) \rightarrow H_{an}^{2i}(X, \mathbf{Z}) = H_{top}^{2i}(X, \mathbf{Z}).$$

**Proof** Since  $\varepsilon_X$  preserves cup products, the splitting principle shows that it suffices to establish the result for  $c_1$ . If  $X$  is smooth then  $c_1$  is the analytic determinant map, and  $\varepsilon_X$  is just the usual map  $\partial_X: H_{an}^1(X, \mathcal{O}_X^*) \rightarrow H_{an}^2(X, \mathbf{Z})$  used to define  $c_1^{an}$  on analytic vector bundles, so it is clear that  $c_1^{an} = \varepsilon_X \circ c_1$ . To deduce the result for general  $X$ , choose a smooth proper hypercover  $u: X_\bullet \rightarrow X$ . Composing  $\partial_X$  (which is  $c_1^{an}$ ) with the descent isomorphism  $H_{an}^{2i}(X, \mathbf{Z}) \cong H_{an}^{2i}(X_\bullet, \mathbf{Z})$  is the descent map  $H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X_\bullet, \mathcal{O}_{X_\bullet}^*)$  (which is  $c_1$ ) composed with  $\partial_{X_\bullet}$ , *i.e.*, with  $\varepsilon_{X_\bullet}$ . •

Reduction of  $\varepsilon_X \bmod n$  yields a map  $\bar{\varepsilon}_X: H_{\mathcal{D}}^{2i}(X, \mathbf{Z}(i)) \rightarrow H_{an}^{2i}(X, \mathbf{Z}/n)$ . Since reduction mod  $n$  is product-preserving and sends  $c_1^{an}$  to the étale Chern class  $c_1^{et}$ , we deduce the

**Corollary 2.2** *The composition of  $c_i$  with  $\bar{\varepsilon}_X$  is the étale Chern class*

$$c_i^{et}: K_0(X) \rightarrow H_{an}^{2i}(X, \mathbf{Z}/n) \cong H_{et}^{2i}(X, \mu_n^{\otimes i}).$$

In this paper we shall be mostly concerned with the class  $c_2: K_0(X) \rightarrow H_{\mathcal{D}}^4(X, \mathbf{Z}(2))$  when  $X$  is a projective surface. Recall from the introduction (or [26]) that the Chow group  $CH_0(X)$  of zero-cycles on  $X$  is isomorphic to the subgroup  $SK_0(X)$  of  $K_0(X)$ .

If  $X$  has  $c$  irreducible components then there is a natural degree map  $CH_0(X) \rightarrow \mathbf{Z}^c$ , and  $A_0(X)$  is defined to be the kernel of this map. The following cohomological interpretation of the degree map will be useful.

**Lemma 2.3** (Beilinson [1, 1.9]) *If  $X$  is a projective surface, the degree map is the same (up to sign) as the classical Chern class*

$$CH_0(X) \hookrightarrow K_0(X) \xrightarrow{c_2^{an}} H_{an}^4(X, \mathbf{Z}) \cong \mathbf{Z}^c.$$

By (4), the Deligne Chern class  $c_2$  induces a natural map  $\rho: A_0(X) \rightarrow J^2(X)$  fitting into the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & A_0(X) & \rightarrow & CH_0(X) & \xrightarrow{\text{deg}} & \mathbf{Z}^c & \rightarrow 0 \\ & & \rho \downarrow & & c_2 \downarrow & & \parallel & \\ 0 & \rightarrow & J^2(X) & \rightarrow & H_{\mathcal{D}}^4(X, \mathbf{Z}(2)) & \xrightarrow{\varepsilon} & H_{an}^4(X, \mathbf{Z}) & \rightarrow 0 \end{array}$$

**Definition:** We shall refer to the map  $\rho$  as the *Abel–Jacobi map*, because if  $X$  is a smooth surface then  $J^2(X)$  is the usual Albanese variety and the map  $\rho$  coincides with the classical Abel–Jacobi map by [1, 1.9.1] or [18, 2.24].

**Proof** Observe that if  $X$  has  $c$  proper irreducible components then  $H^4(X, \mathbf{Z}) \cong \mathbf{Z}^c$ , because the singular locus of  $X$  has real analytic dimension  $\leq 2$ . Given Lemma 2.1, the second assertion follows from the first. If  $X$  is a smooth projective surface the result is classical; one way to see it is to use the product formula for two divisors on  $X$ :

$$c_2^{an}(D \otimes E) = -c_1^{an}(D) \cup c_1^{an}(E) = -(D \cdot E)[X].$$

In general, choose a resolution of singularities  $X' \rightarrow X$ . Since  $X'$  has  $c$  disjoint components, the degree map on  $X$  factors through the degree map on  $X'$  as  $CH_0(X) \rightarrow CH_0(\tilde{X}) \rightarrow \mathbf{Z}^c$ . By naturality of  $c_2^{an}$ , the isomorphism  $H_{an}^4(X, \mathbf{Z}) \cong H_{an}^4(X', \mathbf{Z})$  allows us to deduce the result for  $X$  from the result for  $X'$ .  $\bullet$

(2.4) As observed by Beilinson [1, 2.3] (cf. [18, §5]), the formalism of Deligne cohomology allows us to extend the Chern classes from  $K_0(X)$  to higher  $K$ -theory as well. The higher Deligne Chern classes are homomorphisms

$$c_i: K_q(X) \rightarrow H_{\mathcal{D}}^{2i-q}(X, \mathbf{Z}(i)).$$

Composition with  $\varepsilon_X$  yields the higher analytic Chern classes  $c_i^{an}$ , and reduction mod  $n$  yields the higher étale Chern classes  $c_i^{et}$ . Moreover, the following holds.

(2.4.1) There is a connected simplicial presheaf  $K \simeq \Omega_0 BQP$  and a simplicial sheaf  $\mathcal{D}$  on  $X_{zar}$  such that  $\pi_q K(U) = K_q(U)$  for  $q \geq 1$ , and  $\pi_q \mathcal{D}(U) = H^{2i-q}(U, \mathbf{Z}(i)_{\mathcal{D}})$  for  $q \geq 0$ . Moreover, there is a map of simplicial presheaves  $C_i^{ss}: K \rightarrow \mathcal{D}$  such that  $\pi_q C_i^{ss}(X)$  is the Deligne cohomology Chern class  $c_i$  on  $K_q(X)$ . (cf. [18, 5.4], which differs somewhat from [1] and [19].)

Indeed,  $\mathcal{D}$  is the simplicial sheaf of abelian groups associated by the Dold-Kan theorem ([41, 8.4.1]) to the good truncation  $\tau^{\leq 0} \mathbf{R}\omega_* \mathbf{Z}(i)_{\mathcal{D}}[2i]$  of the total derived direct image of  $\mathbf{Z}(i)_{\mathcal{D}}[2i]$  under  $\omega: X_{.,an} \rightarrow X_{zar}$ .

(2.4.2) Let  $\mathcal{E}$  denote the simplicial sheaf associated by the Dold-Kan theorem to the good truncation  $\tau^{\leq 0} \mathbf{R}\omega_* \mathbf{Z}/n[2i]$  of the total derived direct image of  $\mathbf{Z}/n[2i]$ . Then  $\pi_q \mathcal{E}(U) = H_{an}^{2i-q}(U, \mathbf{Z}/n) \cong H_{\acute{e}t}^{2i-q}(U, \mu_n^{\otimes i})$ . If we define  $L$  to be the homotopy fiber of  $K \xrightarrow{n} K$  then we have  $\pi_q L(U) = K_{q+1}(U; \mathbf{Z}/n)$ . This all gives a homotopy commutative diagram whose rows are homotopy fibration sequences

$$\begin{array}{ccccccc} \Omega K & \rightarrow & L & \rightarrow & K & \xrightarrow{n} & K \\ \downarrow \Omega C_2^{ss} & & \downarrow C_2^{ss} & & \downarrow C_2^{ss} & & \downarrow C_2^{ss} \\ \Omega \mathcal{D} & \rightarrow & \Omega \mathcal{E} & \xrightarrow{\delta} & \mathcal{D} & \xrightarrow{n} & \mathcal{D} \xrightarrow{\bar{\varepsilon}} \mathcal{E}. \end{array} \quad (11)$$

From Corollary 2.2 and a standard argument with  $H_{\acute{e}t}^*(X, G, \mu_n^{\otimes i})$  it is easy to see that not only does  $K \rightarrow \mathcal{E}$  induce the higher étale Chern class  $c_i^{et}$  on  $K_*(X)$  but the map  $L \rightarrow \Omega \mathcal{E}$  induces the usual étale Chern classes on  $K$ -theory with coefficients mod  $n$ .

$$c_i^{et}: K_q(X; \mathbf{Z}/n) \longrightarrow H_{\acute{e}t}^{2i-q}(X, \mu_n^{\otimes i})$$

Applying  $\pi_2$  to (11) with  $i = 2$  and  $U = X$  yields the commutative diagram

$$\begin{array}{ccccccc} K_3(X) & \rightarrow & K_3(X; \mathbf{Z}/n) & \longrightarrow & K_2(X) & \xrightarrow{n} & K_2(X) \\ & & \downarrow c_2^{et} & & \downarrow c_2 & & \downarrow c_2 \\ 0 & \rightarrow & H_{\acute{e}t}^1(X, \mu_n^{\otimes 2}) & \xrightarrow{\delta} & H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) & \xrightarrow{n} & H_{\mathcal{D}}^2(X, \mathbf{Z}(2)). \end{array} \quad (12)$$

By (9) we see that  $c_2^{et}$  vanishes on  $K_3(X)$  and factors through  ${}_nK_2(X)$ .

Applying  $\pi_2$  to (11) with  $i = 2$  and sheafifying yields the commutative diagram of sheaves in which the bottom row is part of (10):

$$\begin{array}{ccccccc} \mathcal{K}_3 & \rightarrow & \mathcal{K}_3(\mathbf{Z}/n) & \longrightarrow & \mathcal{K}_2 & \xrightarrow{n} & \mathcal{K}_2 \\ & & \downarrow c_2^{et} & & \downarrow c_2 & & \downarrow c_2 \\ 0 & \rightarrow & \mathcal{H}^1(\mu_n^{\otimes 2}) & \xrightarrow{\delta} & \mathcal{H}_{\mathcal{D}}^2(2) & \xrightarrow{n} & \mathcal{H}_{\mathcal{D}}^2(2) \end{array} \quad (13)$$

By (10) we see that  $c_2^{et}$  vanishes on  $\mathcal{K}_3$  and factors through the sheaf  ${}_n\mathcal{K}_2$ .

(2.4.3) There is a morphism of spectral sequences between the Brown–Gersten spectral sequence for  $K_{-*}(X)$  and the Leray spectral sequence in (7) converging to  $H_{\mathcal{D}}^{2i+*}(X, \mathbf{Z}(i))$ . At the  $E_2^{pq}$ -level the morphisms are the cohomology of  $c_i$ :

$$H_{zar}^p(X, \mathcal{K}_{-q}) \xrightarrow{c_i} H_{zar}^p(X, \mathcal{H}_{\mathcal{D}}^{2i+q}(i)).$$

Here  $\mathcal{K}_q$  is the sheaf on  $X_{zar}$  associated to the presheaf  $K_q$  and the sheaves  $\mathcal{H}_{\mathcal{D}}^j(i)$  are  $R^j\omega_*\mathbf{Z}(i)_{\mathcal{D}}$ , as in the proof of Proposition 1.1. By [38], the first spectral sequence converges to  $K_{-p-q}(X)$  whenever  $X$  is quasi-projective. The second spectral sequence is an obvious reindexing of (7) and converges to  $H_{\mathcal{D}}^{2i+p+q}(X, \mathbf{Z}(i))$ .

Here are three applications of the morphism of spectral sequences in (2.4.3). First, if  $X$  is a projective surface we have a commutative diagram

$$\begin{array}{ccc} CH_0(X) \cong H^2(X, \mathcal{K}_2) & \hookrightarrow & K_0(X) \\ & & \downarrow c_2 \\ H^2(X, \mathcal{H}_{\mathcal{D}}^2(2)) & \rightarrow & H_{\mathcal{D}}^4(X, \mathbf{Z}(2)). \end{array}$$

where the bottom horizontal map is given by Proposition 1.1(vi).

Second, suppose that  $Y$  is 1-dimensional. Then we may identify the group  $H^1(X, \mathcal{K}_2)$  with the subgroup  $SK_1(X)$  of  $K_1(X)$ , and  $c_2: SK_1(X) \rightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2))$  is identified with the composite  $H^1(X, \mathcal{K}_2) \xrightarrow{c_2} H^1(X, \mathcal{H}_{\mathcal{D}}^2(2)) \rightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2))$ .

Third, suppose that  $X$  is an irreducible projective surface. Then  $c_2$  vanishes on the image of  $H^2(X, \mathcal{K}_3)$  in  $K_1(X)$  because it factors through  $H^2(X, \mathcal{H}_{\mathcal{D}}^1(2)) = H_{zar}^2(X, \mathbf{C}^*)$ , which is zero because  $X$  is irreducible, as we saw in the proof of Proposition 1.1. Since  $SK_1(X)$  is an extension of  $H^1(X, \mathcal{K}_2)$  by this image, we may summarize this as follows.

**Lemma 2.5** *Let  $X$  be an irreducible projective surface over  $\mathbf{C}$ . Then the Chern class  $c_2: SK_1(X) \rightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2))$  factors as:*

$$SK_1(X) \xrightarrow{\text{onto}} H^1(X, \mathcal{K}_2) \xrightarrow{c_2} H^1(X, \mathcal{H}_{\mathcal{D}}^2(2)) \hookrightarrow H_{\mathcal{D}}^3(X, \mathbf{Z}(2)).$$

### 3 Mayer–Vietoris sequences

Since we are going to deal with resolutions of singularities or normalizations we will need some Mayer–Vietoris sequences. In this section we do this for mixed Hodge structures, Deligne cohomology and  $K$ -theory.

Associated to a proper birational morphism  $f: X' \rightarrow X$  of  $\mathbf{C}$ -algebraic schemes, and every closed subscheme  $i: Y \hookrightarrow X$  we have the commutative square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array} \quad (14)$$

where  $Y' = f^{-1}(Y)$  ( $= Y \times_X X'$ ). We shall always assume that  $Y$  is chosen so that the restriction  $f: X' - Y' \xrightarrow{\cong} X - Y$  is an isomorphism.

**Proposition 3.1** (Mayer–Vietoris for mixed Hodge structures) *Associated with any square (14) we have a long exact sequence of mixed Hodge structures*

$$\cdots \rightarrow H^n(X, \mathbf{Z}) \xrightarrow{u} H^n(X', \mathbf{Z}) \oplus H^n(Y, \mathbf{Z}) \xrightarrow{v} H^n(Y', \mathbf{Z}) \xrightarrow{\partial} H^{n+1}(X, \mathbf{Z}) \rightarrow \cdots$$

in which

$$u = \begin{pmatrix} f^* \\ i^* \end{pmatrix} \quad \text{and} \quad v = (i'^*, -f'^*)$$

**Proof** We have a map of long exact sequences

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^n(X \bmod Y, \mathbf{Z}) & \rightarrow & H^n(X, \mathbf{Z}) & \xrightarrow{i^*} & H^n(Y, \mathbf{Z}) & \rightarrow & \cdots \\ & & f^* \downarrow \cong & & f^* \downarrow & & f'^* \downarrow & & \\ \cdots & \rightarrow & H^n(X' \bmod Y', \mathbf{Z}) & \rightarrow & H^n(X', \mathbf{Z}) & \xrightarrow{i'^*} & H^n(Y', \mathbf{Z}) & \rightarrow & \cdots \end{array}$$

where  $H^*(-\bmod \dagger, \mathbf{Z})$  is the relative singular cohomology functor (defined in [11, 8.3.8]). By excision  $f^*: H^*(X \bmod Y, \mathbf{Z}) \cong H^*(X' \bmod Y', \mathbf{Z})$  (cf. [11, 8.3.10]). By [11, 8.3.9 and 8.2.2] the diagram above is a diagram in the abelian category of mixed Hodge structures. The Mayer–Vietoris exact sequence now follows by a standard diagram chase. •

We then have, as well:

**Variante 3.2** (Mayer–Vietoris for Deligne cohomology) *Associated with any square (14) we have a long exact sequence in Deligne cohomology*

$$\cdots \rightarrow H_{\mathcal{D}}^n(X', \mathbf{Z}(r)) \oplus H_{\mathcal{D}}^n(Y, \mathbf{Z}(r)) \rightarrow H_{\mathcal{D}}^n(Y', \mathbf{Z}(r)) \rightarrow H_{\mathcal{D}}^{n+1}(X, \mathbf{Z}(r)) \rightarrow \cdots$$

**Proof** The proof of Proposition 3.1 goes through, once we know that Deligne cohomology satisfies excision. But since we have excision for the mixed Hodge structure on relative singular cohomology, one can see it holds for Deligne cohomology by arguing with the relative cohomology sequence (6).  $\bullet$

**Theorem 3.3** (Mayer–Vietoris for  $K$ -theory) *Let  $X$  be a reduced quasiprojective surface over a field with normalization  $\tilde{X}$ . Then there is a 1-dimensional subscheme  $Y$  with  $Y_{red} = \text{Sing } X$  such that the normalization square (cf. (14))*

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \tilde{X} \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & X \end{array}$$

induces exact sequences in  $K$ -theory:

$$\begin{aligned} K_1(\tilde{X}) \oplus K_1(Y) &\rightarrow K_1(\tilde{Y}) \xrightarrow{\partial} K_0(X) \rightarrow K_0(\tilde{X}) \oplus K_0(Y) \rightarrow K_0(\tilde{Y}) \\ SK_1(\tilde{X}) \oplus SK_1(Y) &\rightarrow SK_1(\tilde{Y}) \xrightarrow{\partial} SK_0(X) \rightarrow SK_0(\tilde{X}) \rightarrow 0 \end{aligned}$$

**Proof** Let  $K_*(X, \tilde{X})$  and  $K_*(Y, \tilde{Y})$  be the relative groups fitting into the long exact sequences in the commutative diagram

$$\begin{array}{ccccccccccc} K_1(X) & \rightarrow & K_1(\tilde{X}) & \rightarrow & K_0(X, \tilde{X}) & \rightarrow & K_0(X) & \rightarrow & K_0(\tilde{X}) & \rightarrow & K_{-1}(X, \tilde{X}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel \\ K_1(Y) & \rightarrow & K_1(\tilde{Y}) & \rightarrow & K_0(Y, \tilde{Y}) & \rightarrow & K_0(Y) & \rightarrow & K_0(\tilde{Y}) & \rightarrow & K_{-1}(Y, \tilde{Y}). \end{array}$$

(The far right terms are isomorphic by [33, A.6].) To establish the existence and exactness of the  $K_1 - K_0$  sequence we must show that “excision” holds for  $K_0$ , *i.e.*, that  $K_0(X, \tilde{X}) \cong K_0(Y, \tilde{Y})$  for some  $Y$  with  $Y_{red} = \text{Sing } X$  (see [16, 5.1]). If  $Y$  is a subscheme of  $X$  defined by an  $\mathcal{O}_{\tilde{X}}$ -ideal  $\mathcal{I} \subset \mathcal{O}_X$  then by [33, A.6] there is a natural exact sequence

$$H^1(Y, \mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\tilde{X}/X}) \xrightarrow{\eta(Y)} K_0(X, \tilde{X}) \rightarrow K_0(Y, \tilde{Y}) \rightarrow 0. \quad (15)$$

We define  $Y_1$  using the conductor ideal  $\mathcal{J}$ , and  $Y$  using the ideal  $\mathcal{I} = \mathcal{J}^2$ . Then  $Y_{red} = \text{Sing } X$ , and the map from  $\mathcal{I}/\mathcal{I}^2$  to  $\mathcal{J}/\mathcal{J}^2$  is zero. By naturality in  $Y \rightarrow Y_1$ , the map  $\eta(Y)$  in (15) is the composite map

$$H^1(Y, \mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\tilde{X}/X}) \xrightarrow{0} H^1(Y, \mathcal{J}/\mathcal{J}^2 \otimes \Omega_{\tilde{X}/X}) \xrightarrow{\eta(Y_1)} K_0(X, \tilde{X})$$

so  $\eta(Y) = 0$  in (15). Hence excision holds for  $Y$ , as claimed.

There is a natural map from the  $K_1 - K_0$  sequence onto the “Units–Pic” sequence, and the kernel is the  $SK_1 - SK_0$  sequence. A standard diagram chase, described in [32, 8.6], shows that the latter sequence is also exact.  $\bullet$

We remark that if  $Y$  is reduced, or 0-dimensional, or even affine, then the obstruction  $H^1(Y)$  in (15) automatically vanishes, and excision is immediate. Theorem 3.3 was proven in these special cases in [32, 7.5] and [33, A.3].

**Corollary 3.4** *With the notation of 3.3, the following diagram commutes.*

$$\begin{array}{ccccccc}
SK_1(\tilde{X}) \oplus SK_1(Y) & \rightarrow & SK_1(\tilde{Y}) & \rightarrow & SK_0(X) & \rightarrow & SK_0(\tilde{X}) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(\tilde{X}) \oplus K_1(Y) & \rightarrow & K_1(\tilde{Y}) & \rightarrow & K_0(X) & \rightarrow & K_0(\tilde{X}) \rightarrow 0 \\
c_2 \downarrow & & c_2 \downarrow & & c_2 \downarrow & & c_2 \downarrow \\
H_{\mathcal{D}}^3(\tilde{X}, \mathbf{Z}(2)) \oplus H_{\mathcal{D}}^3(Y, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^3(\tilde{Y}, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^4(X, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^4(\tilde{X}, \mathbf{Z}(2)) \rightarrow 0
\end{array}$$

**Proof** We use the notation of (2.4.1). For each open  $U$  in  $X$ , let  $F(U)$  denote the homotopy fiber of  $K(U \times_X Y) \times K(U \times_X \tilde{X}) \rightarrow K(U \times_X \tilde{Y})$ . By Proposition 3.2 the corresponding homotopy fiber for Deligne cohomology is  $\mathcal{D}(U)$ . In addition, there is a natural map from  $K(U)$  to  $F(U)$  which is an isomorphism on  $\pi_0$  by Theorem 3.3. Therefore the natural map  $C_2^{ss}$  of (2.4.1) induces a map  $F(U) \rightarrow \mathcal{D}(U)$  on homotopy fibers, making the diagram

$$\begin{array}{ccccc}
K_1(\tilde{Y}) & \xrightarrow{\partial} & \pi_0 F(X) & \xleftarrow{\cong} & K_0(X) \\
C_2^{ss} \downarrow & & \downarrow C_2^{ss} & & \downarrow C_2^{ss} \\
H_{\mathcal{D}}^3(\tilde{Y}, \mathbf{Z}(2)) & \xrightarrow{\partial} & \pi_0 \mathcal{D}(X) & \xleftarrow{\cong} & H_{\mathcal{D}}^4(X, \mathbf{Z}(2))
\end{array}$$

commute. But the top composite is the  $K$ -theory boundary map in Theorem 3.3. •

Using (2.4) and Lemma 2.5, we may refine Corollary 3.4 as follows.

**Variant 3.5** *With the notation of 3.3, the following diagram commutes.*

$$\begin{array}{ccccccc}
SK_1(\tilde{X}) \oplus SK_1(Y) & \rightarrow & SK_1(\tilde{Y}) & \rightarrow & SK_0(X) & \rightarrow & SK_0(\tilde{X}) \rightarrow 0 \\
\text{onto} \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
H^1(\tilde{X}, \mathcal{K}_2) \oplus H^1(Y, \mathcal{K}_2) & \rightarrow & H^1(\tilde{Y}, \mathcal{K}_2) & \rightarrow & H^2(X, \mathcal{K}_2) & \rightarrow & H^2(\tilde{X}, \mathcal{K}_2) \\
c_2 \downarrow & & c_2 \downarrow & & c_2 \downarrow & & c_2 \downarrow \\
H_{\mathcal{D}}^3(\tilde{X}, \mathbf{Z}(2)) \oplus H_{\mathcal{D}}^3(Y, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^3(\tilde{Y}, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^4(X, \mathbf{Z}(2)) & \rightarrow & H_{\mathcal{D}}^4(\tilde{X}, \mathbf{Z}(2)) \rightarrow 0
\end{array}$$

## 4 The Albanese 1-motive of a proper surface

In this section a *surface* will mean a proper reduced 2-dimensional scheme  $X$  of finite type over the complex numbers  $\mathbf{C}$ . We will consider the intermediate jacobian

$$J^2(X) \stackrel{\text{def}}{=} \frac{H^3(X, \mathbf{C})}{F^2 H^3 + H^3(X, \mathbf{Z}(2))}$$

This is the mixed Hodge theoretic generalization of the classical Albanese group variety of a smooth surface.

We begin with an elementary result (cf. [18, Remark 5.5]).

**Lemma 4.1** *Suppose that  $X$  is a proper surface. Then*

$$F^2H^i(X, \mathbf{C}) \cap H^i(X, \mathbf{R}) = 0 \quad \text{for } i = 2, 3.$$

Hence in sequence (2) we have

$$\begin{aligned} H^i(X, \mathbf{Z})_{tors} &= \text{kernel of } H^i(X, \mathbf{Z}) \rightarrow H^i(X, \mathbf{C})/F^2H^i \\ &= \text{image of } H_{\mathcal{D}}^i(X, \mathbf{Z}(2)) \xrightarrow{\varepsilon} H_{an}^i(X, \mathbf{Z}). \end{aligned}$$

**Proof** We will show that  $H^i(X, \mathbf{R})$  injects into  $H^i(X, \mathbf{C})/F^2$ . When  $X$  is smooth, then  $H^i(X)$  has pure weight  $i$ . In this case complex conjugation on  $H^i(X, \mathbf{C})$  fixes  $H^i(X, \mathbf{R})$  but the subspace  $F^2H^i(X, \mathbf{C})$  meets its conjugate in 0.

If  $X$  is a singular surface, choose a resolution of singularities  $X' \rightarrow X$ . If  $Y$  is a curve containing the singular locus of  $X$ , then we are in the situation of square (14). Since  $F^2H^1 = F^2H^2 = 0$  for the curves  $Y$  and  $Y'$ , the Mayer–Vietoris sequence in Proposition 3.1 yields  $F^2H^i(X, \mathbf{C}) = F^2H^i(X', \mathbf{C})$  for  $i = 2, 3$ . Comparing the  $\mathbf{R}$  and  $\mathbf{C}$  structures in the Mayer–Vietoris long exact sequence of Proposition 3.1 yields the following diagram has exact rows:

$$\begin{array}{ccccc} H^1(Y', \mathbf{R}) & \rightarrow & H^2(X, \mathbf{R}) & \rightarrow & H^2(X', \mathbf{R}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(Y', \mathbf{C}) & \rightarrow & \frac{H^2(X, \mathbf{C})}{F^2H^2} & \rightarrow & \frac{H^2(X', \mathbf{C})}{F^2H^2} \end{array}$$

The right-most vertical arrow in the diagram is injective because  $X'$  is smooth. A diagram chase shows that the middle vertical arrow is injective, whence the lemma. •

## Normal surfaces

Consider a surface  $X$  with normal singularities; its singular locus  $\Sigma$  is a finite set of closed points. Choose a desingularization  $f: X' \rightarrow X$  and consider the exceptional divisor  $E = f^{-1}(\Sigma)$ ;  $E$  is the finite disjoint union of the inverse images  $E_\sigma \stackrel{\text{def}}{=} f^{-1}(\sigma)$  of the  $\sigma \in \Sigma$ . Associated to  $f$  is the square (14), with  $Y = \Sigma$  and  $Y' = E$ . Because the fibers  $E_\sigma$  of  $f$  are connected (by Zariski’s Main Theorem), we have  $H^0(\Sigma, \mathbf{Z}) \cong H^0(E, \mathbf{Z})$ . From Proposition 3.1 we get a long exact sequence of mixed Hodge structures:

$$\begin{aligned} 0 \rightarrow H^1(X, \mathbf{Z}) \xrightarrow{f^*} H^1(X', \mathbf{Z}) \rightarrow H^1(E, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z}) \xrightarrow{f^*} \\ H^2(X', \mathbf{Z}) \rightarrow H^2(E, \mathbf{Z}) \rightarrow H^3(X, \mathbf{Z}) \xrightarrow{f^*} H^3(X', \mathbf{Z}) \rightarrow 0 \end{aligned} \quad (16)$$

Recall that if  $X$  is proper then each  $H^n(X')$  has pure weight  $n$ , and that  $W_{n-1}H^n(X, \mathbf{Q})$  is the kernel of  $f^*: H^n(X, \mathbf{Q}) \rightarrow H^n(X', \mathbf{Q})$  by [11, 8.2.5]. That is,  $H^n(X)$  has pure weight  $n$  if and only if  $H^n(X, \mathbf{Q})$  injects into  $H^n(X', \mathbf{Q})$ . There are examples of normal surfaces for which  $H^2(X)$  does not have pure weight 2, i.e., with  $W_1H^2 \neq 0$  (cf. [3],[4]). The following result quantifies this impurity.

**Proposition 4.2** *Let  $X$  be a proper normal surface. If  $n \neq 2$  then  $H^n(X)$  has pure weight  $n$ . If  $n = 2$  and  $E$  is the exceptional divisor in a desingularization  $X'$ , then*

$$W_1 H^2(X, \mathbf{Q}) = \text{coker } H^1(X', \mathbf{Q}) \rightarrow H^1(E, \mathbf{Q}).$$

**Proof** If  $n \neq 3$  this follows from the sequence of mixed Hodge structures (16). For  $n = 3$  we must show that  $H^3(X, \mathbf{Q})$  embeds in  $H^3(X', \mathbf{Q})$ . Nothing is lost if we replace  $X'$  by a quadratic transformation, so we may assume that all the irreducible components  $E_1, \dots, E_n$  of  $E$  are non-singular and that if  $i \neq j$  and  $E_i \cap E_j \neq \emptyset$  then  $E_i$  and  $E_j$  intersect transversally in exactly one point not belonging to any other  $E_k$ . From (16) and the commutative square

$$\begin{array}{ccc} \text{Pic}(X') & \xrightarrow{\cap^E} & \text{Pic}(E) \\ c_1 \downarrow & & \downarrow c_1 \\ H^2(X', \mathbf{Z}) & \rightarrow & H^2(E, \mathbf{Z}) \end{array}$$

we see that it suffices to prove that  $\text{Pic}(X') \otimes \mathbf{Q} \rightarrow H^2(E, \mathbf{Q})$  is a surjection. Now  $H^2(E, \mathbf{Q}) = \oplus H^2(E_i, \mathbf{Q}) \cong \mathbf{Q}^n$ , and  $\text{Pic}(E) \otimes \mathbf{Q} \xrightarrow{c_1} H^2(E, \mathbf{Q})$  is just the degree map. Moreover, the intersection pairing on the divisors on  $X'$  satisfies  $(D \cdot E_i) = \deg(D \cap E_i)$ . Thus if we represent an element of  $\text{Pic}(X')$  by a divisor  $D$ , its image in  $H^2(E, \mathbf{Q}) \cong \mathbf{Q}^n$  is given by the intersection vector  $(D \cdot E_1, \dots, D \cdot E_n)$ . Now each  $E_i$  represents an element of  $\text{Pic}(X')$ , and their intersection vectors form a basis of  $H^2(E, \mathbf{Q})$  because the intersection matrix  $(E_i \cdot E_j)$  is negative definite (see [31, §1] or [28, 14.1]). •

**Corollary 4.3** *Let  $f: X' \rightarrow X$  be a resolution of singularities of a proper normal surface. Then  $J^2(X)$  is an abelian variety, because there is an isomorphism*

$$f^*: J^2(X) \xrightarrow{\cong} J^2(X')$$

**Proof** By (16) and 4.2,  $f^*: H^3(X, \mathbf{Z}) \rightarrow H^3(X', \mathbf{Z})$  is onto with torsion kernel. •

## Normalization

Now let  $X$  be a non-normal surface. The singular locus  $\Sigma$  of  $X$  is 1-dimensional. Letting  $\pi: \tilde{X} \rightarrow X$  denote its normalization, we have  $\pi: \tilde{X} - \tilde{\Sigma} \xrightarrow{\cong} X - \Sigma$ , where  $\tilde{\Sigma} = \pi^{-1}\Sigma$ . By Proposition 3.1,  $\pi$  induces a long exact sequence of mixed Hodge structures.

$$\begin{array}{ccccccc} H^1(X, \mathbf{Z}) & \rightarrow & H^1(\tilde{X}, \mathbf{Z}) \oplus H^1(\Sigma, \mathbf{Z}) & \rightarrow & H^1(\tilde{\Sigma}, \mathbf{Z}) & \rightarrow & H^2(X, \mathbf{Z}) \rightarrow \\ H^2(\tilde{X}, \mathbf{Z}) \oplus H^2(\Sigma, \mathbf{Z}) & \rightarrow & H^2(\tilde{\Sigma}, \mathbf{Z}) & \rightarrow & H^3(X, \mathbf{Z}) & \rightarrow & H^3(\tilde{X}, \mathbf{Z}) \rightarrow 0 \end{array} \quad (17)$$

Since the Hodge structure on  $H^2$  of a curve is pure of type  $(1, 1)$ , the abelian group

$$M = \frac{\text{coker } H^2(\Sigma, \mathbf{Z}) \xrightarrow{\pi^*} H^2(\tilde{\Sigma}, \mathbf{Z})}{\text{coker } H^2(X, \mathbf{Z}) \xrightarrow{\pi^*} H^2(\tilde{X}, \mathbf{Z})}$$

has a mixed Hodge structure which is pure of type  $(1, 1)$ , and there is an extension of mixed Hodge structures

$$0 \rightarrow M \rightarrow H^3(X, \mathbf{Z}) \rightarrow H^3(\tilde{X}, \mathbf{Z}) \rightarrow 0. \quad (18)$$

**Proposition 4.4** *Let  $X$  be a proper surface, with normalization  $\pi: \tilde{X} \rightarrow X$ . Then we have an extension*

$$0 \rightarrow (\mathbf{C}/\mathbf{Z})^s \rightarrow J^2(X) \xrightarrow{\pi^*} J^2(\tilde{X}) \rightarrow 0$$

where  $s$  is the rank of the abelian group  $M$ .

**Proof**  $J^2(X)$  is the cokernel of the natural map  $H^3(X, \mathbf{Z}(2)) \rightarrow H^3(X, \mathbf{C})/F^2H^3$ . Given this, the result is a formal consequence of (18) and the fact that  $F^2M = 0$ , which implies that  $F^2H^3(X, \mathbf{C}) \cong F^2H^3(\tilde{X}, \mathbf{C})$ . We remark that the complex torus  $(\mathbf{C}/\mathbf{Z})^s$  that arises in this extension is a quotient of the complex torus  $(\mathbf{C}/\mathbf{Z}(2))^s = M \otimes (\mathbf{C}/\mathbf{Z}(2))$  by a finite group. •

**Corollary 4.5** *Let  $f: X' \rightarrow X$  be a desingularization of a proper surface  $X$ , obtained by resolving the singularities of its normalization  $\tilde{X}$ . Then there is an exact sequence*

$$0 \rightarrow (\mathbf{C}/\mathbf{Z})^s \rightarrow J^2(X) \xrightarrow{f^*} J^2(X') \rightarrow 0$$

where  $s$  is the rank of  $M$ , as in Proposition 4.4. In particular, if  $X'$  has irregularity  $q$  then the torsion subgroup of  $J^2(X)$  is isomorphic to  $(\mathbf{Q}/\mathbf{Z})^{2q+s}$ .

Recall from [11, 10.1.2] that a “1-motive”  $M = (L, A, T, J, u)$  is defined to be an extension  $J$  of an abelian variety  $A$  by a complex torus  $T$ , a lattice  $L$  and a homomorphism  $L \xrightarrow{u} J$ . Since we may canonically identify the group of  $\mathbf{C}$ -points of the abelian variety  $\text{Alb}(X')$  with  $J^2(X')$ , the conclusion of Corollary 4.5 is just that  $J^2(X)$  is part of a 1-motive  $\text{Alb}(X)$  in which the lattice  $L$  is zero.

**Definition 4.6:** Let  $X$  be a proper surface over  $\mathbf{C}$ . The *Albanese 1-motive* of  $X$  is the 1-motive  $\text{Alb}(X)$  given by

$$(0, \text{Alb}(X'), (\mathbf{C}/\mathbf{Z}(2))^s, J^2(X), \text{zero}).$$

As the construction in 4.4 shows,  $\text{Alb}$  is a functor from proper surfaces to 1-motives.

## Torsion in $J^2(X)$

For simplicity, let us write  $\mathbf{Q}/\mathbf{Z}$  for the torsion subgroup  $\mathbf{Q}(2)/\mathbf{Z}(2)$  of  $\mathbf{C}/\mathbf{Z}(2)$ , so that  $H^i(-, \mathbf{Q}/\mathbf{Z}) \cong H^i(-, \mathbf{C}/\mathbf{Z}(2))_{\text{tors}}$ . The maps  $H^i(-, \mathbf{C}/\mathbf{Z}(2)) \rightarrow H_{\mathcal{D}}^{i+1}(-, \mathbf{Z}(2))$  of (3) induce canonical maps

$$H^i(-, \mathbf{Q}/\mathbf{Z}) \cong H^i(-, \mathbf{C}/\mathbf{Z}(2))_{\text{tors}} \rightarrow H_{\mathcal{D}}^{i+1}(-, \mathbf{Z}(2))_{\text{tors}}.$$

These are the maps in the following Proposition:

**Proposition 4.7** *Let  $Z$  be a proper scheme over  $\mathbf{C}$ . Then*

i)  $H^1(Z, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\cong} H_{\mathcal{D}}^2(Z, \mathbf{Z}(2))_{tors}$

ii) *If  $Z$  is either a curve or a surface then*

$$H^2(Z, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\cong} H_{\mathcal{D}}^3(Z, \mathbf{Z}(2))_{tors}$$

iii) *If  $Z$  is a surface then*

$$H^3(Z, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\cong} H_{\mathcal{D}}^4(Z, \mathbf{Z}(2))_{tors} \cong J^2(Z)_{tors}$$

**Proof** The first assertion was proven in 1.1. If  $Z$  is a curve then  $F^2H^2(Z, \mathbf{C}) = 0$ , so by (3) we have  $H_{\mathcal{D}}^3(Z, \mathbf{Z}(2)) \cong H^2(Z, \mathbf{C}/\mathbf{Z}(2))$ , and the result is immediate.

We may therefore suppose that  $Z$  is a surface, say with  $c$  irreducible components, so that  $H^4(Z, \mathbf{Z}) = \mathbf{Z}^c$ . We deduce from (4) that  $J^2(Z)_{tors} \cong H_{\mathcal{D}}^4(Z, \mathbf{Z}(2))_{tors}$ . Moreover, since  $F^2H^4(Z, \mathbf{C}) = H^4(Z, \mathbf{C}) = \mathbf{C}^c$  the sequence (3) ends in

$$H_{\mathcal{D}}^3(Z, \mathbf{Z}(2)) \rightarrow F^2H^3(Z, \mathbf{C}) \rightarrow H^3(Z, \mathbf{C}/\mathbf{Z}(2)) \rightarrow H_{\mathcal{D}}^4(Z, \mathbf{Z}(2)) \xrightarrow{\varepsilon} \mathbf{Z}^c \rightarrow 0. \quad (19)$$

Lemma 4.1 states that for  $i = 2, 3$  the image of  $\varepsilon$  in the exact sequence

$$H^{i-1}(Z, \mathbf{C})/F^2H^{i-1} \rightarrow H_{\mathcal{D}}^i(Z, \mathbf{Z}(2)) \xrightarrow{\varepsilon} H^i(Z, \mathbf{Z}) \rightarrow H^i(Z, \mathbf{C})/F^2H^i$$

of (2) is the torsion subgroup  $H^i(Z, \mathbf{Z})_{tors}$ . Combining this with the universal coefficient theorem, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & H^2(Z, \mathbf{C})/F^2H^2 \oplus H^2(Z, \mathbf{Z}) & \rightarrow & H_{\mathcal{D}}^3(Z, \mathbf{Z}(2)) & \xrightarrow{\varepsilon} & H^3(Z, \mathbf{Z})_{tors} & \rightarrow 0 \\ & \uparrow & & \uparrow & & \parallel & \\ 0 \rightarrow & H^2(Z, \mathbf{Z}) \otimes \mathbf{C}/\mathbf{Z}(2) & \rightarrow & H^2(Z, \mathbf{C}/\mathbf{Z}(2)) & \rightarrow & H^3(Z, \mathbf{Z})_{tors} & \rightarrow 0. \end{array}$$

By the five-lemma, sequence (3) and (19) we get the extensions

$$0 \rightarrow F^2H^2(Z, \mathbf{C}) \rightarrow H^2(Z, \mathbf{C}/\mathbf{Z}(2)) \rightarrow H_{\mathcal{D}}^3(Z, \mathbf{Z}(2)) \rightarrow 0, \quad (20)$$

$$0 \rightarrow F^2H^3(Z, \mathbf{C}) \rightarrow H^3(Z, \mathbf{C}/\mathbf{Z}(2)) \rightarrow J^2(Z) \rightarrow 0. \quad (21)$$

Since  $F^2H^i(Z)$  is uniquely divisible, we may pass to torsion subgroups. This proves the remainder of the proposition. •

## 5 Curves

The singular locus of a reduced surface is usually an (unreduced) curve. For this reason, we need information about  $K_1$  and  $K_2$  of curves in order to study surfaces. This information is given by theorems 5.1 and 5.3 below. Part *i*) of Theorem 5.1 is of course well-known and almost classical; a reference is [34, 1.1]. Since these results are of independent interest, we have expanded our exposition to include the case of characteristic  $p$ .

By a ‘curve’ over a field  $k$  we mean a 1-dimensional quasiprojective scheme  $Y$  over  $k$ ; a curve is not necessarily reduced. There is a natural map from  $K_1(Y)$  to the group  $H^0(Y, \mathcal{O}_Y)$  of global units of  $Y$ ; the kernel of this map is usually written as  $SK_1(Y)$ . When  $Y$  is a curve there is a natural isomorphism  $SK_1(Y) \cong H^1(Y, \mathcal{K}_2)$ , as well as a natural short exact sequence

$$0 \rightarrow H^1(Y, \mathcal{K}_3) \rightarrow K_2(Y) \rightarrow H^0(Y, \mathcal{K}_2) \rightarrow 0$$

given by the Brown-Gersten spectral sequence [38].

**Theorem 5.1** *Let  $Y$  be a smooth curve over an algebraically closed field  $k$ . Let  $r \geq 0$  denote the number of irreducible components of  $Y$  which are proper. Then*

- i)  $SK_1(Y) \cong (k^*)^r \oplus V_1$  where  $V_1$  is a uniquely divisible group;*
- ii)  $K_2(Y)$  and  $H^0(Y, \mathcal{K}_2)$  are both divisible abelian groups.*

**Proof** If  $Y$  is a smooth connected curve over an algebraically closed field  $k$  then the localization sequence is

$$\coprod_{y \in Y(k)} K_2(k) \rightarrow K_2(Y) \rightarrow K_2(k(Y)) \xrightarrow{\text{tame}} \coprod_{y \in Y(k)} k^* \rightarrow SK_1(Y) \rightarrow 0 \quad (22)$$

and the image of  $K_2(Y)$  in  $K_2(k(Y))$  is  $H^0(Y, \mathcal{K}_2)$ . Since  $\coprod K_2(k)$  is divisible [5, 1.3], the divisibility of  $K_2(Y)$  is equivalent to the divisibility of  $H^0(Y, \mathcal{K}_2)$ . If  $\text{char}(k) = 0$ , the result now follows from Suslin’s exact sequence [37, 4.4] for  $n$  invertible in  $k$ :

$$0 \rightarrow H^0(Y, \mathcal{K}_2)/n \rightarrow H_{\acute{e}t}^2(Y, \mu_n^{\otimes 2}) \rightarrow {}_n SK_1(Y) \rightarrow 0$$

Indeed, if  $Y$  is affine then  $H_{\acute{e}t}^2(Y) = 0$ , and if  $Y$  is projective then the composite  $\mu_n \cong H_{\acute{e}t}^2(Y, \mu_n^{\otimes 2}) \rightarrow SK_1(Y) \rightarrow k^*$  is the standard inclusion.

If  $\text{char}(k) = p > 0$ , we need only a slight additional argument. Because  $k(Y)$  is the function field of a curve, we know from [5, p.391] that  $K_2(k(Y))$  is  $p$ -divisible, and from [37, 1.10] (which is implicit in [5, p.397]) that it has no  $p$ -torsion. Hence both  $K_2(k(Y))$  and  $\coprod k^*$  are uniquely  $p$ -divisible groups, *i.e.*,  $\mathbf{Z}[\frac{1}{p}]$ -modules. It follows that both the kernel  $H^0(Y, \mathcal{K}_2)$  and cokernel  $SK_1(Y)$  of the ‘tame symbol’ map in (22) must be uniquely  $p$ -divisible. This proves Theorem 5.1 in characteristic  $p$ . •

**Lemma 5.2** *Let  $Y$  be a smooth connected projective curve over  $\mathbf{C}$ . Then*

$$c_2: SK_1(Y) \rightarrow H_{\mathcal{D}}^3(Y, \mathbf{Z}(2)) \cong \mathbf{C}^*$$

*is a split surjection. In particular, it is an isomorphism on torsion subgroups.*

**Proof** This is implicit in p.219 of Gillet's paper [18]. The isomorphism  $H_{\mathcal{D}}^3(Y, \mathbf{Z}(2)) \cong \mathbf{C}/\mathbf{Z}(2) \cong \mathbf{C}^*$  follows from (3) or 4.7. If  $y \in Y(\mathbf{C})$  is considered as an element of  $Pic(Y)$  and  $z \in \mathbf{C}^*$  then we can form  $\{y, z\} \in SK_1(Y)$  and the product formula yields  $c_2(\{y, z\}) = -c_1(y) \cup c_1(z) = z^{-1} \in \mathbf{C}^*$ . •

**Theorem 5.3** *Let  $Y$  be any curve over an algebraically closed field  $k$ , and let  $r \geq 0$  denote the number of irreducible components of  $Y$  which are proper. Then*

*i) If  $char(k) = 0$ , or if  $Y$  is reduced, then*

$$SK_1(Y) \cong (k^*)^r \oplus V_1,$$

*where  $V_1$  is a uniquely divisible abelian group;*

*ii) If  $char(k) = p > 0$  then*

$$SK_1(Y) \cong (k^*)^r \oplus V_1 \oplus P,$$

*where  $V_1$  is uniquely divisible and  $P$  is a  $p$ -group of bounded exponent.*

*iii) If  $k = \mathbf{C}$  then the Chern class*

$$c_2: SK_1(Y) \rightarrow H_{\mathcal{D}}^3(Y, \mathbf{Z}(2)) \cong (\mathbf{C}^*)^r$$

*is a split surjection. In particular, it is an isomorphism on torsion subgroups.*

**Proof** We proceed in three steps.

*Step 1.* Suppose that  $Y$  is any reduced curve over  $k$ . If we pick  $r$  smooth points  $y_i$  on  $Y$ , one on each proper component of  $Y$ , then  $Y_0 = Y - \{y_1, \dots, y_r\}$  is affine. The localization sequence for  $Y_0 \subset Y$  is

$$\prod_{i=1}^r K_2(k) \rightarrow K_2(Y) \rightarrow K_2(Y_0) \rightarrow \prod_{i=1}^r k^* \rightarrow SK_1(Y) \rightarrow SK_1(Y_0) \rightarrow 0 \quad (23)$$

If  $\tilde{Y}$  is the normalization of  $Y$ , then we may indentify the  $y_i$  with points on the smooth curve  $\tilde{Y}$ . By the smooth case 5.1, the composition of

$$\prod k^* \rightarrow SK_1(Y) \rightarrow SK_1(\tilde{Y})$$

is an injection. Hence  $SK_1(Y)$  is the direct sum of  $\prod k^*$  and  $SK_1(Y_0)$  while  $K_2(Y)$  is the direct sum of the image of the divisible group  $\prod K_2(k)$  and the group  $K_2(Y_0)$ .

Part (iii) now follows from the above Lemma. This argument also shows that we may replace  $Y$  by  $Y_0$  in proving parts (i) and (ii) of Theorem 5.3 for reduced curves.

*Step 2.* Now suppose that  $Y = \text{Spec}(A)$  is any reduced affine curve over  $k$ . Let  $B$  be the normalization of  $A$ , and  $I$  the conductor ideal from  $B$  to  $A$ . By [15, 3.1 and 4.2], excision holds for  $K_1$  and there is an exact sequence

$$K_2(B) \oplus K_2(A/I) \rightarrow K_2(B/I) \rightarrow SK_1(A) \rightarrow SK_1(B) \rightarrow 0$$

Since  $B$  is a finite product of Dedekind domains,  $B/I$  is a finite principal ideal ring. By Corollary 5.5 below,  $K_2(B/I)$  is uniquely divisible. By Theorem 5.1,  $SK_1(B)$  is uniquely divisible and  $K_2(B)$  is divisible. Finally, since  $A/I$  is finite dimensional, we know from Corollary 5.5 that  $K_2(A/I)$  is uniquely divisible (modulo bounded  $p$ -torsion if  $\text{char}(k) = p \neq 0$ ). A diagram chase shows that  $SK_1(A)$  is uniquely divisible (modulo bounded  $p$ -torsion if  $p \neq 0$ ). This proves theorem 5.3 for reduced curves.

**Lemma 5.4** *Let  $I$  be a nilpotent ideal in an algebra  $A$  over a field  $k$ .*

- (a) *If  $\text{char}(k) = 0$ ,  $K_n(A, I)$  is a uniquely divisible group for every  $n$ .*
- (b) *If  $\text{char}(k) = p > 0$ ,  $K_2(A, I)$  is a  $p$ -group of bounded exponent.*

**Proof** Part (a) is proven in [39, 1.4]. If  $\text{char}(k) = p$ , chose  $m$  such that  $I^{p^m} = 0$ ; we will show that  $p^m K_2(A, I)$ . Indeed,  $K_2(A, I)$  is generated by Steinberg symbols  $\{a, 1+x\}$  with  $a \in A$  and  $x \in I$ , and  $p^m \{a, 1+x\}$  is  $\{a, 1+x^{p^m}\} = \{a, 1\} = 0$ . •

**Corollary 5.5** *Let  $A$  be a finite algebra over an algebraically closed field  $k$ .*

- (a) *If  $\text{char}(k) = 0$  or if  $A$  is a principal ideal ring, the group  $K_2(A)$  is uniquely divisible.*
- (b) *If  $\text{char}(k) = p$ ,  $K_2(A)$  is the sum of the uniquely divisible group  $K_2(A_{red})$  and a  $p$ -group of bounded exponent.*

**Proof** Let  $I$  be the nilradical of  $A$ , so that  $A_{red} = A/I$  is semisimple and hence  $A \rightarrow A_{red}$  splits. Then  $K_2(A) \cong K_2(A_{red}) \oplus K_2(A, I)$ , and  $K_2(A_{red})$  is uniquely divisible by [5, 1.3]. Finally, if  $A$  is a principal ideal ring then  $A$  is a product of truncated polynomial rings  $k[s]/(s^n)$  and a direct calculation ([20, p.485]) shows that  $K_2(k[s]/(s^n)) \cong K_2(k)$ . •

*Step 3.* Finally, suppose that  $Y$  is a curve which is not reduced. Let  $\mathcal{I}$  denote the nilradical ideal sheaf of  $\mathcal{O}_Y$ , and write  $\mathcal{K}_2\mathcal{I}$  for the sheafification of the presheaf  $U \mapsto K_2(\mathcal{O}_Y(U), \mathcal{I}(U))$ . If  $\mathcal{K}_{2,red}$  denotes the sheafification of  $U \mapsto K_2(U_{red})$ , there is an exact sequence of sheaves on  $Y_{Zar}$

$$\mathcal{K}_2\mathcal{I} \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_{2,red} \rightarrow 0 \tag{24}$$

Let  $U$  denote the smooth locus of  $Y_{red}$ . Since  $U_{red}$  is smooth, the ring map  $\mathcal{O}_U \rightarrow \mathcal{O}_{U_{red}}$  splits. Therefore  $\mathcal{K}_2\mathcal{I} \mid_U$  injects into  $\mathcal{K}_2 \mid_U$ , *i.e.*, the kernel of  $\mathcal{K}_2\mathcal{I} \rightarrow \mathcal{K}_2$  is a skyscraper sheaf supported on  $Y - U$ . It follows that we have an exact sequence

$$H^0(Y, \mathcal{K}_{2,red}) \rightarrow H^1(Y, \mathcal{K}_2\mathcal{I}) \rightarrow H^1(Y, \mathcal{K}_2) \rightarrow H^1(Y_{red}, \mathcal{K}_2) \rightarrow 0$$

which we may rewrite as follows

$$K_2(Y_{red}) \rightarrow H^1(Y, \mathcal{K}_2\mathcal{I}) \rightarrow SK_1(Y) \rightarrow SK_1(Y_{red}) \rightarrow 0 \quad (25)$$

By Step 2,  $SK_1(Y_{red})$  is uniquely divisible. If  $\text{char}(k) = p$ , we know by Lemma 5.4(b) that  $H^1(Y, \mathcal{K}_2\mathcal{I})$  is a  $p$ -group of bounded exponent, and this part (ii) of Theorem 5.3 because a uniquely divisible group has no nontrivial extensions by a  $p$ -group. Finally, suppose that  $\text{char}(k) = 0$ . By Lemma 5.4(a),  $H^1(Y, \mathcal{K}_2\mathcal{I})$  is uniquely divisible. By Proposition 5.6 below,  $K_2(Y_{red})$  is divisible. In this case part (i) of Theorem 5.3 follows from Step 2 and the exact sequence (25).  $\bullet$

**Proposition 5.6** *If  $\frac{1}{n} \in k$  and  $Y$  is a curve then  $K_2(Y)$  is  $n$ -divisible.*

**Proof** We consider the  $K$ -theory of  $Y$  with coefficients  $\mathbf{Z}/n$ , which is related to the usual Quillen  $K$ -theory of  $Y$  by exact sequences such as

$$0 \rightarrow K_2(Y) \otimes \mathbf{Z}/n \rightarrow K_2(Y; \mathbf{Z}/n) \rightarrow K_1(Y)_{n-tors} \rightarrow 0$$

We know by [39, 1.4] that  $K_2(Y; \mathbf{Z}/n) \cong K_2(Y_{red}; \mathbf{Z}/n)$ , and hence that  $K_2(Y) \otimes \mathbf{Z}/n$  is a subgroup of  $K_2(Y_{red}) \otimes \mathbf{Z}/n$ . Thus we may assume that  $Y$  is reduced. Let  $\tilde{Y}$  be the normalization of  $Y$ . The conductor ideal defines a zero-dimensional subscheme  $\text{Spec}(C)$  of  $Y$ , and also its preimage  $\text{Spec}(D)$  in  $\tilde{Y}$ . Because excision holds (see [39, 1.2]) we have an exact sequence

$$K_3(D; \mathbf{Z}/n) \rightarrow K_2(Y; \mathbf{Z}/n) \rightarrow K_2(\tilde{Y}; \mathbf{Z}/n) \oplus K_2(C; \mathbf{Z}/n)$$

Now  $K_3(D; \mathbf{Z}/n) \cong K_3(D_{red}; \mathbf{Z}/n)$ , again by [39, 1.4]. Since  $D_{red}$  is a finite product of copies of  $k$ , and  $K_3(k; \mathbf{Z}/n) = 0$  by Suslin [36], we have  $K_3(D; \mathbf{Z}/n) = 0$ . Hence  $K_2(Y; \mathbf{Z}/n)$  injects into  $K_2(\tilde{Y}; \mathbf{Z}/n) \oplus K_2(C; \mathbf{Z}/n)$ . By naturality, the subgroup  $K_2(Y) \otimes \mathbf{Z}/n$  of  $K_2(Y; \mathbf{Z}/n)$  injects into the corresponding subgroup  $K_2(\tilde{Y}) \otimes \mathbf{Z}/n \oplus K_2(C) \otimes \mathbf{Z}/n$  of  $K_2(\tilde{Y}; \mathbf{Z}/n) \oplus K_2(C; \mathbf{Z}/n)$ , but:  $K_2(\tilde{Y})$  is divisible by Theorem 5.1 and  $K_2(C)$  is divisible by Corollary 5.5, so this latter subgroup is zero, hence  $K_2(Y) \otimes \mathbf{Z}/n = 0$  as claimed.  $\bullet$

## 6 $K$ -theory results

In this section we collect some results on the relation between the Zariski sheaves  $\mathcal{K}_2$  and  $\mathcal{H}^q(\mu_n^{\otimes 2})$ , namely 6.1, 6.4 and 6.5, which will be used in the proof of the main theorem. In this section, our field  $k$  will always contain  $\frac{1}{n}$ .

The first result, which we cite without proof, concerns the sheafification of the étale Chern class  $c_2^{et}: K_2(X)/n \rightarrow H_{\acute{e}t}^2(X, \mu_n^{\otimes 2})$ . It is an extension by Hoobler of a well known result for smooth schemes.

**Proposition 6.1** (Hoobler [21]; cf. [33, Thm. 0.2]) *Let  $X$  be a scheme of finite type over a field containing  $\frac{1}{n}$ . Then the étale Chern class  $c_2^{et}$  induces an isomorphism of Zariski sheaves on  $X$ :  $\mathcal{K}_2/n \xrightarrow{\cong} \mathcal{H}^2(\mu_n^{\otimes 2})$ .*

Our other results concern the  $n$ -torsion subsheaf  ${}_n\mathcal{K}_2$  of  $\mathcal{K}_2$ . We begin with the local version.

**Lemma 6.2** *Let  $A$  be a semilocal ring essentially of finite type over a field  $k$ . Assume  $k$  contains a primitive  $n^{\text{th}}$  root of unity  $\zeta$ . Define a map*

$$\varphi: H_{\acute{e}t}^1(A, \mu_n^{\otimes 2}) \cong A^*/A^{*n} \rightarrow {}_nK_2(A)$$

by  $\varphi(a) = \{a, \zeta\}$ ,  $a \in A^*$ . Then  $\varphi$  is surjective. If  $A$  is regular and  $k$  contains an algebraically closed field then  $\varphi$  is an isomorphism.

**Proof**  $\varphi$  is well defined because  $\{a^n, \zeta\} = \{a, 1\} = 0$ . Suppose first that  $A$  is regular. Then  $\varphi$  is onto by the Merkurjev-Suslin Theorem. If in addition its field of fractions  $F$  contains an algebraically closed field, then  $H_{\acute{e}t}^1(F, \mu_n^{\otimes 2}) \cong {}_nK_2(F)$  by [37, 3.7]. Comparing the Bloch-Ogus resolution of  $H_{\acute{e}t}^1(A, \mu_n^{\otimes 2})$  to the Gersten-Quillen resolution of  ${}_nK_2(A)$ , one gets that  $H_{\acute{e}t}^1(A, \mu_n^{\otimes 2}) \cong {}_nK_2(A)$ .

The promotion to any semilocal ring  $A$  follows from the same arguments used by R.Hoobler in [21]. Since  $A$  is a localization of a finitely generated  $k$ -algebra, there exists localization  $B$  of a polynomial ring over  $k$  and an ideal  $I$  in  $B$  such that  $A = B/I$ . Let  $(B^h, I^h)$  be the henselization of the pair  $(B, I)$ . As  $B^h$  is a direct limit of semilocal regular rings finite over  $B$ , the map  $H_{\acute{e}t}^1(B^h, \mu_n^{\otimes 2}) \xrightarrow{\varphi} {}_nK_2(B^h)$  is an isomorphism. By a result of O. Gabber [14, Th.1] we have  $K_3(B^h; \mathbf{Z}/n) \cong K_3(B^h/I^h; \mathbf{Z}/n) \cong K_3(A; \mathbf{Z}/n)$ . By proper base change

$$H_{\acute{e}t}^1(B^h, \mu_n^{\otimes 2}) \cong H_{\acute{e}t}^1(B^h/I^h, \mu_n^{\otimes 2}) \cong H_{\acute{e}t}^1(A, \mu_n^{\otimes 2}).$$

The universal exact sequence for  $K$ -theory with coefficients yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_3(B^h) \otimes \mathbf{Z}/n & \rightarrow & K_3(B^h; \mathbf{Z}/n) & \rightarrow & {}_nK_2(B^h) \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \rightarrow & K_3(A) \otimes \mathbf{Z}/n & \rightarrow & K_3(A; \mathbf{Z}/n) & \rightarrow & {}_nK_2(A) \rightarrow 0 \end{array}$$

Thus the right-most vertical arrow is surjective. We then conclude from commutativity of the diagram:

$$\begin{array}{ccc} H_{\acute{e}t}^1(B^h, \mu_n^{\otimes 2}) & \xrightarrow{\cong} & {}_nK_2(B^h) \\ \parallel & & \downarrow \text{onto} \\ H_{\acute{e}t}^1(A, \mu_n^{\otimes 2}) & \xrightarrow{\varphi} & {}_nK_2(A) \end{array} \bullet$$

**Variation 6.3** Let  $A$  and  $k$  be as in Lemma 6.2. If  $n$  is even, assume that  $k$  contains a square root of  $-1$ . If  $\beta$  is a Bott element in  $K_2(k; \mathbf{Z}/n)$  mapping to  $\zeta \in K_1(k) = k^*$ , then multiplication by  $\beta$  lifts the map  $\varphi$  to a map

$$\bar{\varphi}: H_{\text{ét}}^1(A, \mu_n^{\otimes 2}) \rightarrow K_3(A; \mathbf{Z}/n).$$

This map is a split injection, because the étale Chern class satisfies  $c_2^{\text{ét}} \bar{\varphi} = -1$ .

**Proof** The assumption on  $k$  implies that  $\beta$  exists and has order  $n$ , so  $\bar{\varphi}(a) = \{a, \beta\}$  is well-defined and lifts  $\varphi$ . The product formula (see [40, Theorem 3.2(ii)]) states that  $c_2(\{a, \beta\}) = -[a] \otimes \zeta$  in  $A^*/A^{*n} \otimes \mu_n(k) \cong H_{\text{ét}}^1(\text{Spec}(A), \mu_n^{\otimes 2})$  for every  $a \in A^*$ . Thus up to sign  $c_2$  is a left inverse of  $\bar{\varphi}$ .  $\bullet$

**Theorem 6.4** Let  $X$  be a scheme of finite type over  $\mathbf{C}$ . Then the étale Chern class defines an isomorphism of Zariski sheaves:

$$c_2^{\text{ét}}: {}_n\mathcal{K}_2 \xrightarrow{\cong} \mathcal{H}^1(\mu_n^{\otimes 2}).$$

**Proof** We saw in (2.4.2) that  $c_2^{\text{ét}}: \mathcal{K}_3(\mathbf{Z}/n) \rightarrow \mathcal{H}^1(\mu_n^{\otimes 2})$  vanishes on  $\mathcal{K}_3$ . Hence  $c_2^{\text{ét}}$  is well-defined on  ${}_n\mathcal{K}_2$ . To verify that it is an isomorphism, we check the stalks at a point  $x \in X$ . If  $A = \mathcal{O}_{X,x}$  we see from 6.3 that the surjection  $\varphi: H_{\text{ét}}^1(A, \mu_n^{\otimes 2}) \rightarrow {}_nK_2(A)$  of Lemma 6.2 satisfies  $c_2^{\text{ét}} \varphi = -1$ . Elementary algebra now implies that  $c_2^{\text{ét}}$  is an isomorphism on  ${}_nK_2(A)$  and hence on  ${}_n\mathcal{K}_2$ .  $\bullet$

**Corollary 6.5** By (13), the following diagram commutes:

$$\begin{array}{ccc} {}_n\mathcal{K}_2 & \xrightarrow{\tau} & \mathcal{K}_2 \\ c_2^{\text{ét}} \downarrow \cong & & \downarrow c_2 \\ \mathcal{H}^1(\mu_n^{\otimes 2}) & \xrightarrow{\delta} & \mathcal{H}_{\mathcal{D}}^2(2) \end{array}$$

where  $\tau$  is the obvious inclusion and  $\delta$  is defined in (8) and (10).

**Remark 6.6:** This gives the following explicit formula for  $\delta$ . Given a unit  $a \in A^*$ , where  $U = \text{Spec}(A)$ , write  $[a]$  for the class of  $c_1(a)$  in  $H_{an}^1(U, \mathbf{Z}(1)_{\mathcal{D}}) \cong H_{an}^0(U, \mathcal{O}_U^*)$ . Then the product formula for  $c_2$  shows that  $\delta$  sends  $a \otimes \zeta \in \mathcal{O}_X^*(U) \otimes \mu_n \cong \mathcal{H}^1(\mu_n^{\otimes 2})(U)$  to  $[\zeta] \cup [a] \in H_{an}^2(U, \mathbf{Z}(2)_{\mathcal{D}}) = \mathcal{H}_{\mathcal{D}}^2(U)$ .

## 7 An exact sequence for $\mathcal{K}_2$ -cohomology

We now give some exact sequences relating  $H^1(X, \mathcal{K}_2)$  and  $H^2(X, \mathcal{K}_2)$ . The first is a reinterpretation of [33, Theorem D] in terms of hypercohomology. Let  $\mathcal{K}_2^\bullet$  denote the complex  $\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2$  concentrated in degrees 0 and 1. The short exact sequence

$0 \rightarrow \mathcal{K}_2[-1] \rightarrow \mathcal{K}_2^\bullet \rightarrow \mathcal{K}_2 \rightarrow 0$  gives rise to a long exact sequence, reminiscent of [37, (4.4)]:

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{K}_2)/n \xrightarrow{\iota} \mathbf{H}^1(X, \mathcal{K}_2^\bullet) \rightarrow H^1(X, \mathcal{K}_2) \xrightarrow{\eta} H^1(X, \mathcal{K}_2) \xrightarrow{\iota} \\ \mathbf{H}^2(X, \mathcal{K}_2^\bullet) \rightarrow H^2(X, \mathcal{K}_2) \xrightarrow{\eta} H^2(X, \mathcal{K}_2) \rightarrow \dots \end{aligned} \quad (26)$$

From this we extract short exact ‘‘Kummer’’ sequences, such as

$$0 \rightarrow H^1(X, \mathcal{K}_2)/n \xrightarrow{\iota} \mathbf{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\pi} {}_n H^2(X, \mathcal{K}_2) \rightarrow 0. \quad (27)$$

We also have the exact sequence of low degree terms in the hypercohomology spectral sequence for  $\mathcal{K}_2^\bullet$ , the relevant part of which is:

$$H^0(X, \mathcal{K}_2/n) \xrightarrow{d_2} H^2(X, {}_n \mathcal{K}_2) \rightarrow \mathbf{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\eta} H^1(X, \mathcal{K}_2/n) \rightarrow H^3(X, {}_n \mathcal{K}_2). \quad (28)$$

**Proposition 7.1** ([33, Theorem D]) *Let  $X$  be a quasi-projective over a field  $k$  containing  $\frac{1}{n}$ . Then:*

(a) *The  $d_2$ -differential  $H^0(X, \mathcal{K}_2/n) \xrightarrow{d_2} H^2(X, {}_n \mathcal{K}_2)$  in the hypercohomology spectral sequence (28) is the composite*

$$H^0(X, \mathcal{K}_2/n) \xrightarrow{\partial} H^1(X, n \cdot \mathcal{K}_2) \xrightarrow{\partial} H^2(X, {}_n \mathcal{K}_2)$$

*of the boundary maps in the usual interlocking sequences for  $\mathcal{K}_2$ .*

(b) *If  $X$  is a surface with isolated singularities, the map  $\pi$  in the Kummer sequence (27) for  $\mathbf{H}^2(X, \mathcal{K}_2^\bullet)$  factors through the surjection  $\eta$  in the hypercohomology spectral sequence (28).*

**Proof** Part (a) is a special case of a more general result which we have isolated in Lemma 7.2 below. For part (b), it suffices to show that the following diagram commutes.

$$\begin{array}{ccccccccc} H^0(X, \mathcal{K}_2/n) & \xrightarrow{\gamma} & H^2(X, {}_n \mathcal{K}_2) & \xrightarrow{\beta} & H^1(X, \mathcal{K}_2)/n & \rightarrow & H^1(X, \mathcal{K}_2/n) & \rightarrow & {}_n H^2(X, \mathcal{K}_2) \rightarrow 0 \\ \parallel & & \parallel & & \iota \downarrow & & \parallel & & \\ H^0(X, \mathcal{K}_2/n) & \xrightarrow{d_2} & H^2(X, {}_n \mathcal{K}_2) & \rightarrow & \mathbf{H}^2(X, \mathcal{K}_2^\bullet) & \rightarrow & H^1(X, \mathcal{K}_2/n) & \rightarrow & 0 \end{array}$$

The top row is the exact sequence of [33, Theorem D], the bottom row is the exact sequence of low degree terms (28) and the vertical arrow  $\iota$  comes from the Kummer sequence (27). Since  $\mathcal{K}_2 \rightarrow \mathcal{K}_2/n$  factors through  $\mathcal{K}^\bullet[1]$ , the right square commutes. The left square commutes by part (a). The map  $\beta$  is constructed as follows. Let  $\mathcal{L}^\bullet$  denote the subcomplex  $\mathcal{K}_2 \rightarrow n \cdot \mathcal{K}_2$  of  $\mathcal{K}_2^\bullet$ ;  $\mathcal{L}^\bullet$  is quasi-isomorphic to  ${}_n \mathcal{K}_2$ . The inclusion of  $n \cdot \mathcal{K}_2[-1]$  into  $\mathcal{L}^\bullet$  induces a natural map  $H^1(X, n \cdot \mathcal{K}_2) \rightarrow H^2(X, {}_n \mathcal{K}_2)$ , and we know that this map is onto by [33, 4.8.1]. We showed in [33, Proposition 4.9] that

$H^1(X, n \cdot \mathcal{K}_2) \rightarrow H^1(X, \mathcal{K}_2)$  factors through this surjection, and the induced map is  $\beta$ . Thus  $\iota\beta$  is induced from the composite map  $n \cdot \mathcal{K}_2 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_2^\bullet[1]$  upon taking  $H^1$ . But this composite map factors through the subcomplex  $\mathcal{L}^\bullet[1]$  of  $\mathcal{K}^\bullet[1]$ , so it follows that the left square commutes.  $\bullet$

Here is the general result about hypercohomology spectral sequences used to prove part (a) above. It works for any topos  $X$ .

**Lemma 7.2** *For any sheaf  $\mathcal{F}$ , let  $\mathcal{C}^\bullet$  denote the complex  $\mathcal{F} \xrightarrow{n} \mathcal{F}$  concentrated in degrees 0 and 1. Then up to the sign  $(-1)^p$ , the  $d_2$ -differentials*

$$H^p(X, \mathcal{F}/n) = H^p(X, H^1\mathcal{C}) \longrightarrow H^{p+2}(X, H^0\mathcal{C}) = H^{p+2}(X, {}_n\mathcal{F})$$

*in the hypercohomology spectral sequence of  $\mathcal{C}^\bullet$  are the composites*

$$H^p(X, \mathcal{F}/n) \xrightarrow{\partial} H^{p+1}(X, n \cdot \mathcal{F}) \xrightarrow{\partial} H^{p+2}(X, {}_n\mathcal{F})$$

*of the boundary maps  $\partial$  associated respectively to the exact sequences*

$$0 \rightarrow n \cdot \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/n \rightarrow 0, \quad 0 \rightarrow {}_n\mathcal{F} \rightarrow \mathcal{F} \rightarrow n \cdot \mathcal{F} \rightarrow 0$$

**Proof** Given injective resolutions  ${}_n\mathcal{F} \rightarrow \mathcal{I}^\bullet$ ,  $n \cdot \mathcal{F} \rightarrow \mathcal{J}^\bullet$  and  $\mathcal{F}/n\mathcal{F} \rightarrow \mathcal{K}^\bullet$  we can form injective resolutions  $\mathcal{F} \rightarrow \mathcal{E}^{0\bullet} = \mathcal{I}^\bullet \oplus \mathcal{J}^\bullet$  and  $\mathcal{F} \rightarrow \mathcal{E}^{1\bullet} = \mathcal{J}^\bullet \oplus \mathcal{K}^\bullet$  using the Horseshoe Lemma. These form the two columns of a Cartan-Eilenberg resolution  $\mathcal{E}^{\bullet\bullet}$  of the complex  $\mathcal{F} \xrightarrow{n} \mathcal{F}$ ; by the sign trick, the single horizontal differential in this complex is  $(-1)^p$  times the projection/inclusion  $I^p \oplus \mathcal{J}^p \rightarrow \mathcal{J}^p \rightarrow \mathcal{J}^p \oplus \mathcal{K}^p$ .

Given a class  $[s] \in H^p(X, \mathcal{F}/n)$ , represent it by  $s \in H^0(X, K^p)$  with  $\partial s = 0$  in  $H^0(X, K^{p+1})$ . Applying  $\partial^v$  to  $(0, s) \in H^0(X, J^p \oplus K^p)$  gives an element  $(t, 0)$  of  $H^0(X, J^{p+1} \oplus K^{p+1})$ . Thus  $\partial: H^p(X, \mathcal{F}/n) \rightarrow H^{p+1}(X, n \cdot \mathcal{F})$  sends  $[s]$  to  $[t]$ . Applying  $\partial^v$  to  $(0, t) \in H^0(X, I^p \oplus J^p)$  gives  $(u, 0)$  for some  $u \in H^0(X, I^{p+1})$ . By construction,  $u$  is a cycle in  $I^\bullet$  and  $\partial: H^{p+1}(X, n \cdot \mathcal{F}) \rightarrow H^{p+2}(X, {}_n\mathcal{F})$  sends  $[t]$  to  $[u]$ .

Now the hypercohomology spectral sequence arises from the row filtration on the Cartan-Eilenberg resolution  $\mathcal{E}^{\bullet\bullet}$ . Since the pair  $((0, (-1)^p t), (0, s)) \in \text{Tot}^p(\mathcal{I})$  has  $((-1)^p u, 0), (0, 0)$  for its boundary, the  $d_2$ -differential in the spectral sequence takes  $[s]$  to  $(-1)^p [u]$ .  $\bullet$

We are now going to connect Proposition 7.1 with étale cohomology using  $c_2^{et}$ . For this we need to resort to some standard topological constructions. Our main result will be Theorem 7.7 below.

Recall from (2.4.1) that there is a simplicial presheaf  $K$  on  $X_{zar}$  such that  $\pi_q K(U) = K_q(U)$ . Let  $\tilde{K}(U)$  be the universal covering space of the basepoint component of  $K(U)$ ;  $\tilde{K}$  is a simplicial presheaf by [29, 8.3 or 16.4]. Let  $\tilde{K}^{(2)}(U)$  denote the second layer of the Postnikov tower of  $\tilde{K}(U)$ , defined in [29, 8.1]; it is an Eilenberg-MacLane complex of type  $(K_2 U, 2)$  and  $\tilde{K}^{(2)}$  is a simplicial presheaf. Moreover by [29, 8.2] there are Kan fibrations  $\tilde{K}^{(2)} \leftarrow \tilde{K} \rightarrow K$ .

Now let  $\tilde{L}(U)$  denote the homotopy fiber of the map  $\tilde{K}(U) \xrightarrow{n} \tilde{K}(U)$ , and let  $M(U)$  denote the homotopy fiber of the map  $\tilde{K}^{(2)}(U) \xrightarrow{n} \tilde{K}^{(2)}(U)$ . Each  $\tilde{L}(U)$  is a connected space with  $\pi_1 \tilde{L}(U) = K_2(U)/n$  and  $\pi_q \tilde{L}(U) = K_{q+1}(U; \mathbf{Z}/n)$  for  $q \geq 2$ , while  $M(U)$  has only two nontrivial homotopy groups:  $\pi_1 M(U) = K_2(U)/n$  and  $\pi_2 M(U) = {}_n K_2(U)$ . In fact, it is not hard to see that  $M(U)$  is homotopy equivalent to the simplicial space obtained by applying the Dold-Kan theorem to the presheaf of chain complexes  $K_2 \xrightarrow{n} K_2$  concentrated in degrees 2 and 1.

We can perform the above constructions so that there is a commutative diagram of simplicial presheaves (in which the diagram (11) forms the right side):

$$\begin{array}{ccccccc}
M & \leftarrow & \tilde{L} & \rightarrow & L & \xrightarrow{C_2^{ss}} & \Omega\mathcal{E} \\
\downarrow & & \downarrow & & \downarrow & & \delta \downarrow \\
\tilde{K}^{(2)} & \leftarrow & \tilde{K} & \rightarrow & K & \xrightarrow{C_2^{ss}} & \mathcal{D} \\
n \downarrow & & n \downarrow & & n \downarrow & & n \downarrow \\
\tilde{K}^{(2)} & \leftarrow & \tilde{K} & \rightarrow & K & \xrightarrow{C_2^{ss}} & \mathcal{D}
\end{array} \tag{29}$$

**(7.3)** Given any simplicial presheaf  $F$  on  $X$ , the *generalized sheaf cohomology groups*  $\mathbf{H}^q(X, F)$  were defined for  $q \leq 0$  by Brown and Gersten [6, p.280]. (The homotopy categories of simplicial presheaves and simplicial sheaves are equivalent by [22, 2.8]. In particular, if  $\tilde{F}$  is the simplicial sheaf associated to  $F$  then  $\mathbf{H}^q(X, F) = \mathbf{H}^q(X, \tilde{F})$ .)

If  $F$  is the simplicial sheaf associated by the Dold-Kan theorem to a cochain complex  $\mathcal{F}$  (concentrated in negative degrees), then  $\mathbf{H}^q(X, F) \cong \mathbf{H}_{zar}^q(X, \mathcal{F})$  for  $q \leq 0$  by [6, p.281]. Since the simplicial sheaf associated to  $\mathcal{K}_2^\bullet[2]$  is the sheafification of  $M$  we have  $\mathbf{H}^q(X, M) = \mathbf{H}_{zar}^{q+2}(X, \mathcal{K}_2^\bullet)$ . Similarly, by (2.4.2) we have

$$\mathbf{H}^q(X, \Omega\mathcal{E}) = \mathbf{H}^{q-1}(X, \mathcal{E}) \cong H_{\acute{e}t}^{3-q}(X, \mu_n^{\otimes 2}).$$

In particular, diagram (29) induces maps

$$\mathbf{H}^2(X, \mathcal{K}_2^\bullet) = \mathbf{H}^0(X, M) \xleftarrow{\lambda} \mathbf{H}^0(X, \tilde{L}) \xrightarrow{c_2^{\acute{e}t}} \mathbf{H}^0(X, \Omega\mathcal{E}) = H_{\acute{e}t}^3(X, \mu_n^{\otimes 2}). \tag{30}$$

If  $F$  is a simplicial presheaf on  $X$ , we write  $\tilde{\pi}_q F$  for the sheaf associated to the presheaf  $U \mapsto \pi_q F(U)$ . For example, we have

$$\tilde{\pi}_q M = \begin{cases} \mathcal{K}_2/n & \text{if } q = 1; \\ {}_n \mathcal{K}_2 & \text{if } q = 2; \\ 0 & \text{else.} \end{cases} \quad \tilde{\pi}_q \tilde{L} = \begin{cases} \mathcal{K}_2/n & \text{if } q = 1; \\ \mathcal{K}_{q+1}(\mathbf{Z}/n) & \text{if } q \geq 2; \\ 0 & \text{else.} \end{cases}$$

Now recall that  $X$  is quasi-projective over  $\mathbf{C}$ . By [6, Theorem 3] there is a ‘‘Brown-Gersten’’ spectral sequence in the fourth quadrant:

$$E_2^{pq} = H_{zar}^p(X, \tilde{\pi}_{-q} F) \Rightarrow \mathbf{H}^{p+q}(X, F).$$

In general, this spectral sequence is ‘‘fringed’’ [6, p.285], but since all the  $F$  we consider here are infinite loop spaces this fringing does not affect  $\mathbf{H}^q(X, F)$  for  $q \leq 0$ .

**Example 7.4:** Here is an example of the fringing phenomenon. If  $F$  is associated to a cochain complex  $\mathcal{F}$ , with  $\mathcal{F}^q = 0$  for  $q > 0$ , then it is well known that the Brown-Gersten spectral sequence for  $F$  is the same as the hypercohomology spectral sequence for  $\mathcal{F}$ . For example, the simplicial sheaf  $\mathcal{E}$  was defined in (2.4.2) as being associated to  $\tau^{\leq 0} \mathbf{R}\omega_* \mathbf{Z}/n[2i]$ . The hypercohomology spectral sequence of this complex coincides with the Leray spectral sequence for  $H_{\acute{e}t}^{2i+*}(X, \mu_n^{\otimes i})$  in the region  $q \leq 0$ . Thus it is a fringed spectral sequence converging in the region  $p + q \leq 0$ . The line  $p + q = +1$  converges to the kernel of  $H_{\acute{e}t}^{2i+1}(X, \mu_n^{\otimes 2}) \rightarrow H^0(X, \mathcal{H}^{2i+1}(\mu_n^{\otimes i}))$ .

On the other hand, the sheafification  $\tilde{M}$  of  $M$  is associated to the complex of sheaves  $\mathcal{K}_2^\bullet[2]$ . Hence the Brown-Gersten spectral sequence for  $M$  is the same as the hypercohomology spectral sequence for  $\mathcal{K}_2^\bullet[2]$ , and there is no fringing effect.

Any morphism  $E \rightarrow F$  of simplicial presheaves induces a morphism of Brown-Gersten spectral sequences. Thus (30) gives us a commutative diagram:

$$\begin{array}{ccccc}
H^0(X, \mathcal{K}_2/n) & \xrightarrow{c_2^{et} \cong} & & H^0(X, \mathcal{H}^2(\mu_n^{\otimes 2})) & \\
\parallel & & & \parallel & \\
H^0(X, \tilde{\pi}_1 M) & \xleftarrow{\cong} H^0(X, \tilde{\pi}_1 \tilde{L}) & \xrightarrow{c_2^{et} \cong} & H^0(X, \tilde{\pi}_2 \mathcal{E}) & \\
d_2 \downarrow & & d_2 \downarrow & & d_2 \downarrow & (31) \\
H^2(X, \tilde{\pi}_2 M) & \leftarrow H^2(X, \tilde{\pi}_2 \tilde{L}) & \xrightarrow{c_2^{et}} & H^2(X, \tilde{\pi}_3 \mathcal{E}) & \\
\parallel & & & \parallel & \\
H^2(X, {}_n \mathcal{K}_2) & \xrightarrow{c_2^{et} \cong} & & H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})) & 
\end{array}$$

(The bottom square of (31) commutes because, as noted in (2.4.2), the Chern class map  $c_2^{et}: \mathcal{K}_3(\mathbf{Z}/n) \rightarrow \mathcal{H}^1(\mu_n^{\otimes 2})$  factors through  ${}_n \mathcal{K}_2$ .)

The following description of the differential in the Leray spectral sequence was suggested in [33, (0.4)].

**Proposition 7.5** *If we identify  $\mathcal{K}_2/n$  with  $\mathcal{H}^2(\mu_n^{\otimes 2})$  by 6.1 and  ${}_n \mathcal{K}_2$  with  $\mathcal{H}^1(\mu_n^{\otimes 2})$  by 6.4, then the differential  $d_2: H^0(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \rightarrow H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2}))$  in the Leray spectral sequence for  $H_{\acute{e}t}^*(X, \mu_n^{\otimes 2})$  becomes identified with the differential in 7.1(a), i.e.,*

$$H^0(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \cong H^0(X, \mathcal{K}_2/n) \xrightarrow{\partial} H^1(X, n \cdot \mathcal{K}_2) \xrightarrow{\partial} H^2(X, {}_n \mathcal{K}_2) \cong H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})).$$

**Proof** The left vertical map in (31) is the differential in the hypercohomology spectral sequence for  $\mathbf{H}^*(X, \mathcal{K}_2^\bullet)$  by Example 7.4, and was described in Proposition 7.1(a). Again by Example 7.4, the right vertical map in (31) is the corresponding differential in the Leray spectral sequence for  $H_{\acute{e}t}^{4+*}(X, \mu_n^{\otimes 2})$ . A diagram chase on (31), starting at  $H^0(X, \tilde{\pi}_1 \tilde{L})$ , yields the result.  $\bullet$

**Definition 7.6:** Following Suslin [37], we define  $NH_{\acute{e}t}^3(X)$  to be the kernel of the natural map  $H_{\acute{e}t}^3(X, \mu_n^{\otimes 2}) \rightarrow H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2}))$ . Here  $X$  can be any scheme in which  $n$  is

invertible. Of course, when  $X$  is a surface over an algebraically closed field the sheaf  $\mathcal{H}^3(\mu_n^{\otimes 2})$  vanishes and we have  $NH_{\acute{e}t}^3(X) = H_{\acute{e}t}^3(X, \mu_n^{\otimes 2})$ .

The following result was proven by Suslin [37, p. 19] for smooth varieties. It is a partial answer to [2, Question 2] and was conjectured in [33, (0.4)].

**Theorem 7.7** *Let  $X$  be a surface with isolated singularities over a field  $k$  containing an algebraically closed field and  $\frac{1}{n}$ . Then*

$$NH_{\acute{e}t}^3(X) \cong \mathbf{H}^2(X, \mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2).$$

In particular, by (27) there is a functorial short exact sequence:

$$0 \rightarrow H^1(X, \mathcal{K}_2)/n \rightarrow NH_{\acute{e}t}^3(X) \rightarrow {}_nCH_0(X) \rightarrow 0.$$

**Proof** Since  $X$  is a surface, the Brown-Gersten spectral sequences associated to the simplicial presheaves in (30) have only three nonzero columns. Using the computations given in (7.3) for  $\tilde{\pi}_q M$  and  $\tilde{\pi}_q \tilde{L}$ , the resulting exact sequences form the rows of a commutative diagram.

$$\begin{array}{ccccccc} H^0(X, \mathcal{K}_2/n) & \xrightarrow{d_2} & H^2(X, {}_n\mathcal{K}_2) & \rightarrow & \mathbf{H}^2(X, \mathcal{K}_2^\bullet) & \rightarrow & H^1(X, \mathcal{K}_2/n) \rightarrow 0 \\ \uparrow \cong & & \uparrow \text{ onto} & & \uparrow \lambda & & \uparrow \cong \\ H^0(X, \mathcal{K}_2/n) & \xrightarrow{d_2} & H^2(X, \mathcal{K}_3(\mathbf{Z}/n)) & \rightarrow & \mathbf{H}^0(X, \tilde{L}) & \rightarrow & H^1(X, \mathcal{K}_2/n) \rightarrow 0 \\ \downarrow \cong & & c_2^{et} \downarrow \text{ onto} & & \downarrow & & \downarrow \cong \\ H^0(X, \mathcal{H}^2(\mu_n^{\otimes 2})) & \xrightarrow{d_2} & H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})) & \rightarrow & NH_{\acute{e}t}^3(X) & \rightarrow & H^1(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \rightarrow 0 \end{array} \quad (32)$$

The outside vertical maps are isomorphisms by 6.1. The two vertical maps marked ‘onto’ in (32) are actually split surjections with the same kernel, and are identified by Lemma 6.2 since  $\varphi: \mathcal{H}^1(\mu_n^{\otimes 2}) \rightarrow {}_n\mathcal{K}_2$  yields an isomorphism on  $H^2$ . Indeed, by 6.3 we know that the map  $c_2^{et}: \mathcal{K}_3(\mathbf{Z}/n) \rightarrow \mathcal{H}^1(\mu_n^{\otimes 2})$  is a surjection, split up to sign by  $\bar{\varphi}: \mathcal{H}^1(\mu_n^{\otimes 2}) \rightarrow \mathcal{K}_3(\mathbf{Z}/n)$ . A diagram chase on (32) shows that the two maps  $\mathbf{H}^0(X, \tilde{L}) \xrightarrow{\lambda} \mathbf{H}^2(X, \mathcal{K}_2^\bullet)$  and  $\mathbf{H}^0(X, \tilde{L}) \xrightarrow{c_2^{et}} NH_{\acute{e}t}^3(X)$  are both onto with the same kernel. Thus the quotients  $\mathbf{H}^2(X, \mathcal{K}_2^\bullet)$  and  $NH_{\acute{e}t}^3(X)$  are isomorphic. •

**Corollary 7.8** *If  $k$  is algebraically closed then the short exact sequence is:*

$$0 \rightarrow H^1(X, \mathcal{K}_2)/n \rightarrow H_{\acute{e}t}^3(X, \mu_n^{\otimes 2}) \rightarrow {}_nCH_0(X) \rightarrow 0.$$

**Theorem 7.9** *Let  $X$  be a normal projective surface over an algebraically closed field  $k$ . Let  $\ell$  a prime number,  $\ell \neq \text{char}(k)$ . Then*

$$H^1(X, \mathcal{K}_2) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell = 0 \quad \text{and} \quad H_{\acute{e}t}^3(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \cong CH_0(X)_{\ell\text{-tors}}$$

**Proof** Choose a resolution of singularities  $\pi: X' \rightarrow X$ . Passing to the direct limit as  $\nu \rightarrow \infty$ , with  $n = \ell^\nu$ , the short exact sequences of Corollary 7.8 become the rows of the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & H^1(X, \mathcal{K}_2) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell & \rightarrow & H_{\acute{e}t}^3(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell) & \rightarrow & CH_0(X)_{\ell\text{-tors}} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow \cong & \\ 0 \rightarrow & H^1(X', \mathcal{K}_2) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell & \rightarrow & H_{\acute{e}t}^3(X', \mathbf{Q}_\ell/\mathbf{Z}_\ell) & \rightarrow & CH_0(X')_{\ell\text{-tors}} & \rightarrow 0. \end{array}$$

The right-hand vertical map is an isomorphism by the Collino-Levine Theorem [8] [24]. By [10], we have  $H^1(X', \mathcal{K}_2) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell = 0$ . Therefore it suffices to show that

$$H_{\acute{e}t}^3(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \cong H_{\acute{e}t}^3(X', \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

There is a Mayer–Vietoris sequence for  $\ell$ -adic cohomology similar to (16) for the square (14). This yields an exact sequence

$$0 \rightarrow T \rightarrow H_{\acute{e}t}^3(X, \mathbf{Z}_\ell) \rightarrow H_{\acute{e}t}^3(X', \mathbf{Z}_\ell) \rightarrow 0$$

with  $T = H_{\acute{e}t}^2(E, \mathbf{Z}_\ell)/\text{im}(H_{\acute{e}t}^2(X', \mathbf{Z}_\ell))$ . The proof of Proposition 4.2 goes through in the  $\ell$ -adic setting as well to show that  $T$  is a torsion group (*cf.* [8, 2.1]). Since we also have  $H_{\acute{e}t}^4(X, \mathbf{Z}_\ell) \cong H_{\acute{e}t}^4(X', \mathbf{Z}) \cong \mathbf{Z}_\ell^c$ , the universal coefficient theorem yields the result:

$$H_{\acute{e}t}^3(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \cong H_{\acute{e}t}^3(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \cong H_{\acute{e}t}^3(X', \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \cong H_{\acute{e}t}^3(X', \mathbf{Q}_\ell/\mathbf{Z}_\ell). \bullet$$

## 8 Proof of the Main Theorem

Let  $X$  be a complex projective surface. In Lemma 2.3 we constructed the Abel–Jacobi map  $\rho: A_0(X) \rightarrow J^2(X)$ . Our Main Theorem, stated in the Introduction, states that  $\rho$  induces an isomorphism  $A_0(X)_{\text{tors}} \cong J^2(X)_{\text{tors}}$ . We now proceed to prove the Main Theorem.

If  $X$  is a normal surface then the result  $A_0(X)_{\text{tors}} \cong J^2(X)_{\text{tors}}$  is a paraphrase of the theorem of Levine and Collino (see [8], [24]) that  $A_0(X)_{\text{tors}} \cong J^2(\tilde{X})_{\text{tors}}$  for any resolution of singularities  $\tilde{X} \rightarrow X$ , because  $J^2(X) \cong J^2(\tilde{X})$  by Corollary 4.3.

Granting the normal case, we shall establish the general case of our Main Theorem by comparing a singular surface  $X$  with its normalization  $\tilde{X}$ . For this, we need the following crucial Lemma. Let  $\mathcal{H}_{an}^2(\mathbf{Z})$  denote the Zariski sheaf on  $X$  associated to the presheaf  $U \mapsto H_{an}^2(U, \mathbf{Z})$

**Lemma 8.1** *Let  $X$  be an irreducible proper surface over  $\mathbf{C}$ . Then the following composition is zero.*

$$H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) \xrightarrow{\varepsilon} H_{an}^2(X, \mathbf{Z}) \rightarrow H^0(X, \mathcal{H}_{an}^2(\mathbf{Z}))$$

**Proof** By Lemma 4.1 the image of  $\varepsilon$  is the torsion subgroup of  $H_{an}^2(X, \mathbf{Z})$ . However, the sheaf  $\mathcal{H}_{an}^2(\mathbf{Z})$  and hence its global sections are torsion free by [3, Cor. 3].  $\bullet$

**Proposition 8.2** *If  $X$  is an irreducible proper surface over  $\mathbf{C}$ , the following natural map is zero.*

$$H^0(X, \mathcal{K}_2) \rightarrow H^0(X, \mathcal{K}_2/n)$$

**Proof** By Proposition 1.1 the natural map  $H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) \rightarrow H_{zar}^0(X, \mathcal{H}_{\mathcal{D}}^2(2))$  is an isomorphism. The Proposition follows from Lemma 8.1 and a chase on the following diagram, the left part of which commutes by (2.4.1).

$$\begin{array}{ccccccc} K_2(X) & \xrightarrow{c_2} & H_{\mathcal{D}}^2(X, \mathbf{Z}(2)) & \xrightarrow{\varepsilon} & H_{an}^2(X, \mathbf{Z}) & \rightarrow & H_{an}^2(X, \mathbf{Z}/n) \\ \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\ H^0(X, \mathcal{K}_2) & \xrightarrow{c_2} & H^0(X, \mathcal{H}_{\mathcal{D}}^2(2)) & \rightarrow & H^0(X, \mathcal{H}_{an}^2(\mathbf{Z})) & \rightarrow & H^0(X, \mathcal{H}^2(\mathbf{Z}/n)). \end{array} \bullet$$

**Remark 8.3:** When  $X$  is a *smooth* proper variety over an algebraically closed field of characteristic zero, Proposition 8.2 was proven by Colliot-Thélène and Raskind [10], and also by H. Esnault [12] over  $\mathbf{C}$ .

**Proposition 8.4** *Let  $Z$  be a scheme which is proper over  $\mathbf{C}$ . If  $Z$  is either a curve or a normal surface then*

*i)  $c_2: H^1(Z, \mathcal{K}_2)_{tors} \cong H^2(Z, \mathbf{Q}/\mathbf{Z})$*

*ii)  $H^1(Z, \mathcal{K}_2) \otimes \mathbf{Q}/\mathbf{Z} = 0$*

**Proof** The hypothesis on  $Z$  allows us to use 4.7 for the isomorphism  $H^2(Z, \mathbf{Q}/\mathbf{Z}) \cong H_{\mathcal{D}}^3(X, \mathbf{Z}(2))_{tors}$ . When  $Z$  is a curve both assertions follow from Theorem 5.3 and this remark. When  $Z$  is a normal surface, part *ii)* was proven in Theorem 7.9. In order to prove part *i)* for a normal surface  $Z$ , we apply  $H^1$  to Corollary 6.5 and combine with the diagram of Corollary 1.2 to get a commutative diagram for each  $n$ :

$$\begin{array}{ccc} H^1(Z, {}_n\mathcal{K}_2) & \xrightarrow{\tau_n} & H^1(Z, \mathcal{K}_2)_{n-tors} \\ c_2^{ét} \downarrow \cong & & \downarrow c_2 \\ H^1(Z, \mathcal{H}^1(\mu_n^{\otimes 2})) & \xrightarrow{\delta} & H^1(Z, \mathcal{H}_{\mathcal{D}}^2(2))_{n-tors} \\ \downarrow & & \downarrow \\ H_{ét}^2(Z, \mu_n^{\otimes 2}) & \xrightarrow{\delta} & H_{\mathcal{D}}^3(Z, \mathbf{Z}(2))_{n-tors}. \end{array} \quad (33)$$

Taking the direct limit as  $n \rightarrow \infty$  turns  $\mu_n^{\otimes 2}$  into  $\mathbf{Q}/\mathbf{Z}$ . Since  $H^2(Z, \mathbf{Q}/\mathbf{Z})$  is the torsion subgroup of  $H_{\mathcal{D}}^3(Z, \mathbf{Z}(2))$  by Proposition 4.7, we have a commutative diagram

$$\begin{array}{ccccccc} H^1(Z, \mathcal{K}_{2,tors}) & \xrightarrow{\tau} & H^1(Z, \mathcal{K}_2)_{tors} & \rightarrow & \text{coker}(\tau) & \rightarrow & 0 \\ c_2^{ét} \downarrow \cong & & \downarrow c_2 & & \downarrow & & \\ 0 \rightarrow H^1(Z, \mathcal{H}^1(\mathbf{Q}/\mathbf{Z})) & \rightarrow & H_{ét}^2(Z, \mathbf{Q}/\mathbf{Z}) & \rightarrow & H^0(Z, \mathcal{H}^2(\mathbf{Q}/\mathbf{Z})) & \xrightarrow{d_2} & H^2(Z, \mathcal{H}^1(\mathbf{Q}/\mathbf{Z})) \end{array}$$

in which the bottom row is exact by Corollary 1.2. Therefore in order to prove (i) we are reduced to the claim that

$$\text{coker } \tau \cong \ker H^0(Z, \mathcal{H}^2(\mathbf{Q}/\mathbf{Z})) \xrightarrow{d_2} H^2(Z, \mathcal{H}^1(\mathbf{Q}/\mathbf{Z}))$$

For each  $n$ , let  $\gamma_n$  denote the composition  $H^0(Z, \mathcal{K}_2/n) \xrightarrow{\partial} H^1(Z, n \cdot \mathcal{K}_2) \xrightarrow{\partial} H^2(Z, n \mathcal{K}_2)$  in the usual interlocking long exact sequences

$$\begin{array}{ccccccc} H^1(Z, n \mathcal{K}_2) & \xrightarrow{\tau_n} & H^1(Z, \mathcal{K}_2) & \rightarrow & H^1(Z, n \cdot \mathcal{K}_2) & \xrightarrow{\partial} & H^2(Z, n \mathcal{K}_2) \\ & & & & \parallel & & \\ H^0(Z, \mathcal{K}_2) & \xrightarrow{0} & H^0(Z, \mathcal{K}_2/n) & \xrightarrow{\partial} & H^1(Z, n \cdot \mathcal{K}_2) & \rightarrow & H^1(Z, \mathcal{K}_2). \end{array} \quad (34)$$

The arrow marked ‘0’ in this diagram is the zero map by Proposition 8.2. The other zig-zag composition in (34), from  $H^1(Z, \mathcal{K}_2)$  to  $H^1(Z, \mathcal{K}_2)$ , is multiplication by  $n$ . It follows from (34) that

$$\ker(\gamma_n) \cong H^0(Z, \mathcal{K}_2/n) \cap \text{im}(H^1(Z, \mathcal{K}_2)) \cong \frac{H^1(Z, \mathcal{K}_2)_{n\text{-tors}}}{H^1(Z, n \mathcal{K}_2)} = \text{coker}(\tau_n).$$

By Proposition 7.1(a),  $\gamma_n$  is the differential  $d_2$  in the hypercohomology spectral sequence for  $\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2$ . By Proposition 7.5, we may also identify  $\gamma_n$  with the  $d_2$ -differential in the Leray spectral sequence for  $H_{\acute{e}t}^*(Z, \mu_n^{\otimes 2})$ . Passing to the direct limit we obtain the claimed formula:  $\text{coker } \tau = \lim_{n \rightarrow \infty} \text{coker } \tau_n \cong \lim_{n \rightarrow \infty} \ker \gamma_n = \ker(d_2)$ .  $\bullet$

We are now ready to prove our Main Theorem for an arbitrary projective surface  $X$ . Letting  $\tilde{X}$  be its normalization and  $Y$  a subscheme chosen as in Theorem 3.3, we have a Mayer-Vietoris Sequence in  $K$ -theory, and also for Deligne cohomology by 3.2. Taking the torsion subgroups of the diagram in Corollary 3.5 yields the following commutative diagram (in which we have abbreviated the left-hand terms for legibility).

$$\begin{array}{ccccccc} \left\{ \begin{array}{c} SK_1(\tilde{X})^\oplus \\ SK_1(Y) \end{array} \right\}_{tors} & \rightarrow & SK_1(\tilde{Y})_{tors} & \rightarrow & SK_0(X)_{tors} & \rightarrow & SK_0(\tilde{X})_{tors} \rightarrow 0 \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \left\{ \begin{array}{c} H^1(\tilde{X})^\oplus \\ H^1(Y, \mathcal{K}_2) \end{array} \right\}_{tors} & \rightarrow & H^1(\tilde{Y}, \mathcal{K}_2)_{tors} & \rightarrow & H^2(X, \mathcal{K}_2)_{tors} & \rightarrow & H^2(\tilde{X}, \mathcal{K}_2)_{tors} \rightarrow 0 \\ c_2 \downarrow \cong & & c_2 \downarrow \cong & & c_2 \downarrow & & c_2 \downarrow \cong \\ \left\{ \begin{array}{c} H_{\mathcal{D}}^3(\tilde{X}) \\ H_{\mathcal{D}}^3(Y) \end{array} \right\}_{tors} & \rightarrow & H_{\mathcal{D}}^3(\tilde{Y}, \mathbf{Z}(2))_{tors} & \rightarrow & H_{\mathcal{D}}^4(X, \mathbf{Z}(2))_{tors} & \rightarrow & H_{\mathcal{D}}^4(\tilde{X}, \mathbf{Z}(2))_{tors} \rightarrow 0 \end{array} \quad (35)$$

Some discussion of diagram (35) is in order. The 3 isomorphisms between the terms in the top two rows come from 3.5. The 2 vertical maps in the lower left of (35) are isomorphisms by Proposition 8.4. The lower right vertical map  $H^2(\tilde{X}, \mathcal{K}_2)_{tors} \cong J^2(\tilde{X})_{tors} \cong H_{\mathcal{D}}^4(\tilde{X}, \mathbf{Z}(2))_{tors}$  is an isomorphism because  $\tilde{X}$  is normal.

The bottom row of (35) is exact, because by Proposition 4.7 it is isomorphic to

$$H^2(\tilde{X}, \mathbf{Q}/\mathbf{Z}) \oplus H^2(Y, \mathbf{Q}/\mathbf{Z}) \rightarrow H^2(\tilde{Y}, \mathbf{Q}/\mathbf{Z}) \rightarrow H^3(X, \mathbf{Q}/\mathbf{Z}) \rightarrow H^3(\tilde{X}, \mathbf{Q}/\mathbf{Z}) \rightarrow 0.$$

The top two rows of (35) are exact except at  $SK_1(\tilde{Y})_{tors}$  and  $H^1(\tilde{Y}, \mathcal{K}_2)_{tors}$  by Proposition 8.4 and the elementary lemma 8.6 below, whose proof is left as an exercise. The 5-lemma implies that we have an isomorphism

$$c_2: H^2(X, \mathcal{K}_2)_{tors} \cong H^4_{\mathcal{D}}(X, \mathbf{Z}(2))_{tors}$$

and this finishes the proof of our Main Theorem. •

**Remark 8.5:** In order for the diagram chase of (35) to work, it suffices to know the crude surjectivity of the left vertical map as  $n \rightarrow \infty$ :

$$H^1(\tilde{X}, \mathcal{K}_2)_{tors} \oplus H^1(Y, \mathcal{K}_2)_{tors} \xrightarrow{c_2} H^2(\tilde{X}, \mathbf{Q}/\mathbf{Z}) \oplus H^2(Y, \mathbf{Q}/\mathbf{Z}).$$

**Lemma 8.6** *Let  $A \rightarrow B \rightarrow C \rightarrow D$  be an exact sequence of abelian groups. If  $A \otimes \mathbf{Q}/\mathbf{Z} = 0$  then the following sequence is exact.*

$$B_{tors} \rightarrow C_{tors} \rightarrow D_{tors}$$

Here is a motivic version of our Main Theorem. For a 1-motive  $M = (L, A, T, J, u)$  we let  $M_{tors}$  denote the extension of torsion subgroups.

$$0 \rightarrow T_{tors} \rightarrow J_{tors} \rightarrow A_{tors} \rightarrow 0$$

Then our Main Theorem says that  $Alb(X)_{tors}$  can be described via *algebraic* zero-cycles, *i.e.*, that  $J^2(X)_{tors}$  is isomorphic to  $A_0(X)_{tors}$  in a way compatible with normalization and desingularization.

**Variante 8.7** *Let  $Alb(X)$  the Albanese 1-motive of a projective surface. We then have the following identification of  $Alb(X)_{tors}$ :*

$$\begin{array}{ccccccc} 0 & \rightarrow & (\mathbf{Q}/\mathbf{Z})^s & \rightarrow & A_0(X)_{tors} & \rightarrow & A_0(\tilde{X})_{tors} \rightarrow 0 \\ & & \parallel & & \downarrow \cong & & \downarrow \cong \\ 0 & \rightarrow & (\mathbf{Q}/\mathbf{Z})^s & \rightarrow & J^2(X)_{tors} & \rightarrow & J^2(\tilde{X})_{tors} \rightarrow 0. \end{array}$$

**Remark 8.8:** If  $X$  is an *affine* surface over  $\mathbf{C}$  then  $CH_0(X) = A_0(X)$  is uniquely divisible. Indeed, the fact that  $A_0(X)_{tors} = 0$  was proven in [25, Theorem 2.6]. And divisibility of  $CH_0(X) = SK_0(X)$  is classical, probably attributable to Murthy: Every smooth point  $x$  on  $X$  is in the image of a map  $j: C \rightarrow X$  in which  $C$  is a smooth affine curve. The group  $\text{Pic}(C)$  is divisible, and the class of  $x$  is in the image of the map  $j_*: \text{Pic}(C) \rightarrow SK_0(X)$ .

Since  $H^3(X, \mathbf{C}) = 0$  as well, we also have  $J^2(X) = 0$ . Thus Roitman's Theorem holds by default in the affine case.

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