ON HIGHER CLASS GROUPS OF ORDERS

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Abstract. The purpose of this paper is to study the torsion in odd-dimensional higher class groups of orders in semi-simple algebras over number fields. We show that for a prime number q these higher class groups can have q-torsion only if the order is *not* maximal at some prime ideal above q, and we determine part of the structure of this torsion. As an application to integral group rings we show that in dimensions $4n+1$ the class groups of the symmetric group S_r have at most 2-torsion and that in dimensions $4n-1$ the possible odd torsion can only occur for primes q such that $\frac{q-1}{2}$ divides r. In dimensions $4n+1$ the same result also holds for Dihedral groups, provided we assume the validity of the local Quillen-Lichtenbaum Conjecture. In a final section we relate the structure of the higher odd-dimensional class groups of a group ring of a finite group G to homomorphisms on the representation ring of G with values in twisted roots of unity and - for G abelian - also to homogeneous functions on G.

Introduction

One of the high points in the study of the K-theory of group rings and orders was R. Oliver's investigation of $SK_1(\mathbb{Z}[G])$ for finite groups G, in a series of deep papers, summarized in [Ol]. He defined the higher class group

$$Cl_1(\mathbb{Z}[G]) = ker\left(SK_1(\mathbb{Z}[G]) \longrightarrow \bigoplus_p SK_1(\widehat{\mathbb{Z}_p}[G])\right),$$

which measures the obstruction to a local-global principle for $SK_1(\mathbb{Z}[G])$. In many cases all the local groups vanish. Some of Oliver's principal tools were conductor squares and the Moore sequence.

In this paper we use arithmetic squares and higher dimensional analogues of the Moore sequence to study the odd dimensional class groups $Cl_{2n-1}(\Lambda)$, $n \geq 1$, of an $\mathcal{O}_F$-order Λ in a semi-simple algebra over a number field F with ring of integers $\mathcal{O}_F$. These are defined as

$$Cl_m(\Lambda) = ker\left(SK_m(\Lambda) \longrightarrow \bigoplus_\wp SK_m(\Lambda_\wp)\right),$$

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where $\wp$ runs through all maximal ideals of $\mathcal{O}_F$. Little is known about these groups beyond the fact that they are finite [Ku2], and that they vanish for maximal orders [Ke1].

In this paper we obtain results about the possible torsion that can appear. Not surprisingly, the only p-torsion possible in $Cl_{2n-1}(\Lambda)$ is for those rational primes p which lie under a prime of $\mathcal{O}_F$ at which Λ is not maximal. For group rings this means that the only torsion possible is for primes dividing the order of the group. For most of our general results we need to assume that for the rational primes q lying under primes of $\mathcal{O}_F$ at which Λ is not maximal the skewfields appearing in the simple factors of the semi-simple algebra $A = \prod_{i=1}^r M_{n_i}(D_i)$ are fields $D_i = E_i$, and that the local Quillen-Lichtenbaum Conjecture holds for the local fields that appear. Under these assumptions we obtain a surjection

$$\prod_{i=1}^{r} H^0(E_i, \mathbb{Q}_q/\mathbb{Z}_q(n)) \twoheadrightarrow Cl_{2n-1}(\Lambda)(q),$$

for odd rational primes q lying under primes of $\mathcal{O}_F$ at which Λ is not maximal. Furthermore, each factor in the above product is a finite cyclic group. The Quillen-Lichtenbaum Conjecture about the bijectivity of the local étale Chern characters of Soulé is known to hold for unramified extensions of $\mathbb{Q}_q$ [BM], and odd primes q, and seems to be in reach in general. If skewfields appear in the simple factors of A, then our methods provide less information, since so far no global reduced norm for the higher K-theory of skewfields is available.

The main idea behind the proofs is to replace the Moore exact sequence and its generalization to skewfields, which is used in dimension 1, by an analogous sequence in étale cohomology, which comes from the Poitou-Tate duality sequence [Sc].

As an application, we show that without any assumptions the class groups $Cl_{4n+1}(\mathbb{Z}[S_r])$ of the symmetric groups S_r on r letters can contain at most 2-torsion, and in dimensions $4n-1$ the only odd torsion which can appear is for odd primes q such that $\frac{q-1}{2}|n$. For Dihedral groups we also obtain that in dimensions $4n+1$ only 2-torsion is possible, but with the extra hypothesis that the local Quillen-Lichtenbaum Conjecture holds for the local fields that appear.

In a final section we study the groups $\prod_{i=1}^r H^0(E_i, \mathbb{Q}_q/\mathbb{Z}_q(n))$ for group rings $\Lambda = \mathbb{Z}[G]$. For abelian groups we give a Hom-description of this product in the style of Fröhlich, which reduces to a description in terms of n-homogeneous functions on G, defined in [DL], for n odd. If $n = 1$, we recover in this way one of the main results in [ADOS].

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I. Torsion in Higher Class Groups

Let F be a number field with ring of integers $\mathcal{O}_F$, and let Λ be an $\mathcal{O}_F$-order in a semi-simple F-algebra A. The group

$$SK_m(\Lambda) = ker\,(K_m(\Lambda) \longrightarrow K_m(A))$$

is known to be finite for all $m \geq 1$ [Ku2, Theorem 1.1]. Let

$$\widehat{\Lambda} = \prod_{\wp} \Lambda_{\wp} \quad \text{and} \quad \widehat{A} = \prod\nolimits_{\wp} (A_{\wp}, \Lambda_{\wp})$$

denote the adele rings of Λ and A respectively. Here $\wp$ runs through all the maximal ideals of $\mathcal{O}_F$, $\Lambda_{\wp}$ and $A_{\wp}$ denote the completions of Λ and A, respectively, at the prime $\wp$, and $\prod_{\wp}(A_{\wp}, \Lambda_{\wp})$ denotes the restricted direct product of the $A_{\wp}$ with respect to the $\Lambda_{\wp}$. The arithmetic square

$$\begin{array}{ccc} \Lambda & \longrightarrow & A \\ \downarrow & & \downarrow \\ \widehat{\Lambda} & \longrightarrow & \widehat{A}, \end{array}$$

introduced by C.T.C. Wall [Wal], was shown in [Ba, Lemma 7.21] to be isomorphic to a localization-completion square, introduced by Karoubi [Ka]. Consequently, we obtain a long exact Mayer-Vietoris sequence for all $m \geq 1$ in K-theory [Vo, Prop. 1.5]:

$$(1.1) \qquad \cdots \longrightarrow K_{m+1}(\widehat{A}) \longrightarrow K_m(\Lambda) \longrightarrow K_m(\widehat{\Lambda}) \oplus K_m(A) \longrightarrow K_m(\widehat{A}) \longrightarrow \cdots .$$

In particular we obtain a surjection

$$SK_m(\Lambda) \twoheadrightarrow \bigoplus_{\wp} SK_m(\Lambda_{\wp}),$$

induced by the inclusion $\Lambda \longrightarrow \Lambda_{\wp}$ in each coordinate.

Let $A = \prod_{i=1}^{r} A_i$ be the decomposition of A into simple factors A_i, which are then isomorphic to $M_{n_i}(D_i)$, D_i a skewfield with center a finite extension E_i of F. Let Γ be a maximal order in A containing Λ. Then there is a similar decomposition of Γ into $\prod_{i=1}^{r} \Gamma_i$, and $\Gamma_i \cong M_{n_i}(M_i)$, where M_i is a maximal order in D_i. Localizing at a prime ideal of $\mathcal{O}_F$ we obtain similar decompositions in the local case.

Following R. Oliver [Ol, p. 6], we now define higher dimensional class groups.

Definition 1.1. Let $m \geq 1$ be an integer.

(1) Define the m-dimensional class group $Cl_m(\Lambda)$ as

$$Cl_m(\Lambda) = ker\left(SK_m(\Lambda) \twoheadrightarrow \bigoplus_{\wp} SK_m(\Lambda_{\wp}) \right).$$

(2) If R is any ring, we will denote the quotient of $K_m(R)$ by its maximal divisible subgroup by $K_m^c(R)$.

(3) Let D be any skewfield over F, and let M be a maximal order in D. Given a finite set S of prime ideals in $\mathcal{O}_F$, we define $Cl_m(M, S)$ to be the cokernel of the map

$$K_{m+1}^c(D) \longrightarrow \bigoplus_{\wp \in S} K_{m+1}^c(D_{\wp}) \oplus \bigoplus_{\wp \notin S} K_{m+1}^c(D_{\wp})/im(K_{m+1}^c(M_{\wp})).$$

Let P_S denote the set of rational primes lying under the prime ideals in S.

As before, let Γ be a maximal order in A containing Λ, and let S_Λ be the finite set of prime ideals $\wp \subset \mathcal{O}_F$, such that $\Lambda_\wp \neq \Gamma_\wp$, that is, such that $\Lambda_\wp$ is not maximal. We will always assume from now on that the set S_Λ is closed under conjugation, as is the case for instance for group rings. This is not an essential restriction. It allows us, however, to formulate the results in a uniform manner. The set of prime numbers q below prime ideals $\wp \in S_\Lambda$ will be denoted by P_Λ.

Clearly, $Cl_m(\Lambda)$ is the cokernel of the map

$$K_{m+1}(\widehat{\Lambda}) \oplus K_{m+1}(A) \longrightarrow K_{m+1}(\widehat{A}),$$

which results in the following description.

Lemma 1.2. *For all $m \geq 1$, there is an isomorphism*

$$Cl_m(\Lambda) \cong coker \left(\bigoplus_{\wp \in S_\Lambda} K^c_{m+1}(\Lambda_\wp) \longrightarrow \prod_{i=1}^{r} Cl_m(M_i, S_\Lambda) \right).$$

Proof. From the Mayer-Vietoris sequence (1.1) we obtain that

$$\begin{aligned} Cl_m(\Lambda) &\cong coker \left(K_{m+1}(\widehat{\Lambda}) \oplus K_{m+1}(A) \longrightarrow K_{m+1}(\widehat{A}) \right) \\ &\cong coker \left(K_{m+1}(\widehat{\Lambda}) \longrightarrow coker \left(K_{m+1}(A) \longrightarrow K_{m+1}(\widehat{A}) \right) \right). \end{aligned}$$

Using Morita invariance of K-theory and the fact that $Cl_m(\Lambda)$ is finite, so that we can disregard divisible subgroups, the lemma follows. ∎

From now on we shall assume that $m = 2n-1$ is odd. The next result determines the possible torsion in $Cl_{2n-1}(\Lambda)$.

Theorem 1.3. *For all $n \geq 1$, $Cl_{2n-1}(\Lambda)(q) = 0$ for $q \notin P_\Lambda$.*

Proof. By Lemma 1.2 it is sufficient to show that in the situation of Part 3 of Definition 1.1 we have for general S and $q \notin P_S$ that $Cl_{2n-1}(M, S)(q) = 0$. For a given prime ideal $\wp \subset \mathcal{O}_F$ let $\mathcal{K}_\wp$ denote the residue field of $M_\wp$. Then for any prime number $q \neq char(\mathcal{K}_\wp)$ we obtain [SY, Lemma 2] from Suslin's Rigidity Theorem that

$$K^c_{2n}(M_\wp) \otimes \mathbb{Z}_q \cong K_{2n}(\mathcal{K}_\wp) \otimes \mathbb{Z}_q = 0.$$

This is true in particular for $\wp \in S_\Lambda$ and $q \notin P_\Lambda$. Therefore, the q-torsion in $Cl_{2n-1}(M, S)$ coincides with the q-torsion in the cokernel of the map

$$K^c_{2n}(D) \longrightarrow \bigoplus_{\wp} K^c_{2n}(D_\wp)/K^c_{2n}(M_\wp).$$

It suffices therefore to show that this cokernel is zero.

Consider the following commutative diagram of localization sequences:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
K^c_{2n}(D) & \longrightarrow & \bigoplus_\wp K^c_{2n}(D_\wp)/K^c_{2n}(M_\wp) & & & & \\
\downarrow = & & \downarrow & & & & \\
K^c_{2n}(D) & \longrightarrow & \bigoplus_\wp K_{2n-1}(\mathcal{K}_\wp) & \longrightarrow & SK_{2n-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \bigoplus_\wp SK_{2n-1}(M_\wp) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

It was shown by Keating [Ke1] (see [Ke2] for an addendum) that

$$SK_{2n-1}(M) \cong \bigoplus_\wp SK_{2n-1}(M_\wp),$$

under the hypothesis that certain transfer maps in the localization sequence of a number ring are zero. This fact was proved later by Soulé [So1] (see [So2] for the correction of an error in [So1]). The result now follows from the Snake-Lemma. ■

Corollary 1.4. *Let G be any finite group. For all $n \geq 1$, the only possible p-torsion in $Cl_{2n-1}(\mathcal{O}_F[G])$ can occur for those p dividing the order of G.*

Now consider the case that $q \in P_\Lambda$. The information we provide on the q-torsion in $Cl_{2n-1}(\Lambda)$ comes from the general computation of the q-torsion in $Cl_{2n-1}(M, S)$. Again, by rigidity, we know that $K^c_{2n}(M_\wp) \otimes \mathbb{Z}_q = 0$ for all $\wp \notin S$. Hence we need to study the q-primary part of the cokernel of the map

$$K^c_{2n}(D) \longrightarrow \bigoplus_\wp K^c_{2n}(D_\wp),$$

which is the higher skewfield analogue of Moore's exact sequence [Mo, Mi]. We consider three scenarios:

(1) $n = 1$: In this case the cokernel is related to the congruence subgroup problem for D and has essentially been computed in [BR, Theorem 5.2] and [PR]; see also [Re]. We will give a slightly different approach below.
(2) $D = E$ is a field, n arbitrary. For q odd we use the étale analogue of Moore's exact sequence and compute a direct summand of $Cl_{2n-1}(\mathcal{O}_E, S)(q)$, $q \in P_S$ odd, which yields the full group if the local Quillen-Lichtenbaum Conjecture is true for all $\mathcal{O}_{E,\wp}$ with $\wp \in S$ such that $char(\mathcal{O}_F/\wp) = q$.
(3) D a skewfield, q odd. Here we obtain partial results by assuming that q does not divide the degree of D.

II. The one-dimensional Case

Theorem 2.1. *(Bak-Rehmann* [BR]*) Let D be a global skewfield with center E. Then the cokernel of the map*

$$K_2(D) \longrightarrow \bigoplus_{\wp} K_2(D_{\wp})$$

is isomorphic to $\mu(E)$ or $\mu(E)/\{\pm 1\}$, except in the case that D_v is split at some real place v of E. In this case the cokernel is trivial.

Remark. The ambiguity of the factor $\{\pm 1\}$ in the Bak-Rehmann Theorem was removed in some cases by R. Oliver [Ol, Theorem 4.13]. In particular, for group rings one obtains that the cokernel is either isomorphic to $\mu(E)$ or is trivial.

As remarked in [MS, Remark 17.5] the ambiguity can be completely removed if the degree of D is square-free. We give a proof in this case since it apparently is not in the literature.

2.2. Proof for square-free degree.

In the square-free degree case, Merkurjev and Suslin define a reduced norm $Nrd : K_2(D) \longrightarrow K_2(E)$ [MS, Theorem 7.3] and show that the following sequence is exact:

$$0 \to K_2(D) \xrightarrow{Nrd} K_2(E) \to \coprod_R \mu_2 \to 0,$$

where R is the set of real places v of E such that the algebra $D_v = D \otimes_E E_v$ is non-split. (The map Nrd was defined later by Suslin without the square-free assumption on the degree of D [Su, Corollary 5.7].)

Also without any assumption on the degree of D we have that for each prime ideal $\wp$ of $\mathcal{O}_E$ the reduced norm

$$Nrd : K_2^c(D_{\wp}) \longrightarrow K_2^c(E_{\wp}) \ (\cong \mu\,(E_{\wp}))$$

is an isomorphism [Yu, Theorem 1]. Now consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & WK_2'(D) & \longrightarrow & K_2(D) & \longrightarrow & \bigoplus_{\wp} K_2^c(D_{\wp}) \\
 & & \cong\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & WK_2'(E) & \longrightarrow & K_2(E) & \longrightarrow & \bigoplus_{\wp} \mu(E_{\wp}) \oplus \bigoplus_R \mu_2 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_R \mu_2 & \xrightarrow{\cong} & \bigoplus_R \mu_2 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

The left vertical isomorphism follows from the fact that $WK_2'(E)$ is contained in the image of $K_2(D)$. The diagram implies that the cokernels of the two horizontal sequences are isomorphic. But Moore's exact sequence [Mo] shows that the cokernel of the bottom row is isomorphic to $\mu(E)$, except in the case where D_v splits at some real place v of E, where the cokernel is trivial. ■

We note the following consequence of this proof, which should be viewed as the "true" analogue of the Moore sequence.

Corollary 2.3. *For any global skewfield D with center E and square-free degree there is an exact sequence*

$$0 \longrightarrow WK_2(E) \longrightarrow K_2(D) \longrightarrow \bigoplus_{\wp} \mu(E_\wp) \oplus \bigoplus_{\substack{v \text{ real} \\ D_v \text{ split}}} \mu_2 \longrightarrow \mu(E) \longrightarrow 0,$$

where $WK_2(E)$ is the wild kernel of E.

III. The Field Case

We now consider higher class groups in the case that all skewfields occuring in the decomposition of the semi-simple algebra A are commutative. Furthermore, we will ignore 2-torsion, however the theorem below should hold for the prime 2 as well, as long as the fields in question contain $\sqrt{-1}$.

First we prove a result which allows us to avoid the use of continuous cohomology.

Lemma 3.1. *The group $Cl_{2n-1}(\mathcal{O}_E, S)$ is isomorphic to the cokernel of the map*

$$K_{2n}(\mathcal{O}_E) \longrightarrow \bigoplus_{\wp \in S} K^c_{2n}(\mathcal{O}_{E,\wp}).$$

Proof. This result follows immediately from applying the Snake Lemma to the following commutative diagram and observing that $K^c_{2n}(\mathcal{O}_E) = K_{2n}(\mathcal{O}_E)$ is finite [Bo], and so is $K^c_{2n}(\mathcal{O}_{E,\wp})$ [Wa, Theorem]:

$$\begin{array}{ccccccccc}
0 & \to & K^c_{2n}(\mathcal{O}_E) & \to & K^c_{2n}(E) & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \to & \bigoplus_{\wp \in S} K^c_{2n}(\mathcal{O}_{E,\wp}) & \to & \bigoplus_{\wp \in S} K^c_{2n}(E_\wp) \oplus \bigoplus_{\wp \notin S} K^c_{2n}(E_\wp)/K_{2n}(\mathcal{O}_\wp) & \to & B & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & Cl_{2n-1}(\mathcal{O}_E, S_\Lambda) & & & &
\end{array}$$

Here, $k_\wp$ denotes the residue field of $\mathcal{O}_{E,\wp}$, and $B = \bigoplus_\wp K_{2n-1}(k_\wp)$. ■

The next result involves the local Quillen-Lichtenbaum Conjecture which asserts that for $q = char(k_\wp)$ there are isomorphisms

$$K_{2n}(\mathcal{O}_{E,\wp})(q) \cong H^0(E_\wp, \mathbb{Q}_q/\mathbb{Z}_q(n))^*.$$

(Here, for any abelian group A, A^* denotes the dual $Hom(A, \mathbb{Q}/\mathbb{Z})$.) It has been proven recently for q odd, if $E_\wp/\mathbb{Q}_q$ is unramified [BM], and it seems that the ramified case as well as the prime 2 are within reach – in contrast to the global Quillen-Lichtenbaum Conjecture.

Theorem 3.2. *Let q be an odd rational prime lying below a prime ideal in S_Λ. Then the group $Cl_{2n-1}(\mathcal{O}_E, S_\Lambda)(q)$ contains a direct summand which is isomorphic to $H^0(E, \mathbb{Q}_q/\mathbb{Z}_q(n))$. If the local Quillen-Lichtenbaum Conjecture is true for all $E_\wp$ with $char(k_\wp) = q$, for instance if $E_\wp/\mathbb{Q}_q$ is unramified, then*

$$Cl_{2n-1}(\mathcal{O}_E, S_\Lambda)(q) \cong H^0(E, \mathbb{Q}_q/\mathbb{Z}_q(n)).$$

Proof. Since $K_{2n}(\mathcal{O}_{E,\wp})(q) = 0$ if $q \neq char(k_\wp)$, we have that

$$Cl_{2n-1}(\mathcal{O}_E, S_\Lambda) \cong coker \left(K^c_{2n}(\mathcal{O}_E)(q) \longrightarrow \bigoplus_{\substack{\mathfrak{q}\in S \\ \mathfrak{q}|q}} K^c_{2n}(\mathcal{O}_{E,\mathfrak{q}})(q) \right).$$

By [So1] and [DF] there are surjective global and local Chern class characters

$$K_{2n}(\mathcal{O}_F)(q) \twoheadrightarrow H^2_{et}(\mathcal{O}_E, \mathbb{Z}_q(n+1))$$

where the target group is the étale cohomology of $Spec\left(\mathcal{O}_E[\frac{1}{q}]\right)$, and

$$K_{2n}(\mathcal{O}_{E,\wp})(q) \twoheadrightarrow H^2(E_\wp, \mathbb{Z}_q(n+1)),$$

for primes $\wp$ dividing q. By local duality we obtain isomorphisms

$$H^2(E_\wp, \mathbb{Z}_q(n+1)) \cong H^0(E_\wp, \mathbb{Q}_q/\mathbb{Z}_q(-n))^* \cong H^0(E_\wp, \mathbb{Q}_q/\mathbb{Z}_q(n))^*.$$

Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
K_{2n}(\mathcal{O}_E)(q) & \to & \bigoplus_{\substack{\wp\in S \\ \wp|q}} K_{2n}(\mathcal{O}_{E,\wp})(q) & \to & Cl_{2n-1}(\mathcal{O}_E, S_\Lambda) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2_{et}(\mathcal{O}_E, \mathbb{Z}_q(n+1)) & \to & \bigoplus_{\substack{\wp\in S \\ \wp|\mathfrak{q}}} H^0(E_\wp, \mathbb{Q}_q/\mathbb{Z}_q(n+1))^* & \to & T & \to & 0,
\end{array}$$

in which all the vertical maps are surjections. Since $H^2_{et}(\mathcal{O}_E, \mathbb{Z}_q(n+1))$ is finite, it is isomorphic to $H^1_{et}(\mathcal{O}_E, \mathbb{Z}_q(n+1))/(\text{max. div. subgroup})$. Furthermore, we have $T \cong H^0(E, \mathbb{Q}_q/\mathbb{Z}_q(n))^*$ [Sc]. Note that for $n = 1$ this group is just $\mu(E)^*(q)$.

Finally, observe that the above Chern class characters are in fact split surjective [K]. This completes the proof of the theorem. ∎

Remark. The q-primary part of the étale version of a higher Moore sequence has been derived from Schneider's result by Banaszak [Bn] and Nguyen Quang Do [Ng].

IV. The General Case

We now consider the general case of skewfields. Here we make use of the following result by Suslin and Yufryakov.

Theorem 4.1. *(*[SY, Lemma 2 and Theorem 3]*) Let $M \subset D$ be a maximal order in a skewfield, as above, $\wp$ a prime of $\mathcal{O}_E$, and $\mathcal{K}_\wp$ the residue field of $M_\wp$. Then*

$$K_{2n}^c(M_\wp) \otimes \mathbb{Z}_q \cong \begin{cases} K_{2n}^c(D_\wp) \otimes \mathbb{Z}_q & \text{if } q = char(\mathcal{K}_\wp) \\ 0 & \text{if } q \neq char(\mathcal{K}_\wp). \end{cases}$$

Furthermore, if $char(\mathcal{K}_\wp)$ is relatively prime to the degree of $D_\wp$, then $K_{2n}(D_\wp) \cong K_{2n}(E_\wp)$ is finite.

Theorem 4.2. *Let q be an odd prime lying below some prime ideal in S_Λ. Assume that for all prime ideals $\wp \in S_\Lambda$ the degree $d_\wp$ of $D_\wp$ is not divisible by $char(\mathcal{K}_\wp)$. Then*

$$Cl_{2n-1}(\mathcal{O}_E, S_\Lambda)(q) \longrightarrow Cl_{2n-1}(M, S_\Lambda)(q)$$

is surjective for all $n \geq 1$.

Proof. From the hypotheses and Theorem 4.1 it follows that $Cl_{2n-1}(M, S_\Lambda) \otimes \mathbb{Z}_q$ is torsion, hence isomorphic to $Cl_{2n-1}(M, S_\Lambda)(q)$.

Since for $n > 1$ no global reduced norm has been defined we cannot proceed as before, but have to use the natural maps induced from inclusions instead. Consider the commutative diagram:

$$\begin{array}{ccccccc} K_{2n}^c(D) \otimes \mathbb{Z}_q & \longrightarrow & \bigoplus_\wp K_{2n}^c(D_\wp) \otimes \mathbb{Z}_q & \longrightarrow & Cl_{2n-1}(M, S_\Lambda) \otimes \mathbb{Z}_q & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ K_{2n}^c(E) \otimes \mathbb{Z}_q & \longrightarrow & \bigoplus_\wp K_{2n}^c(E_\wp) \otimes \mathbb{Z}_q & \longrightarrow & Cl_{2n-1}(\mathcal{O}_E, S_\Lambda) \otimes \mathbb{Z}_q & \longrightarrow & 0. \end{array}$$

To prove the theorem it is sufficient to show now that the middle vertical map is an isomorphism. By our assumption this is clear if $char(\mathcal{K}_\wp) = q$, since the composition

$$K_{2n}^c(E_\wp) \to K_{2n}^c(D_\wp) \xrightarrow{Nrd} K_{2n}^c(E_\wp)$$

is multiplication by $d_\wp$, which is relatively prime to q. If $char(\mathcal{K}_\wp) \neq q$, then we have the following commutative diagram [SY]:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_{2n}^c(D_\wp)(q) & \longrightarrow & K_{2n-1}(\mathcal{K}_\wp)(q) \\ & & \uparrow & & \uparrow \\ & & K_{2n}^c(E_\wp)(q) & \xrightarrow{\cong} & K_{2n-1}(k_\wp)(q). \end{array}$$

Since the right-hand vertical map is injective [Qu,Theorem 8], so is the left-hand one, hence is an isomorphism since both groups are finite of the same order. ∎

Remark. For odd q one would expect the map in Theorem 4.2 to be an isomorphism in general. If we assume the local Quillen-Lichtenbaum Conjecture, then the above proof shows that at least $Cl_{2n-1}(M, S_\Lambda)(q)$ is still cyclic.

We mention for the record that the results in [Y] and [SY] give the following quick way to compute $SK_{2n-1}(M)$ for a maximal order in a local skewfield.

Proposition 4.3. [Ke1] *Let M be the maximal order in a local skewfield D of degree d over a local field F. Let k be the residue field of F, $\mathcal{K}$ that of M. Furthermore, let $char(k) = p$, and $|k| = q$. Assume that either $n = 1$ or p does not divide d. Then $SK_{2n-1}(M)$ is cyclic of order $\frac{q^{nt}-1}{q^n-1}$.*

Proof. We have an exact sequence

$$0 \longrightarrow K_{2n}(M) \longrightarrow K_{2n}(D) \longrightarrow K_{2n-1}(\mathcal{K}) \longrightarrow SK_{2n-1}(M) \longrightarrow 0.$$

Under the assumptions of the theorem we have an isomorphism $K_{2n}(D) \xrightarrow{Nrd} K_{2n}(E)$. Thus, the p-primary torsion in $K_{2n}(D)$ is equal to $K_{2n-1}(k)$. Quillen's computation of the K-groups of a finite field [Qu] and the fact that $|\mathcal{K}| = q^t$ now finish the proof. ■

Combining Lemma 1.2 with the computations in Theorem 2.1, Proposition 3.2 and Proposition 4.2 we can now summarize our main result about the torsion in $Cl_{2n-1}(\Lambda)$:

Theorem 4.4.. *Let $\mathcal{O}_F$ be the ring of integers in a number field F, and let Λ be an $\mathcal{O}_F$-order in a semi-simple F-algebra $A \cong \prod_{i=1}^r M_{n_i}(D_i)$, D_i skewfields with center E_i. Let S_Λ be the set of all primes $\wp \subset \mathcal{O}_F$ at which Λ is not maximal, and let P_Λ be the set of all rational primes which lie under primes in S_Λ. Assume futhermore that S_Λ is closed under conjugation.*

For each odd prime q in P_Λ there is a surjection

$$\prod_{i=1}^r H^0(E_i, \mathbb{Q}_q/\mathbb{Z}_q(n)) \twoheadrightarrow Cl_{2n-1}(\Lambda)(q)$$

in the following cases:

1. *$n = 1$ (here $q = 2$ is also allowed);*
2. *all the skewfields $D_i = E_i$ are commutative, and the local Quillen-Lichtenbaum Conjecture holds for all local fields $E_{i\wp}$, $\wp$ above q;*
3. *for all $\wp \in S_\Lambda$ and for all i, the local degree $deg(D_{i\wp})$ is not divisible by the residue characteristic, and the local Quillen-Lichtenbaum Conjecture holds for all local fields $E_{i\wp}$, $\wp$ above q.*

V. Class Groups of Group Rings

We now apply the results of the previous section to the case of group rings of finite groups. Let G be a finite group and let - as before - $\mathcal{O}_F$ denote the ring of integers in a number field F. Let C_F be a set of representatives of irreducible F-characters. Under the assumptions of Theorem 4.4 we obtain a surjection

$$\prod_{\chi \in C_F} H^0(F(\chi), \mathbb{Q}_q/\mathbb{Z}_q(n)) \longrightarrow Cl_{2n-1}(\mathcal{O}_F[G])(q)$$

for each odd prime number q dividing $|G|$. We note that for a skewfield D occurring in the decomposition of $F[G]$ the local degrees are prime to the residue characteristic except in the case of a dyadic prime, where the local degree may be 2 ([O1, Theorem 1.10 ii]). The latter case does not occur if for instance the 2-Sylow subgroup of G is elementary abelian.

We give some examples:

Theorem 5.1. *Let S_r be the symmetric group on r letters, and let $n \geq 0$. Then $Cl_{4n+1}(\mathbb{Z}[S_r])$ is a finite 2-torsion group, and the only possible odd torsion in $Cl_{4n-1}(\mathbb{Z}[S_r])$ can occur for odd primes q such that $\frac{q-1}{2} \mid n$.*

Proof. This follows from Theorem 4.4.2, since the simple factors of the group algebra $\mathbb{Q}[S_r]$ are matrix rings over $\mathbb{Q}$ [CR2, Theorem 75.19].

If we assume the local Quillen-Lichtenbaum Conjecture, then we obtain complete results for Dihedral groups as well.

Theorem 5.2. *Let D_{2r} be the Dihedral group with $2r$ elements. If the local Quillen-Lichtenbaum Conjecture is true, then $Cl_{4n+1}(\mathbb{Z}[D_{2r}])$ is a finite 2-torsion group.*

Proof. This follows from Theorem 4.4.2, since all skewfields appearing in the simple components of the group algebra of D_{2r} are commutative and totally real [CR1, Example 7.39].

Remark. In the previous two theorems one can replace $\mathbb{Z}$ by the ring of integers in a number field $F/\mathbb{Q}$ which is unramified at all primes dividing the order of the group.

VI. Class Groups of Group Rings, Homomorphisms, and Homogeneous Functions

Let G be a finite group. In this final section we give some alternative descriptions of the group

$$\prod_{\chi \in C_F} H^0(F(\chi), \mathbb{Q}_q/\mathbb{Z}_q(d)),$$

where q divides $|G|$, and $d \geq 1$.

First we prove a general result about modules over the absolute Galois group $\Omega_F = Gal(\bar{F}/F)$ of a number field F.

Theorem 6.1. *Let F be a number field with algebraic closure $\bar{F}$. Let $\Omega = \Omega_F = Gal(\bar{F}/F)$, and let M be an Ω-module. Let G be a finite group, and denote its representation ring by*

$$RG = \bigoplus_{\chi \text{ abs irr}} \mathbb{Z}\chi.$$

There is an isomorphism

$$\phi : Hom_\Omega(RG, M) \xrightarrow{\cong} \prod_{\chi \in C_F} M^{Gal(\bar{F}/F(\chi))}.$$

Proof. Define $\phi(f) = (f(\chi))_\chi$. It is clear that ϕ is well-defined. Since RG is generated by the absolutely irreducible characters, it follows immediately that ϕ is one-to-one. To see that it is onto as well, let $x \in M^{Gal(\bar{F}/F(\chi))}$. Define $f \in Hom_\Omega(RG, M)$ by

$$f(\eta) = \begin{cases} x^\sigma & \text{if } \eta = \chi^\sigma \\ 0 & \text{otherwise,} \end{cases}$$

and extend linearly. It is straightforward to check that f is well-defined and $\phi(f) = x$. This completes the proof. ■

If we take $M = \mathbb{Q}_q/\mathbb{Z}_q(d)$, then

$$M^{Gal(\overline{F}/F(\chi))} \cong H^0(F(\chi), \mathbb{Q}_q/\mathbb{Z}_q(d)),$$

and we obtain the following corollary.

Corollary 6.2. *Let G be a finite group and $d \geq 1$. There is an isomorphism*

$$Hom_{\Omega_F}(RG, \mathbb{Q}_q/\mathbb{Z}_q(d)) \cong \prod_{\chi \in C_F} H^0(F(\chi), \mathbb{Q}_q/\mathbb{Z}_q(d)).$$

This description is reminiscent of Fröhlich's description of the class group of an integral group ring [Fr, p. 20].

We would like to point out that in general

$$\prod_{q||G|} H^0(F(\chi), \mathbb{Q}_q/\mathbb{Z}_q(d)) \neq H^0(F(\chi), \mathbb{Q}/\mathbb{Z}(d)).$$

However, if $F = \mathbb{Q}$ and G is abelian, then for *odd* d we have equality up to 2-torsion. In this case we can give another description of the cohomology groups in terms of homogeneous functions on the dual of G analogous to the case $d = 1$ treated in [DL]. First we recall some definitions.

Definition 6.3. [DL] Let G be a finite abelian group, d a non-negative integer. A function $f : G \longrightarrow \mathbb{Q}/\mathbb{Z}$ is *homogeneous of degree d* if $f(nx) = n^d f(x)$ for all $x \in G$ and all $n \in \mathbb{N}$ such that $(n, o(x)) = 1$. Denote by $\mathrm{Hmg}^d(G)$ the (finite) abelian group (under pointwise addition) of all homogeneous functions of degree d on G.

A subgroup K of G is called *cocyclic* if the quotient G/H is cyclic. Let $\phi : K \longrightarrow \mathbb{Q}/\mathbb{Z}$ be a character. Then the induced character

$$\phi_K : G \longrightarrow \mathbb{Q}/\mathbb{Z},$$

defined by

$$\phi_K(x) = \begin{cases} \phi(x) & \text{if } x \in K \\ 0 & \text{otherwise,} \end{cases}$$

is homogeneous of degree 1, and is called a *cocyclic function.* Let $\mathrm{Coc}(G)$ denote the subgroup of $\mathrm{Hmg}(G) = \mathrm{Hmg}^1(G)$ generated by all cocyclic functions.

In [DL, Theorem 4.1] it is shown that if G is a finite abelian group of odd order, then there is a surjection from $\mathrm{Hmg}(\widehat{G})$ onto $Cl_1(\mathbb{Z}G)$ with kernel $\mathrm{Coc}(\widehat{G})$.

Theorem 6.4. *Let G be a finite group. For all odd $d > 0$, there is an isomorphism*

$$Hmg^d(\widehat{G}) \xrightarrow{\cong} \prod_{\chi \in C_0} H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d)),$$

given by the evaluation map. The index set C_0 is a set of representatives of the non-trivial $\mathbb{Q}$-irreducible characters of G.

Proof. First observe that $H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d))$ is equal to the fixed points of $\mathbb{Q}/\mathbb{Z}$, viewed as roots of unity, under the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\chi))$, twisted d times. Since $\mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/n$ and $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\chi))$ is an inverse limit of the Galois groups of finite extensions, we obtain that $H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d))$ is a direct limit of the cohomology of finite extensions of $\mathbb{Q}(\chi)$. Thus, $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\chi))$ acts on any given element of $\mathbb{Q}/\mathbb{Z}(d)$ via multiplication with u^d for some integer u. Since $im(\chi)$ is fixed under this action, it follows that $u^d \equiv 1 \pmod{o(\chi)}$. Since d is assumed to be odd, we get that

$$H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d)) = \{a \in \mathbb{Q}/\mathbb{Z} | u^d a = a \text{ for all } u \equiv 1 \pmod{o(\chi)}\}.$$

Now define

$$\Phi : \mathrm{Hmg}^d(\widehat{G}) \longrightarrow \prod_{\chi \in C_0} H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d))$$

by $f \mapsto (f(\chi))_\chi$. To see that Φ is well-defined, observe that, if $u \equiv 1 \pmod{o(\chi)}$, then

$$f(\chi) = f(u\chi) = u^d f(\chi),$$

hence $f(\chi) \in H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d))$. It is clearly a homomorphism and one-to-one.

Let $x \in H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d))$ for $\chi \in \widehat{G}$. Define $f \in \mathrm{Hmg}(\widehat{G})$ by

$$f(\eta) = \begin{cases} n^d x & \text{if } \eta = n\chi \\ 0 & \text{otherwise.} \end{cases}$$

Then f is well-defined, homogeneous of degree d, and $f(\chi) = x$ so that $\Phi(f) = x$. This shows that Φ is onto, and the proof of the theorem is complete. ∎

If $d = 1$, then

$$\prod_{\chi \in C_0} H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(1)) \cong \prod_{\chi \in C_0} im(\chi),$$

and we recover part of [DL, Theorem 4.3].

Corollary 6.5. *For any finite abelian group G and any odd $d > 0$ there is an isomorphism*

$$Hmg^d(\widehat{G}) \longrightarrow Hom_\Omega(RG, \mathbb{Q}/\mathbb{Z}(d))/H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(d)),$$

given by $f \mapsto \tilde{f}$, where $\tilde{f}(\chi) = f(\chi)$. Furthermore, the factor $H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(d))$, arising from the trivial character, has order two.

We now define cocyclic functions of degree d.

Definition 6.6. [DL] Let G be a finite abelian group, $d \geq 0$ an integer.

(1) Let $G[d]$ be the (finite) abelian group with generators $[x]$, $x \in G$, and relations $n^d[x] - [nx]$ for all integers n such that $(n, o(x)) = 1$.

(2) Let K be a cocyclic subgroup of $G[d]$, that is, $G[d]/K$ is cyclic. Let $\phi : K \longrightarrow \mathbb{Q}/\mathbb{Z}$ be a character of K, and let $\phi_K : G[d] \longrightarrow \mathbb{Q}/\mathbb{Z}$ be the induced character on $G[d]$. Now define

$$\varphi_K : G \longrightarrow \mathbb{Q}/\mathbb{Z}$$

by $\varphi_K(\eta) = \phi_K([\eta])$. If $(n, o(\eta)) = 1$, then

$$\varphi_K(n\eta) = \phi_K([n\eta]) = \phi_K(n^d[\eta]) = n^d\phi_K([\eta]) = n^d\varphi_K(\eta).$$

Hence $\varphi_K \in \mathrm{Hmg}^d(G)$. Call φ_K a *cocyclic function of degree* d. Denote by $\mathrm{Coc}^d(G)$ the subgroup of $\mathrm{Hmg}^d(G)$ generated by all cocyclic functions of degree d.

It was shown in [DL], that there is an isomorphism

$$\mathrm{Hmg}^d(G) \cong \mathrm{Hom}(G[d], \mathbb{Q}/\mathbb{Z}).$$

Proposition 6.7. *Let G be a finite abelian group. There is a homomorphism*

$$\Psi : G[d] \otimes_{\mathbb{Z}} \mathbb{Z}[G[d]] \xrightarrow{\Psi'} \prod_{\chi \in C_0} H^0(\mathbb{Q}(\chi), \mathbb{Q}/\mathbb{Z}(d)) \xrightarrow{\cong} Hmg^d(\widehat{G})$$

whose image in $Hmg^d(\widehat{G})$ is exactly $Coc^d(\widehat{G})$.

Proof. To begin with, observe that there is a (non-canonical) isomorphism

$$\widehat{G}[d] \cong G[d] \cong \widehat{G[d]},$$

which we shall denote by $[\chi] \mapsto \widehat{[\chi]}$. Let Ψ'_χ denote the image of Ψ' in the component indexed by χ. Then define

$$\Psi'_\chi(x \otimes y) = \begin{cases} \widehat{[\chi]}(x) & \text{if } \widehat{[\chi]}([y]) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To see that Ψ'_χ is well defined observe that Ψ'_χ is invariant under multiplication with u^d for any integer u which is congruent to 1 modulo the order of χ, since $\widehat{[\chi]} = u^d\widehat{[\chi]}$.

Define K to be the kernel of the evaluation map $\widehat{G[d]} \longrightarrow \mathbb{Q}/\mathbb{Z}$ given by $\widehat{[\eta]} \mapsto \widehat{[\eta]}(y)$. Then K is a cocylic subgroup of $\widehat{G[d]}$. Define a character $\phi : K \longrightarrow \mathbb{Q}/\mathbb{Z}$ by $\widehat{[\eta]} \mapsto \widehat{[\eta]}(x)$. The induced character

$$\phi_K : \widehat{G[d]} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is given by

$$\phi_K(\widehat{[\eta]}) = \begin{cases} \widehat{[\eta]}(x) & \text{if } \widehat{[\eta]} \in K \\ 0 & \text{otherwise,} \end{cases}$$

that is, $\phi_K = \Psi_\chi(x \otimes y)$. To see that the corresponding function φ_K lies in $\mathrm{Hmg}^d(\widehat{G})$, observe that, if $(n, o(\eta)) = 1$ and $\widehat{[n\eta]} \in K$, then

$$\varphi_K(n\eta) = \phi_K(\widehat{[\eta]}) = \widehat{[n\eta]}(x) = n^d \widehat{[\eta]}(x) = n^d \varphi_K(\eta),$$

since $\widehat{[\eta]} \in K$ as well. This shows that the image of Ψ lies in $\mathrm{Coc}^d(\widehat{G})$.

Given any cocyclic function φ_K of degree d on $\widehat{G}$, then K is the kernel of a character on the dual of $\widehat{G[d]}$, which is isomorphic to $G[d]$. Hence K is the kernel of the evaluation map of a group element $x \in G[d]$. Using the isomorphism between $G[d]$ and its double dual once again, we obtain $y \in G[d]$ such that $\phi_K(\eta) = \eta(y)$. Then $\phi_K = \Psi(x \otimes y)$. This shows that all the generators of $\mathrm{Coc}^d(\widehat{G})$ are in $im(\Psi)$, hence $im(\Psi) = \mathrm{Coc}^d(\widehat{G})$.■

Remark. For $d = 1$, it follows from [ADS, Theorem 2.10], [ADOS, Theorem 1.8] and [DL, Theorem 4.1], that the image of the map Ψ' in Proposition 6.7 is equal to the image of a certain K_2-group, and this image in turn is equal to $\mathrm{Coc}^1(\widehat{G})$. It would be interesting to see if for general odd d one can relate $\mathrm{Coc}^d(\widehat{G})$ to K-theory.

References

[ADS] R. C. Alperin, R. K. Dennis and M. R. Stein, *SK_1 of Finite Abelian Groups*, Inventiones Math (1985), 1–18.

[ADOS] R. C. Alperin, R. K. Dennis, R. Oliver and M. R. Stein, *SK_1 of Finite Abelian Groups II*, Inventiones Math (1987), 253–302.

[Ba] A. Bak, *K-theory of Forms*, Annals of Math. Studies 98, Princeton University Press, Princeton, NJ, 1981.

[BR] A. Bak and U. Rehmann, *The Congruence Subgroup and Metaplectic Problems for $SL_{n\geq 2}$ of Division Algebras*, J. Algebra **78** (1982), 475–547.

[Bn] G. Banaszak, *Generalization of the Moore Exact Sequence and the Wild Kernel for Higher K-groups*, Compositio Math. **86** (1993), 281–305.

[BM] M. Bökstedt and I. Madsen, *Algebraic K-theory of Local Number Fields: The Unramified Case*, preprint.

[Bo] A. Borel, *Cohomologie Reelle Stable de Groupes S-arithmetiques Classiques*, Comptes Rendus de l'Academie des Sci. **274** (1972), 1700–1703.

[CR1] C. W. Curtis and I. Reiner, *Methods of Representation Theory I*, Wiley-Interscience, New York, 1981.

[CR2] ———, *Methods of Representation Theory II*, Wiley-Interscience, New York, 1987.

[DL] R. K. Dennis and R. C. Laubenbacher, *Homogeneous Functions and Algebraic K-theory*, preprint.

[DF] W. G. Dwyer and E. M. Friedlander, *Algebraic and Etale K-theory*, Trans. Amer. Math. Soc. **292** (1985), 247–280.

[Fr] A. Fröhlich, *Galois Module Structure of Algebraic Integers*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 1, Springer Verlag, New York, 1983.

[K] B. Kahn, *On the Lichtenbaum-Quillen Conjecture*, Algebraic K-theory and Algebraic Topology, Nato Proceedings, Lake Louise, vol. 407, Kluwer, 1993, pp. 147–166.

[Ka] M. Karoubi, *Localisation de Formes Quadratiques I*, Ann. Scient. École Norm. Sup. **7** (1974), 359–404.

[Ke1] M. E. Keating, *A Transfer Map in K-Theory*, J. London Math. Soc. **16** (1977), 38–42.

[Ke2] ———, *Corrigendum: A Transfer Map in K-Theory*, J. London Math. Soc. **18** (1978), 14.

[Ku1] A. O. Kuku, *SK_n of Orders and G_n of Finite Rings*, Algebraic K-Theory (M. R. Stein, ed.), Springer Lecture Notes 551, Springer Verlag, New York, 1976, pp. 60–68.

[Ku2] ———, *Some Finiteness Results in the Higher K-theory of Orders and Group-Rings*, Topology and its Applications **25** (1987), 185–191.

[MS] A. S. Merkurjev and A. A. Suslin, *K-Cohomology of Severi-Brauer Varieties and the Norm Residue Homomorphism*, Math. USSR Izv. **21** (1983), 307–340.

[Mi] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Mathematics Studies Nr. 72, Princeton University Press, Princeton NJ, 1971.

[Mo] C. Moore, *Group Extensions of p-adic and Adelic Linear Groups*, I.H.E.S. Publications **35** (1968), 5–70.

[Ng] T. Nguyen Quang Do, *Analogues Supérieures du Noyeau Sauvage*, Sém. de Théorie des Nombres, Bordeaux **4** (1992), 263–271.

[Ol] R. Oliver, *Whitehead Groups of Finite Groups*, Cambridge University Press, New York, 1988.

[PR] G. Prasad and M. S. Raghunathan, *On the Congruence Subgroup Problem: Determination of the "Metaplectic" Kernel*, Invent. Math. **71** (1983), 21–42.

[Qu] D. Quillen, *On the Cohomology and K-theory of the General Linear Groups Over a Finite Field*, Ann. of Math **96** (1972), 552–586.

[Re] U. Rehmann, *A Survey of the Congruence Subgroup Problem*, Algebraic K-theory (R. K. Dennis, ed.), Springer Lecture Notes 966, Springer Verlag, New York, 1982.

[Sc] P. Schneider, *Über Gewisse Galoiscohomologiegruppen*, Math. Zeit **168** (1979), 181–205.

[So1] C. Soulé, *K-théorie des Anneaux d'Entiers de Corps de Nombres et Cohomologie Étale*, Invent. Math. **55** (1979), 251–295.

[So2] ———, *Groupes de Chow et K-théorie de Varietés sur un Corps Fini*, Math. Ann. **268** (1984), 317–345.

[Su] A. A. Suslin, *Torsion in K_2 of Fields*, K-Theory **1** (1987), 5–29.

[SY] A. A. Suslin and A. V. Yufryakov, *K-theory of Division Algebras*, Soviet Math. Dokl. **33** (1986), 794–798.

[Vo] T. Vorst, *Localization of the K-theory of Polynomial Extensions*, Math. Ann. **244** (1979), 33–53.

[Wa] J. B. Wagoner, *Continuous Cohomology and p-adic K-theory*, Algebraic K-Theory (M. R. Stein, ed.), Springer Lecture Notes 551, Springer Verlag, New York, 1976, pp. 241–248.

[Wal] C. T. C. Wall, *On the Classification of Hermitian Forms V. Global Rings*, Invent. Math **23** (1974), 261–288.

[Yu] A. V. Yufryakov, *On the Group K_2 of a Local Division Ring*, Soviet Math. Dokl. **34** (1987), 451–453.

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