ALGEBRAIC K-THEORY OF MONOID RINGS

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Are all finitely generated projective $k[t_1,\dots,t_d]$-modules free for an arbitrary field k and arbitrary $d \in \mathbf{N}$? This question, set in Serre's famous paper FAC in 1955, inspired an enormous activity of algebraists worldwide. The activity culminated in two independent confirmations of the question in 1976 by Quillen and Suslin. In the meanwhile the algebraic K-theory was created, in which one of the central topics was the so called homotopy properties of algebraic K-functors. Grothendieck-Serre's classical theorem on K_0-regularity of a regular ring was a starting point here. These two mentioned results, concerning respectively unstable and stable homotopic behavior of the classical functor K_0, proved to be a fruitful domain for further search and generalizations. Almost the whole of this activity, save few exceptional cases, was concentrated on the consideration of polynomial ring extensions. On the other hand polynomial (and Laurent polynomial) rings are simplest (and the only regular) representatives of monoid rings and we could ask what results, established previuosly for polynomial rings, generalize to the latter ones. Below we shall survey the progress made in this direction during last years. A brief account of this search looks as follows.

All the considered monoids and rings we deal with are supposed to be commutative. In addition monoids are cancellative and, if the contrary is not explicitly stated, torsion free (that is the corresponding groups of fractions are so). A monoid M is called normal if (writing additively) $nx \in M$ for $n \in \mathbf{N}$ and $x \in K(M)$ (the group of fractions) imply $x \in M$; M is called seminormal if $2x \in M$ and $3x \in M$ imply $x \in M$; we shall say that M is c-divisible for some $c \in \mathbf{N}$ if for any $x \in M$ there exists $y \in M$ for which $cy = x$ (observe that a c-divisible monoid is always seminormal). Later on $\mathbf{Z}_+$ will denote the additive monoid of nonnegative rational integers and $\mathbf{Q}_+$ that of nonnegative rationals. It is well known that a monoid domain $R[M]$ is normal (seminormal) if and only if the domain R and the monoid M are normal (seminormal respectively). We remark that analogous statement for a completion of a monoid domain (with respect to the natural augmentation ideal) is also valid [G5].

In order to present a complete picture we start with the following relatevely old (1986) result, which confirms Anderson's conjecture:

THEOREM A [G1]. *For any principal ideal domain (PID) R and a monoid M the following conditions are equivalent*

(a) $\mathrm{Pic}(R[M]) = 0$,

(b) $K_0(R[M]) = \mathbf{Z}$,

(c) *Finitely generated projective $R[M]$-modules are all free*,

(d) *M is seminormal.*

THEOREM B [G1+G5]. *For any regular ring R and a monoid M we have $SK_0(R) = SK_0(R[M])$ and $K_{-i}(R) = K_{-i}(R[M]) = 0$ (Bass negative K-groups); the following conditions are equivalent*

(a) $\mathrm{Pic}(R) = \mathrm{Pic}(R[M])$,

(b) $K_0(R) = K_0(R[M])$,

(c) *M is seminormal.*

The above equality concerning SK_0 is still valid for monoids M for which $K(M)_{tor}$ is a p-group (for some prime p) if in addition $p = 0$ in R [G5].

R.G.Swan deduced from [G1] the most general non-stable result:

THEOREM C [S]. (a) *For a monoid M and a Dedekind domain R all finitely generated projective $R[M]$-modules are of type* free $\oplus$ rank1. (*This confirms a conjecture of Murthy.*)

(b) *For any affine regular domain R and a seminormal monoid M without nontrivial units all finitely generated projective $R[M]$-modules are extended from R.*

By Popescu's theorem on regular domains containing a field we immediately obtain that the statement (b) genenarizes to arbitrary regular rings just containing a subfield (we recall that the case when M is free corresponds to Bass-Quillen conjecture, proven for geometric case by Lindel).

THEOREM D [G2]. *For a not necessarily torsion free monoid M the following conditions are equivalent*:

(a) $\mathrm{Pic}(R[M]) = 0$ *for all PID's R,*

(b) *M is torsion free and seminormal.*

REMARK. Recently M.Masuda, L.Moser-Jauslin and T.Petrie succeeded in establishing positive answer to the equivariant Serre Problem for abelian groups (that every G-vector bundle over a G-module is trivial whenever G is abelian) by connecting it with the corresponding Quotient Problem, which in its turn reduces to the special case of Th.A.

The situation changes radically when we consider higher K-groups (we always mean Quillen's K-groups):

THEOREM E [G6]. *For any K_2-regular ring R and any intermediate finitely generated monoid $\mathbf{Z}_+^r \subset M \subset \mathbf{Q}_+^r$, where r is an arbitrary natural number, the following conditions are equivalent*:

(a) $M \approx \mathbf{Z}_+^r$,

(b) *$R[M]$ is K_1-regular*,

(c) *M is seminormal and $SK_1(R) = SK_1(R[M])$,*

and, if in addition, $\Omega^1_{R/\mathbf{Z}} \neq 0$

(d) $SK_1(R) = SK_1(R[M])$.

First explicit examples of nontrivial elements in $SK_1(R[M])$ for certain rank 2 monoids (i.e. when $r = 2$) were constructed by V. Srinivas. Actually, as it follows from [G6], Theorem E is valid almost for all (and conjecturally for all) finitely generated monoids M without nontrivial units (and we explicitly construct the corresponding nontrivial elements in SK_1-groups). This theorem implies that rings of type $R[M]$ for M as above are not K_i-regular for $i > 0$. However there still exists a wide class of monoids, for which we can establish "positive" results:

THEOREM F [G2+G5]. *Let $c > 1$ be a natural number and M* a *c-divisible monoid. Then:*

(a) *$SL_r(R[M]) = E_r(R[M])$ for any Euclidean domain R whenever $r > 2$,*

(b) *$K_1(R) = K_1(R[M])$ for any regular ring R whenever M has no nontrivial units,*

(c) *$K_i(R) = K_i(R[M])$ for any regular ring R whenever $\mathbf{Z}_+^r \subset M \subset \mathbf{Q}_+^r,\ i, r \in \mathbf{N}$.*

THEOREM G [M]. *Let M be a c-divisible monoid without nontrivial units for some natural $c > 1$. Then $K_2(R) = K_2(R[M])$ for any regular ring R.*

Linking up the aforementioned "negative" and "positive" results concerning K_1 and using transfer maps in algebraic K-theory (and Th. A) one gets the following

COROLLARY. *Let M be an arbitrary finitely generated intermediate seminormal monoid $\mathbf{Z}_+^r \subset M \subset \mathbf{Q}_+^r$ ($r \in \mathbf{N}$) and $c > 1$ a natural number. Assume R is a domain of characterisitic 0. Consider $R[M]$ as an $R[M]$-module via the R-algebra homomorphism $R[M] \to R[M]$ induced by $m \mapsto m^c$. Then homological dimension of this module is finite if and only if M is free and, clearly, in this case the mentioned dimension is equal to 0.*

We should like to have a "pure commutative algebraic" proof of this result (probably valid for all rings and all finitely generated monoids without nontrivial units)

The following result concerns certain class of monoids: s.c. monoids of Φ-simplicial growth. Without describing the details we only mention here that this class generalizes the class of intermediate monoids $\mathbf{Z}_+^r \subset M \subset \mathbf{Q}_+^r$ exactly in the same way as the class of simplicial growth convex polyhedra generalizes the class of arbitrary simplices (of arbitrary dimensions), where a finite convex polyhedron P is said to be of simplicial growth if there exists a sequence of convex polyhedra $P_1 \subset P_2 \subset \ldots \subset P_n = P$ (for some natural n) such that P_1 and the closures (in the sense of Euclidean topology) of $P_i \backslash P_{i-1} (i \in [2, n])$ are all simplices.

THEOREM H [G3+G4]. *Let R be a noetherian ring of finite Krull dimension d and M a monoid of Φ-simplicial growth. Then the group of elementary matrices $E_n(R[M])$ acts transitively on the set of unimodular n-rows $Um_n(R[M])$ whenever $n \geq \max(d+2, 3)$.*

The classical case of this theorem (i.e. when $M = \mathbf{Z}_+^r$) is due to Suslin. We remark that the case of monoids M of type $\mathbf{Z}_+^r \subset M \subset \mathbf{Q}_+^r$ is also nontrivial and that in order to involve all monoids one has to treat the monoids to whom correspond arbitrary finite convex polyhedra. It should be also mentioned that Theorem H naturally generalizes to the monoids for which $K(M)_{tor}$ is either cyclic or a p-group for some prime p (providing $p = 0$ in R) [G5]. Theorem H evidently implies (for monoid rings) an improvement of Bass-Vaserstein general estimate for surjective K_1-stabilization. In view of results of Bhatwadekar-Lindel-Rao, concerning K_0-stabilizations for polynomial rings, we can hope for such an improvement of K_0-stabilizations for monoid rings (generalizing Serre's Unimodular element and Bass' Cancellation theorems respectively).

Basing on [G2] and the technique developed by Lindel's group G. Schabhuser proved that the expected injective K_0-stabilization for a monoid ring (i.e. the corresponding cancellation property for projective modules) actually is the case whenever the monoid is c-divisible for some $c > 1$.

It turns out that we can involve "generalized Discrete Hodge Algebras" in the above story as follows.

THEOREM I [G5]. (a) *For a ring R and a not necessarily torsion free monoid M the equality $K_0(R) = K_0(R[M])$ ($SK_1(R) = SK_1(R[M])$) implies $K_0(R) = K_0(R[M]/RI)$ ($SK_1(R) = SK_1(R[M]/RI)$ respectively), where I is an arbitrary proper ideal of M; the corresponding implications in the non-stable situation also hold.*

(b) *For a not necessarily commutative ring R and a (commutative, cancellative, torsion free) c-divisible monoid M for some $c > 1$ the equality $K_i(R) = K_i(R[M])$ implies $K_i(R) =$*

$K_i(R[M]/RI)$, where $i \in \mathbf{N}$ and I is an arbitrary proper radical ideal of M ($\sqrt{I} = I$).

The special case of (a) for "ordinary" Discrete Hodge Algebras (i.e. M is free) is due to Vorst.

The proofs of these results involve the corresponding algebraic tools (Quillen's local-to-global principle, various generalizations of Horrocks' monic inversion theorem, symbols, excision in algebraic K-theory, special automorphisms of polynomial and monoid rings etc.) combined with purely convex geometrical constructions (relative interiors, homothetic transformations, special decompositions of polyhedra etc.).

Appendix. In view of the results presented above it is natural to ask whether one can always distinguish monoid rings corresponding to nonisomorphic monoids. Here is a relevant issue (for low rank monoids).

THEOREM J [G7]. *Let M and N be arbitrary finitely generated submonoids of $\mathbf{Z}^2$ and R a (commutative) ring. Then M and N are isomorphic whenever $R[M]$ and $R[N]$ are isomorphic as R-algebras.*

In the talk we shall demonstrate on one example of pure categorial approach (without any K-theoretical work) how the monoid rings can be treated. If time permits a program for further search along with several comments will be outlined.

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