ON THE p-TYPICAL CURVES IN QUILLEN'S K-THEORY

by Lars Hesselholt

Contents

Introduction

Twenty years ago Bloch, [Bl], introduced the complex $C_*(A;p)$ of p-typical curves in K-theory and outlined its connection to the crystalline cohomology of Berthelot-Grothendieck. However, to prove this connection Bloch restricted his attention to the symbolic part of K-theory, since only this admitted a detailed study at the time. In this paper we evaluate $C_*(A;p)$ in terms of Bökstedt's topological Hochschild homology. Using this we show that for any smooth algebra A over a perfect field k of positive characteristic, $C_*(A;p)$ is isomorphic to the de Rham-Witt complex of Bloch-Deligne-Illusie. This confirms the outlined relationship between p-typical curves in K-theory and crystalline cohomology in the smooth case. In the singular case, however, we get something new. Indeed, we calculate $C_*(A;p)$ for the ring $k[t]/(t^2)$ of dual numbers over k and show that in contrast to crystalline cohomology, its cohomology groups are finitely generated modules over the Witt ring $W(k)$.

Let A be a ring, by which we shall always mean a commutative ring, and let $K(A)$ denote the algebraic K-theory spectrum of A. Here we use the term spectrum in the sense of topology. More generally, if $I \subset A$ is an ideal, $K(A,I)$ denotes the relative algebraic K-theory, that is, the homotopy theoretical fiber of the map $K(A) \to K(A/I)$. We define the *curves on* $K(A)$ to be the homotopy limit of spectra

$$C(A) = \operatorname*{holim}_{\overleftarrow{n}} \Omega K(A[X]/(X^n),(X)).$$

The homotopy groups $C_*(A) = \pi_* C(A)$ are given by Milnor's exact sequence

$$0 \to \operatorname*{\underleftarrow{lim}}_{n}{}^{(1)} K_{*+2}(A[X]/(X^n),(X)) \to C_*(A) \to \operatorname*{\underleftarrow{lim}}_{n} K_{*+1}(A[X]/(X^n),(X)) \to 0,$$

so in particular, $C_0(A) = \mathbf{W}(A)$, the big Witt ring of A. We note that originally Bloch defined the curves on $K(A)$ to be the inverse limit on the right hand side. So our definition differs from his in that we include a possible $\lim^{(1)}$-term. Furthermore, Bloch defined a pairing on $C(A)$ which makes $C(A)$ a homotopy associative ring spectrum. In particular, this gives a ring homomorphism from $\mathbf{W}(A)$ to the ring of cohomology operations in $C(A)$. So when A is a $\mathbb{Z}_{(p)}$-algebra, the idempotents of $\mathbf{W}(A)$ gives a splitting

$$C(A) \simeq \prod_{(k,p)=1} C(A;p),$$

as a product of copies of a spectrum $C(A;p)$, the *p-typical* curves on $K(A)$.

For any ring A, Bökstedt, [Bö], has defined its topological Hochschild homology $T(A)$. This is a genuine S^1-equivariant spectrum, and moreover, there are two maps

$$R, F\colon T(A)^{C_{p^n}} \to T(A)^{C_{p^{n-1}}}$$

of the fixed sets under the cyclic groups of order p^n and p^{n-1}. The map F is the obvious inclusion while the map R, introduced by Bökstedt, Hsiang and Madsen, [BHM], is given by the *cyclotomic* structure of $T(A)$. We write

$$\mathrm{TR}(A;p) = \underset{\overleftarrow{R}}{\mathrm{holim}}\, T(A)^{C_{p^n}}.$$

Theorem A. *Let A be a $\mathbb{Z}/p^j$-algebra, then $C(A;p) \simeq \mathrm{TR}(A;p)$.*

The proof is based on a recent result of McCarthy, [Mc], which states that the cyclotomic trace of [BHM],

$$\mathrm{trc}\colon K(A) \to \mathrm{TC}(A),$$

from K-theory of a certain topological version of Connes' cyclic homology, induces an equivalence of the relative theories $K(A, I)$ and $\mathrm{TC}(A, I)$ after profinite completion , provided that the ideal $I \subset A$ is nilpotent. We note that for any $\mathbb{F}_p$-algebra A, $\mathrm{TR}(A;p)$ is a generalized Eilenberg-MacLane spectrum, and therefore, determined up to homotopy by its homotopy groups $\mathrm{TR}_*(A;p)$. In higher characteristic, however, this is not likely to be the case.

Let k be a perfect field of characteristic $p > 0$ and let $W_n(k)$ be its ring of p-typical Witt vectors of length n, $1 \leqslant n \leqslant \infty$. For any k-algebra A we have the de Rham-Witt complex $W_n\Omega^*_A$ of Bloch, Deligne and Illusie, [I]. It is a differential graded algebra over $W_n(k)$ whose restriction modulo p is equal to the de Rham complex Ω^*_A. The restriction, Frobenius and Verschiebung maps of Witt vectors extend to operations

$$R, F\colon W_n\Omega^*_A \to W_{n-1}\Omega^*_A, \qquad V\colon W_n\Omega^*_A \to W_{n+1}\Omega^*_A,$$

suitably compatible with the differential structure, and $W_n\Omega^*_A$ may be characterized as the universal example of such a structure. We show in paragraph 1 below that topological Hochschild homology provides another example. In particular, there are maps

$$I\colon W_n\Omega^*_A \to \pi_* T(A)^{C_{p^{n-1}}}.$$

The differential d on $\pi_* T(A)^{C_{p^{n-1}}}$ is induced from the S^1-action. In the basic case $n = 1$, it corresponds to Connes' B-operator under linearization $\pi_* T(A) \to \mathrm{HH}_*(A)$. In paragraph 2 we prove

Theorem B. *Suppose that A is a smooth k-algebra. Then the map I extends to an isomorphism*

$$I\colon W_n\Omega^*_A \otimes_{W_n(k)} S_{W_n}\{\sigma_n\} \to \pi_* T(A)^{C_{p^{n-1}}}, \qquad \deg \sigma_n = 2.$$

Moreover, $F(\sigma_n) = \sigma_{n-1}$, $V(\sigma_n) = p\sigma_{n+1}$ and $R(\sigma_n) = p\lambda_n\sigma_{n-1}$, where λ_n is a unit of $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$.

The basic case $A = k$ was proved in [HM]. The bulk of paragraph 2 is the explicit calculation of the right hand side in the case where A is a polynomial algebra. We find that it is abstractly isomorphic to the left hand side, which is known from [I], and prove that the map I is an isomorphism. The general case follows by a covering argument. If we take the limit over the restriction maps, the extra generator σ_n vanishes, and hence

Theorem C. *If A is a smooth k-algebra, then $W\Omega^*_A \cong \mathrm{TR}_*(A;p) \cong C_*(A;p)$.*

We note that Bloch proved that the same result holds if $C_*(A;p)$ is replaced by its symbolic part $SC_*(A;p)$, provided that A is local of Krull dimension less than p. The restriction on the dimension was later removed by Kato, [K].

For any scheme X over $\operatorname{Spec} k$, Berthelot, [B], has defined its crystalline cohomology,

$$H^*(X/W_n), \qquad H^*(X/W) = \varprojlim_n H^*(X/W_n).$$

It is good cohomology theory when X is proper and smooth. In particular, the cohomology groups are finitely generated $W(k)$-modules. However, if X is either not smooth or not proper, the theory behaves rather pathologically. The main theorem of [I] states that there are natural isomorphisms

$$H^*(X/W_n) \cong \mathbb{H}^*(X; W_n\Omega^*_X), \qquad H^*(X/W) \cong \mathbb{H}^*(X; W\Omega^*_X),$$

provided that X be smooth and smooth and proper, respectively. Here the right hand sides denote the hypercohomology of X with coefficients in the complexes $W_n\Omega^*_X$ and $W\Omega^*_X$ of Zariski sheaves on X. It follows that

$$H^*(X/W) \cong \mathbb{H}^*(X; C_*(-;p))$$

when X is smooth and proper. However, in the non-reduced case we get something new: in section 3.5 below we evaluate $C_*(A;p)$ for the ring $k[t]/(t^2)$ of dual numbers over k. The argument is based on [HM]. Let $m(i,j)$ be the unique natural number such that $p^{m(i,j)-1}j \leqslant 2i+1 < p^{m(i,j)}j$, then

Theorem D. *Let k be a perfect field of positive characteristic and let $X = \operatorname{Spec} k[t]/(t^2)$. Then*

i) $p > 2$: $\mathbb{H}^{2i}(X; C_*(-;p)) = \mathbb{H}^{2i+1}(X; C_*(-;p)) = \bigoplus_{1 \leqslant j \leqslant 2i+1, j \text{ odd}} W(k)/(j, p^{m(i,j)})$,

ii) $p = 2$: $\mathbb{H}^{2i}(X; C_*(-;p)) = \mathbb{H}^{2i-1}(X; C_*(-;p)) = k^{\oplus i}$,

and in both cases $\mathbb{H}^0(X; C_*(-;p)) = W(k)$.

We note that in comparison the crystalline cohomology $H^*(X/W)$ is concentrated in degree 0 and 1 and the latter is infinitely generated as a $W(k)$-module. This suggests that $\mathbb{H}^*(X; C(-;p))$ might be a more well-behaved theory than crystalline cohomology for non-reduced schemes. The proof of theorem A involves the following result, which is also of interest in its own right.

Theorem E. *Let A be a $\mathbb{Z}/p^j$-algebra. Then $K_i(A[X]/(X^n), (X))$ is a bounded p-group, i.e. any element is annihilated by p^N for some number N which may depend on i.*

The fact that these groups are p-groups has previously been proved by Weibel, [We], by quite different methods. However, the result that they are bounded is new.

In paragraphs 1 and 3 below we use the notion of G-spectrum. The reader is referred to [LMS] for this material. We shall use the term equivalence to mean a weak homotopy equivalence, *i.e.* a map of spectra which induces isomorphism of all homotopy groups, and a G-equivalence will mean a G-equivariant map which induces an equivalence of H-fixed spectra for all closed subgroups $H \subset G$. Throughout the paper, G will denote circle groups and rings will be assumed commutative without further notice.

It is a pleasure to thank Ib Madsen for helpful discussions and for his strong encouragement. I also want to thank John Klein for helpful remarks. I am particularly grateful to Teimuraz Pirashvili who suggested to me the possible link between topological cyclic homology and crystalline cohomology. The majority of the work presented here was done during my postdoctoral fellowship at the Mittag-Leffler Institute in Stockholm, Sweden, and I would like to thank the institution for its hospitality and support.

1. The complex $\mathrm{TR}_*(A;p)$.

1.1. We briefly recall the ring of p-typical Witt vectors associated to a ring A. The standard reference is [S] but see also G.M.Bergman's lecture in [M]. As a set $W(A) = A^{\mathbb{N}}$ and the ring structure is defined by the requirement that the ghost map

$$w\colon W(A) \to A^{\mathbb{N}}$$

given by the Witt polynomials

$$\begin{aligned} w_0 &= a_0 \\ w_1 &= a_0^p + pa_1 \\ w_2 &= a_0^{p^2} + pa_1^p + p^2 a_2 \\ &\vdots \end{aligned} \tag{1.1.1}$$

be a natural transformation of functors from rings to rings. We call a_i and w_i the Witt and ghost coordinates of the Witt vector $a = (a_0, a_1, \dots)$, respectively. If A is a $\mathbb{Z}[1/p]$-algebra then obviously w is a bijection. It follows that for $a, b \in W(A)$

$$\begin{aligned} a + b &= (s_0(a,b), s_1(a,b), \dots) \\ a \cdot b &= (p_0(a,b), p_1(a,b), \dots) \end{aligned}$$

for certain polynomials s_i and p_i which depend only on $a_0, \dots, a_i$. A priori, these polynomials will have coefficients in $\mathbb{Z}[1/p]$, but the Kummer congruences $x^{p^n} \equiv x^{p^{n-1}} (\bmod\ p^n)$, $x \in \mathbb{Z}$, show that they are integral. Hence $W(A)$ is well-defined for any ring. We have $1 = (1, 0, \dots) \in W(A)$.

There are natural operators, Frobenius and Verschiebung, on $W(A)$ characterized by the formulas

$$\begin{aligned} &F\colon W(A) \to W(A), \qquad F(w_0, w_1, \dots) = (w_1, w_2, \dots), \\ &V\colon W(A) \to W(A), \qquad V(a_0, a_1, \dots) = (0, a_0, a_1, \dots). \end{aligned} \tag{1.1.2}$$

A general principle asserts that a relation which holds true in ghost coordinates also holds in Witt coordinates. For a $\mathbb{Z}[1/p]$-algebra this is obvious. But a relation which holds for some ring also holds for any subring and any quotient. It follows that F is a ring homomorphism, that V is additive and that the following relations hold

$$x \cdot V(y) = V(F(x) \cdot y), \qquad FV = p, \qquad VF = \mathrm{mult}_{V(1)}\,. \tag{1.1.3}$$

When A is an $\mathbb{F}_p$-algebra, one has in addition $V(1) = p$ and $F = W(\varphi)$ where φ is the Frobenius endomorphism of A. The Teichmüller character $\omega\colon A \to W(A)$ is the multiplicative map given by $\omega(x) = (x, 0, 0, \dots)$. We will also write $\underline{x} = \omega(x)$.

The additive subgroup $V^n W(A)$ of $W(A)$ is an ideal by (1.1.3) and the quotient

$$W_n(A) = W(A)/V^n W(A)$$

is called the ring of Witt vectors of lenght n in A. The elements in $W_n(A)$ are in 1-1 correspondance with tuples $(a_0, \dots a_{n-1})$ with addition and multiplication given by the same polynomials s_i, p_i as in $W(A)$. Hence, $W(A)$ is the limit of the $W_n(A)$ over the restriction maps

$$R\colon W_n(A) \to W_{n-1}(A), \qquad R(a_0, \dots, a_{n-1}) = (a_0, \dots, a_{n-2}). \tag{1.1.4}$$

It follows that $W(A)$ is complete and seperated in the topology defined by the ideals $V^n W(A)$, $n \geqslant 1$. If k is a perfect field, then $W(k)$ is the unique complete discrete valuation ring with maximal ideal $VW(k) = VFW(k) = pW(k)$. In particular, $W(\mathbb{F}_p) = \mathbb{Z}_p$.

1.2. This section recalls the de Rham-Witt pro-complex of Bloch-Deligne-Illusie. We shall only give a very brief account of the construction and refer to [I] for details. Suppose A is an $\mathbb{F}_p$-algebra. The de Rham-Witt pro-complex associated to A is a limit system of commutative DGA's over $\mathbb{Z}$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
W_3\Omega^0_A & \xrightarrow{d} & W_3\Omega^1_A & \xrightarrow{d} & W_3\Omega^2_A & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
W_2\Omega^0_A & \xrightarrow{d} & W_2\Omega^1_A & \xrightarrow{d} & W_2\Omega^2_A & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
W_1\Omega^0_A & \xrightarrow{d} & W_1\Omega^1_A & \xrightarrow{d} & W_1\Omega^2_A & \xrightarrow{d} & \cdots
\end{array}$$

such that $W_n\Omega^0_A = W_n(A)$ and such that R extends the restriction map (1.1.4). Furthermore, there are additive maps

$$F\colon W_n\Omega^i_A \to W_{n-1}\Omega^i_A, \qquad V\colon W_n\Omega^i_A \to W_{n+1}\Omega^i_A$$

which extend the Frobenius and Verschiebung maps (1.1.2) and the following set of relations hold

$$\begin{aligned}
&RF = FR, \qquad RV = VR, \\
&FV = VF = p, \qquad FdV = d, \qquad Fd\underline{a} = \underline{a}^{p-1}d\underline{a}, \\
&F(xy) = F(x)F(y), \qquad xV(y) = V(F(x)y).
\end{aligned} \tag{1.2.1}$$

Moreover, $W_\bullet\Omega^*_A$ is *universal* with these properties, *cf.* [I], I.1.3 and I.2.17. The de Rham-Witt complex of A is defined as the limit

$$W\Omega^*_A = \lim_{\overleftarrow{R}} W_n\Omega^*_A. \tag{1.2.2}$$

We note a few easy facts. The canonical map

$$\Omega^*_{W_n(A)} \to W_n\Omega^*_A$$

is a surjection for any $n \geqslant 1$ and an isomorphism when $n = 1$. Similarly, the maps $R\colon W_n\Omega^*_A \to W_{n-1}\Omega^*_A$ are surjective for all $n \geqslant 1$. Finally, consider the composite

$$\mathbf{F}\colon W_n\Omega^i_A \xrightarrow{\varphi} W_n\Omega^i_A \xrightarrow{R} W_{n-1}\Omega^i_A$$

where φ is induced from the Frobenius endomorphism of A. One has $\mathbf{F} = p^iF$. In paragraph 2 below we recall the concrete description of the de Rham-Witt complex of a polynimial algebra.

1.3. We next recall some facts about the topological Hochschild spectrum $T(A)$. The reader is refered to [HM] where this material is treated in detail. We adopt the notation of [HM] and write G for the circle group. Finite and hence cyclic subgroups of G are denoted by C or C_r if we want to specify the order.

Firstly, $T(A)$ is a G-spectrum indexed on a complete universe $\mathcal{U}$ in the sense of [LMS]. This implies in particular that the inclusion maps

$$F_r\colon T(A)^{C_{rs}} \to T(A)^{C_s} \tag{1.3.1}$$

are companioned by 'transfer' maps going in the opposite direction,

$$V_r\colon T(A)^{C_s} \to T(A))^{C_{rs}}. \tag{1.3.2}$$

We call these maps the r'th *Frobenius* and *Verschiebung*, respectively. Secondly, $T(A)$ is a cyclotomic spectrum in the sense of [HM] §1. In particular there is an additional map

$$R_r: T(A)^{C_{rs}} \to T(A)^{C_s},$$

called the r'th *restriction*. It has the following equivariance property: Let $C \subset G$ be a subgroup of order r. The r'th root defines an isomorphism of groups $\rho_C: G \to G/C$, and consequently, we may view the G/C-spectrum $T(A)^C$ as a G-spectrum $\rho_C^{\#} T(A)^C$ via ρ_C. Then R_r is a map of G-spectra

$$R_r: \rho_{C_{rs}}^{\#} T(A)^{C_{rs}} \to \rho_{C_s}^{\#} T(A)^{C_s}. \tag{1.3.3}$$

We also define G-spectra

$$\mathrm{TR}(A) = \operatorname*{holim}_{\overleftarrow{R}} \rho_C^{\#} T(A)^C, \qquad \mathrm{TR}(A;p) = \operatorname*{holim}_{\overleftarrow{R}} \rho_{C_{p^n}}^{\#} T(A)^{C_{p^n}} \tag{1.3.4}$$

and note that F_r and V_r induce selfmaps of these which we again denote F_r and V_r. Finally, $T(A)$ is a ring spectrum in a very strong sense, see [HM], proposition 1.7.1. In particular, for any $C \subset G$, the homotopy groups $\pi_* T(A)^C$ form a graded commutative ring. The following relations are proved in [HM] lemma 2.3,

$$\begin{array}{ll} (1) & R_r(xy) = R_r(x)R_r(y) \\ (2) & F_r(xy) = F_r(x)F_r(y), \qquad V_r(F_r(x)y) = xV_r(y) \\ (3) & F_rV_r = r, \qquad V_rF_r = V_r(1) \\ (4) & F_rV_s = V_sF_r, \qquad \text{if } (r,s)=1. \\ (5) & R_rF_s = F_sR_r, \qquad R_rV_s = V_sR_r \end{array} \tag{1.3.5}$$

We will mainly be interested in the fixed point spectra of $T(A)$ for the cyclic p-groups. When there is no danger of confusion we shall write R, F and V instead of F_p, R_p and V_p. We note that by [HM], theorem 2.3,

$$\pi_0 T(A)^{C_{p^n}} \cong W_{n+1}(A), \tag{1.3.6}$$

such that $\pi_0 R$, $\pi_0 F$ and $\pi_0 V$ corresponds to the restriction, Frobenius and Verschiebung of Witt vectors, respectively. We also recall from [HM], theorem 1.2 that there is a cofibration sequence of non-equivariant spectra

$$T(A)_{hC_{p^n}} \xrightarrow{N} T(A)^{C_{p^n}} \xrightarrow{R} T(A)^{C_{p^{n-1}}}, \tag{1.3.7}$$

where the spectrum on the left is the homotopy orbit spectrum or Borel construction. The spectra in this sequence are all $T(A)^{C_{p^n}}$-modules, and therefore by (1.3.6), the associated homotopy long exact sequence is a seqence of $W_{n+1}(A)$-modules. Moreover, there is a first quadrant homology type spectral sequence of $W_{n+1}(A)$-modules

$$E^2 = H_*(C_{p^n}; (F^n)^*(\pi_* T(A))) \Rightarrow \pi_* T(A)_{hC_{p^n}}. \tag{1.3.8}$$

The homotopy groups $\pi_* T(A)$ are A-modules which we view as $W_{n+1}(A)$-modules $(F^n)^*(\pi_* T(A))$ via the iterated Frobenius $F^n: W_{n+1}(A) \to A$. This specifies the $W_{n+1}(A)$-module structure on the E^2-term. The differentials are $W_{n+1}(A)$-linear.

1.4. We proceed to define a differential on $\pi_*(T(A)^C)$. The construction works for any G-spectrum T indexed on a complete universe $\mathcal{U}$. We shall make use of concepts and results from equivariant stable homotopy theory. The reader if refered to [LMS] for this material.

We let T be as a above and consider the Tate spectrum from [GM]

$$\hat{\mathbb{H}}(G;T) = [\tilde{E}G \wedge F(EG_+, T)]^G,$$

where $\tilde{E}G$ is the unreduced suspension of EG. We let $\mathbb{C}$ be the standard G-representation and take $S(\mathbb{C}^\infty)$ as our model for EG. Then $\tilde{E}G = S^{\mathbb{C}^\infty}$ and we have Greenlees' 'filtration'

$$\ldots \to S^{-\mathbb{C}^2} \xrightarrow{i_{-1}} S^{-\mathbb{C}} \xrightarrow{i_0} S^0 \xrightarrow{i_1} S^{\mathbb{C}} \xrightarrow{i_2} S^{\mathbb{C}^2} \to \ldots \to S^{\mathbb{C}^\infty}.$$

When we smash with the G-spectrum $F(EG_+, T)$ this makes perfectly good sense and defines a whole plane spectral sequence

$$\hat{E}^1 = P[s, s^{-1}] \otimes \pi_*(T) \Rightarrow \pi_*(\hat{\mathbb{H}}(G; T)); \qquad \deg s = (-1, -1).$$

Moreover, if T is a G-homotopy associative and commutative G-ring spectrum then this is a multiplicative spectral sequence in the sense that $(\hat{E}^r, d^r)$ is a differential bi-graded algebra, $r \geqslant 1$. The spectral sequence (indexed slightly differently) is treated in great detail in [BM] and [HM1]. Now recall that the obvious collaps maps gives an isomorphism

$$\pi_*^S(S^1_+) \cong \pi_*^S(S^1) \oplus \pi_*^S(S^0) \tag{1.4.1}$$

and let $\sigma, \eta \in \pi_1^S(G_+)$ denote the generators which reduce to $(\mathrm{id}, 0)$ and $(0, \eta)$ respectively.

Lemma 1.4.2. *The differential $d^1_{n,m}\colon \hat{E}^1_{n,m} \to \hat{E}^1_{n-1,m}$ is given by the composition*

$$\pi_{m-n}(T) \xrightarrow{\sigma + n\eta} \pi_{m-n+1}(G_+ \wedge T) \xrightarrow{\mu} \pi_{m-n+1}(T),$$

where the first map is exterior multiplication by $\sigma + n\eta$ and the second map is induced by action map.

Proof. We first show that $\hat{E}^1$ is as claimed. The cofiber of $i_1\colon S^0 \to S^{\mathbb{C}}$ may be identified with ΣG_+. Indeed, for any representation $S^V \cong S(\mathbb{R} \oplus V)$ and we have the cofibration sequence

$$S(\mathbb{R}) \times D(\mathbb{C}) \hookrightarrow S(\mathbb{R}) \times D(\mathbb{C}) \cup D(\mathbb{R}) \times S(\mathbb{C}) \to \big(D(\mathbb{R}) \times S(\mathbb{C})\big)/\big(S(\mathbb{R}) \times S(\mathbb{C})\big).$$

In general, $i_n = \mathrm{id}_{S^{(n-1)\mathbb{C}}} \wedge i_1$ so the cofiber is $\Sigma G_+ \wedge S^{(n-1)\mathbb{C}}$. The map which collapses EG to a point induces an equivariant equivalence

$$\Sigma G_+ \wedge S^{(n-1)\mathbb{C}} \wedge F(EG_+, T) \simeq_G \Sigma G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T,$$

and hence an equivalence of G-fixed spectra. Finally, we recall from [LMS], p. 97, that the equivariant transfer induces an equivalence

$$\tau : \Sigma^2 G_+ \wedge_G S^{(n-1)\mathbb{C}} \wedge T \simeq [\Sigma G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T]^G,$$

and $\Sigma^2 G_+ \wedge_G S^{(n-1)\mathbb{C}} \wedge T \cong \Sigma^2 S^{(n-1)\mathbb{C}} \wedge T$. The isomorphism given by the (diagonal) G-action.

In order to evaluate the d^1-differential we consider the diagram

$$\begin{array}{ccccc}
[\Sigma G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T]^G & \xrightarrow{\partial} & [\Sigma S^{(n-1)\mathbb{C}} \wedge T]^G & \xrightarrow{\mathrm{pr}} & [\Sigma^2 G_+ \wedge S^{(n-2)\mathbb{C}} \wedge T]^G \\
\simeq \uparrow \tau & & & & \simeq \uparrow \tau \\
\Sigma^2 G_+ \wedge_G S^{(n-1)\mathbb{C}} \wedge T & \xrightarrow{\partial} & \Sigma^2 S^{(n-1)\mathbb{C}} \wedge_G T & \xrightarrow{\mathrm{pr}} & \Sigma^3 G_+ \wedge_G S^{(n-2)\mathbb{C}} \wedge T \\
\cong \downarrow \bar{\phi} & & & & \cong \downarrow \bar{\phi} \\
\Sigma^2 S^{(n-1)\mathbb{C}} \wedge T & & & & \Sigma^3 S^{(n-2)\mathbb{C}} \wedge T.
\end{array}$$

If we apply $\pi_{n+m}(-)$, then $d^1_{n,m}$ is the composite of the maps from the lower left hand corner to the lower right hand corner. The unit map $S^0 \to G_+$ induces a map

$$\iota\colon \Sigma^2 S^{(n-1)\mathbb{C}} \wedge T \to \Sigma^2 G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T$$

which composed with the projection onto the orbit spectrum is the inverse of $\bar{\phi}$, and it is not hard to see that the composite

$$\Sigma^2 S^{(n-1)\mathbb{C}} \wedge T \xrightarrow{\iota} \Sigma^2 G_+ \wedge S^{(n-1)\mathbb{C}} \wedge T \xrightarrow{\partial} \Sigma^2 S^{(n-1)\mathbb{C}} \wedge T \xrightarrow{\mathrm{pr}} \Sigma^3 G_+ \wedge S^{(n-2)\mathbb{C}} \wedge T$$

represents exterior multiplication by $\sigma \in \pi_1^S(G_+)$. Next, we may write the diagonal action map ϕ as the composition

$$G_+ \wedge S^{(n-2)\mathbb{C}} \wedge T \xrightarrow{\zeta\wedge 1} G_+ \wedge S^{(n-2)\mathbb{C}} \wedge T \xrightarrow{\mathrm{tw}\wedge 1} S^{(n-2)\mathbb{C}} \wedge G_+ \wedge T \xrightarrow{1\wedge\mu} S^{(n-2)\mathbb{C}} \wedge T,$$

where ζ is given by

$$\zeta\colon G_+ \wedge S^{(n-2)\mathbb{C}} \xrightarrow{\Delta\wedge 1} G_+ \wedge G_+ \wedge S^{(n-2)\mathbb{C}} \xrightarrow{1\wedge\mu} G_+ \wedge S^{(n-2)\mathbb{C}}.$$

We claim that under the isomorphism in (1.4.1)

$$\zeta = \begin{pmatrix} 1 & 0 \\ (n-2)\eta & 1 \end{pmatrix}.$$

It suffices to consider the case $n = 3$, *i.e.* the representation is $\mathbb{C}$. For the case $n > 3$ follows by composition and the case $n < 3$ follows by smashing with $S^{N\mathbb{C}}$ for some $N > 3 - n$. Now, the map which we claim represents η may be identified with the composite

$$D(\mathbb{C}) \times S(\mathbb{C}) \cup S(\mathbb{C}) \times D(\mathbb{C}) \xrightarrow{\mathrm{pr}} \big(S(\mathbb{C}) \times D(\mathbb{C})\big)/\big(S(\mathbb{C}) \times S(\mathbb{C})\big) \xrightarrow{\mu} D(\mathbb{C})/S(\mathbb{C}).$$

Hence the claim. □

The lemma leaves us two choices of differentials on $\pi_*(T)$. For the identification $\pi_*(T) \cong \hat{E}^1_{n,n-*}$ in general gives different differentials for n even and n odd.

Definition 1.4.3. For any G-spectrum T indexed on $\mathcal{U}$ let d denote the degree one operator on $\pi_*(T)$ given by

$$d\colon \pi_n(T) \xrightarrow{\sigma} \pi_{n+1}(G_+ \wedge T) \xrightarrow{\mu} \pi_{n+1}(T),$$

where the first map is exterior multiplication by $\sigma \in \pi_1^S(G_+)$ and the second map is induced by the G-action.

One might also want to replace σ by η. However, the operator which results is just multiplication by $\eta \in \pi_1^S(S^0)$. We note the formula

$$dd = \eta d = d\eta. \tag{1.4.4}$$

In particular, the two choices of differentials coincide and equals d when η acts trivially on $\pi_*(T)$. For example this is the case if multiplication by 2 on $\pi_*(T)$ is an isomorphism or if the underlying non-equivariant spectrum of T is a generalized Eilenberg-MacLane spectrum, *e.g.* $T = T(A)$.

Let T be a G-spectrum indexed on a complete universe $\mathcal{U}$ and consider the composition

$$\xi\colon T \wedge G_+ \xrightarrow{\mathrm{id}\wedge\Delta} T \wedge G_+ \wedge G_+ \xrightarrow{\mu\wedge\mathrm{id}} T \wedge G_+,$$

where μ is the action map. For later use we record the effect of ξ on homotopy groups.

Lemma 1.4.5. *Under the decomposition $\pi_*(T \wedge G_+) \cong \pi_*(T) \oplus \pi_{*-1}(T)$ induced from (1.4.1),*

$$\xi = \begin{pmatrix} \mathrm{id} & 0 \\ d & \mathrm{id} \end{pmatrix},$$

where $d\colon \pi_{*-1}(T) \to \pi_*(T)$ is the map from (1.4.3).

Proof. The only map which requires proof is $\xi_{22}\colon \pi_{*-1}(T) \to \pi_{*-1}(T)$. We need to explicite ξ as a map of G-spectra indexed on $\mathcal{U}$. The inclusion $i\colon \mathcal{U}^G \to \mathcal{U}$ of the trivial universe gives rise to a forgetful functor

$i^*: G\mathcal{S}U \to G\mathcal{S}U^G$ which, we recall, has a left adjoint i_*. Here $G\mathcal{S}U$ and $G\mathcal{S}U^G$ denotes the category of G-spectra indexed on $\mathcal{U}$ and $\mathcal{U}^G$, respectively, *cf.* [LMS] or [HM], the proof of proposition 1.1.

We let $|T|$ denote i^*T with the G-action forgotten. The action of G gives a map $\tilde{\mu}: |T| \wedge G_+ \to i^*T$ in $G\mathcal{S}U^G$ and hence a map $\mu: i_*|T| \wedge G_+ \to T$ in $G\mathcal{S}U$, and then ξ is the composite

$$\xi: i_*|T| \wedge G_+ \xrightarrow{\mathrm{id} \wedge \Delta} i_*|T| \wedge G_+ \wedge G_+ \xrightarrow{\mu} T \wedge G_+.$$

Now recall from [LMS] or [HM], (1.7.1) the duality

$$\Sigma F(G_+, D) \simeq_G D \wedge G_+,$$

valid for any $D \in G\mathcal{S}U$. We have in particular

$$(D \wedge G_+)^G \simeq \Sigma F(G_+, D)^G \cong \Sigma D.$$

Hence we may identify ξ_{22} with the map induced by ξ on G-fixed spectra. Now consider the diagram

$$\begin{array}{ccccc}
(i_*|T| \wedge G_+)^G & \xrightarrow{\mathrm{id} \wedge \Delta} & (i_*|T| \wedge G_+ \wedge G_+)^G & \xrightarrow{\mu \wedge \mathrm{id}} & (T \wedge G_+)^G \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
\Sigma|T| & \xrightarrow{(\mathrm{id},1)} & \Sigma|T \wedge G_+| & \xrightarrow{\mu} & \Sigma|T|.
\end{array}$$

The left hand square is strictly commutative, as one may readily verify from the definition of the duality map, and the right hand square is homotopy commutative by naturality of the transfer. Now the composition of the maps in the bottom row is the identity. □

We end this section with a comparison with ordinary Hochschild homology. Recall from [HM], proposition 1.4, that the zero'th space of $T(A)$ is naturally equivalent to Bökstedt's topological Hochschild space $\mathrm{THH}(A)$. This is the realization of a cyclic space with k-simplices

$$\mathrm{THH}(A)_k = \operatorname*{holim}_{\overrightarrow{I^{k+1}}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, A(S^{i_0}) \wedge \ldots \wedge A(S^{i_k})),$$

where $A(S^i)$ is an Eilenberg-MacLane space for A concentrated in degree i. The set of components of $\mathrm{THH}(A)_k$ is equal to the iterated tensor product $A^{\otimes(k+1)}$, that is, the k-simplices in the cyclic abelian group $\mathrm{HH}(A)_\bullet$ which defines ordinary Hochschild homology. Moreover, the cyclic structure maps are such that we get a map of cyclic spaces $l_\bullet: \mathrm{THH}(A)_\bullet \to \mathrm{HH}(A)_\bullet$ and hence a G-equivariant map of their realizations,

$$l: \mathrm{THH}(A) \to \mathrm{HH}(A),$$

called *linearization*. We note that it induces an isomorphism of $\pi_n(-)$ for $n \leqslant 1$.

Proposition 1.4.5. *The following diagram commutes*

$$\begin{array}{ccc}
\pi_n(T(A)) & \xrightarrow{l} & \mathrm{HH}_n(A) \\
\downarrow d & & \downarrow B \\
\pi_{n+1}(T(A)) & \xrightarrow{l} & \mathrm{HH}_{n+1}(A).
\end{array}$$

Here l is linearization and B is Connes' operator.

Proof. Recall that a simplicial abelian group $X_\bullet$ determines a (generalized Eilenberg-MacLane) spectrum $\mathbf{X}$ with $|X_\bullet|$ as its zero'th space. As a model for the n'th deloop we may take the realization of the n-fold iterated nerve

$$\mathbf{X}(n) = |N \ldots N X_{\bullet,\ldots,\bullet}|.$$

The realization may either be formed by realizing the diagonal simplicial set or inductively realizing the simplicial directions one after one; the spaces which result are canonically homeomorphic. From this description it is clear that if $X_\bullet$ is a cyclic abelian group then each $\mathbf{X}(n)$ is a G-space and the spectrum structure maps are G-equivariant. In particular, the G-action gives a map of spectra $\mu\colon G_+ \wedge \mathbf{X} \to \mathbf{X}$ and we proceed to give the induced map on homotopy groups

$$\mu_*\colon H_*(G;\mathbb{Z}) \otimes \pi_*(\mathbf{X}) \to \pi_*(\mathbf{X}),$$

an explicite description.

Let $\mathbf{C}_\bullet$ be the standard cyclic model for the circle with n-simplices the cyclic group $\mathbf{C}_n = \{1, \tau_n, \dots, \tau_n^n\}$ and cyclic structure maps

$$\begin{aligned} d_i\tau_n^s &= \tau_{n-1}^s && \text{, if } i+s \leqslant n \\ &= \tau_{n-1}^{s-1} && \text{, if } i+s > n \\ s_i\tau_n^s &= \tau_{n+1}^s && \text{, if } i+s \leqslant n \\ &= \tau_{n+1}^{s+1} && \text{, if } i+s > n \\ t_n\tau_n^s &= \tau_n^{s-1}. \end{aligned}$$

Then we may identify the zero'th space of the smash product spectrum $G_+ \wedge \mathbf{X}$ with the realization of the simplicial abelian group $\mathbb{Z}[\mathbf{C}_\bullet] \otimes X_\bullet$. Let us write X_* for the chain complex associated with $X_\bullet$ and recall that the Eilenberg-MacLane shuffle map provides an explicit chain homotopy equivalence

$$\theta\colon \mathbb{Z}[\mathbf{C}_\bullet]_* \otimes X_* \xrightarrow{\simeq} (\mathbb{Z}[\mathbf{C}_\bullet] \otimes X_\bullet)_*.$$

The forgetful functor from cyclic abelian groups to simplicial abelian groups has a left adjoint. It assigns to a simplicial abelian group $Y_\bullet$ the cyclic abelian group $FY_\bullet$ with n-simplices

$$FY_n = \mathbb{Z}[\mathbf{C}_n] \otimes Y_n,$$

and cyclic structure maps

$$\begin{aligned} d_i(\tau_n^s \otimes y) &= \tau_{n-1}^s \otimes d_{i+s}y && \text{, if } i+s \leqslant n \\ &= \tau_{n-1}^{s-1} \otimes d_{i+s}y && \text{, if } i+s > n \\ s_i(\tau_n^s \otimes y) &= \tau_{n+1}^s \otimes s_{i+s}y && \text{, if } i+s \leqslant n \\ &= \tau_{n+1}^{s+1} \otimes s_{i+s}y && \text{, if } i+s > n \\ t_n(\tau_n^s \otimes y) &= \tau_n^{s-1} \otimes y, \end{aligned}$$

where all indicices are to be understood as the principal representatives modulo $n+1$. Although $FY_\bullet$ and $\mathbb{Z}[\mathbf{C}_\bullet] \otimes Y_\bullet$ have the same n-simplices they are not isomorphic as simplicial abelian groups. However, their associated chain complexes are canonically isomorphic, the isomorphism given by

$$h\colon FY_* \to (\mathbb{Z}[\mathbf{C}_\bullet] \otimes Y_\bullet)_*, \qquad h(\tau_n^s \otimes y) = (-1)^{ns}\tau_n^{-s} \otimes y.$$

Finally, if $X_\bullet$ is a cyclic abelian group, then μ_* is given by the composite

$$\mu_*\colon \mathbb{Z}[\mathbf{C}_\bullet]_* \otimes X_* \xrightarrow{h\otimes 1} \mathbb{Z}[\mathbf{C}_\bullet]_* \otimes X_* \xrightarrow{\theta} (\mathbb{Z}[\mathbf{C}_\bullet] \otimes X_\bullet)_* \xrightarrow{h^{-1}} FX_* \xrightarrow{\epsilon} X_*,$$

where $\epsilon\colon FX_\bullet \to X_\bullet$ is the counit of the adjunction.

When $X_\bullet = \mathrm{HH}(A)_\bullet$ is the cyclic abelian group which defines Hochschild homology of A one easily calculates

$$\mu_*(\tau_1 \otimes (a_0 \otimes \dots \otimes a_n)) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1}.$$

This is the usual formula for Connes' B-operator, *cf.* [LQ]. □

We note that by [HKR], $(\mathrm{HH}_*(A/k), B) \cong (\Omega^*_{A/k}, d)$ when A is a smooth k-algebra.

1.5. Suppose that A is an $\mathbb{F}_p$-algebra. We may take $T = \rho^{\#}_{C_{p^n}} T(A)^{C_{p^n}}$ in 1.4.3 and get a map

$$d\colon \pi_* T(A)^{C_{p^n}} \to \pi_{*+1} T(A)^{C_{p^n}}.$$

We know from [HM], proposition 4.4, that $\mathrm{TR}(\mathbb{F}_p; p) = H\mathbb{Z}_p$, the Eilenberg-MacLane spectrum for the p-adic integers concentrated in degree zero. Therefore $T(A)^{C_{p^n}}$ is also an Eilenberg-MacLane spectrum. For it is a module spectrum over $\mathrm{TR}(\mathbb{F}_p; p)$ and any module spectrum over an Eilenberg-MacLane spectrum is again Eilenberg-MacLane. If follows that $d \circ d = 0$ such that d makes $\pi_* T(A)^{C_{p^n}}$ a graded commutative DGA over $\mathbb{Z}$. Moreover, (1.3.3) shows that the restrition map R is a map of DGA's. We get a new limit system of DGA's over $\mathbb{Z}$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
\pi_0 T(A)^{C_{p^2}} & \xrightarrow{d} & \pi_1 T(A)^{C_{p^2}} & \xrightarrow{d} & \pi_2 T(A)^{C_{p^2}} & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
\pi_0 T(A)^{C_{p}} & \xrightarrow{d} & \pi_1 T(A)^{C_{p}} & \xrightarrow{d} & \pi_2 T(A)^{C_{p}} & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \downarrow R & & \\
\pi_0 T(A) & \xrightarrow{d} & \pi_1 T(A) & \xrightarrow{d} & \pi_2 T(A) & \xrightarrow{d} & \cdots
\end{array}$$

and $\pi_0 T(A)^{C_{p^{n-1}}} = W_n(A)$. It remains to prove two of the relations in (1.2.1).

Lemma 1.5.1. *Let A be a ring. Then $F_r dV_r = d + (r-1)\eta$.*

Proof. We prove this in the case where $s = 1$, *cf.* (1.3.1). The general case is similar. Let us write $T = T(A)$ and $C = C_r$. We use the equivalence of G/C-spectra discussed in [HM], §7.1,

$$T^C \cong F(G/C_+, T)^C \simeq_{G/C} \Sigma^{-1}(T \wedge G/C_+)^G.$$

Let ξ denote the composition

$$G/C_+ \wedge (T \wedge G_+)^G \xrightarrow{\mathrm{pr}} G/C_+ \wedge (T \wedge G/C)^G \xrightarrow{\mu} (T \wedge G/C_+)^G \xrightarrow{\mathrm{pr}^!} (T \wedge G_+)^G.$$

Then by [HM], lemma 7.1, $F_r dV_r$ is equal to

$$\begin{aligned}\pi_*((T \wedge G_+)^G) &\xrightarrow{\sigma} \pi_{*+1}(G_+ \wedge (T \wedge G_+)^G) \xrightarrow[\cong]{\rho_C \wedge 1} \pi_{*+1}(G/C_+ \wedge (T \wedge G_+)^G) \\ &\xrightarrow{\xi} \pi_{*+1}((T \wedge G_+)^G).\end{aligned}$$

We claim that ξ is homotopic to the composition

$$G/C_+ \wedge (T \wedge G_+)^G \xrightarrow{\tau \wedge 1} G_+ \wedge (T \wedge G_+)^G \xrightarrow{\mu} (T \wedge G_+)^G,$$

where $\tau\colon \Sigma^\infty_+ G/C \to \Sigma^\infty_+ G$ is the ordinary non-equivariant transfer of the r-fold covering $G \to G/C$. Granting this for the moment, the fact that under (1.4.1),

$$\tau = \begin{pmatrix} 1 & 0 \\ (r-1)\eta & r \end{pmatrix}$$

shows that $F_r dV_r = d + (r-1)\eta$ as stated. To prove the claim we consider the two subgroups of $G \times G$ given by

$$C_1 = \{(x, 1) | x \in C\}, \qquad \Delta = \{(x, x^{-1}) | x \in C\}$$

and note that $\Delta \cap C_1 = 1$ and $\Delta C_1 = C \times C$. We shall write $|X|$ for the underlying non-equivariant space of a G-space X. The diagram

$$(1.5.2)\qquad \begin{array}{ccccc} \mathbf{\Sigma}^\infty_+(|G| \times G)/C_1 & \xrightarrow{\pi^{C\times C}_{C_1}} & \mathbf{\Sigma}^\infty_+(|G| \times G)/(C \times C) & \xrightarrow{\mu} & \mathbf{\Sigma}^\infty_+ G/C \\ \downarrow{\scriptstyle (\pi^{C_1}_1)^!} & & \downarrow{\scriptstyle (\pi^{C\times C}_\Delta)^!} & & \downarrow{\scriptstyle (\pi^C_1)^!} \\ \mathbf{\Sigma}^\infty_+|G| \times G & \xrightarrow{\pi^\Delta_1} & \mathbf{\Sigma}^\infty_+(|G| \times G)/\Delta & \xrightarrow{\bar\mu} & \mathbf{\Sigma}^\infty_+ G \end{array}$$

commutes up to G-homotopy. For, in the diagram

$$\begin{array}{ccccc} (|G| \times G)/C_1 & \xrightarrow{\pi^{C\times C}_{C_1}} & (|G| \times G)/(C \times C) & \xrightarrow{\mu} & G/C \\ \uparrow{\scriptstyle \pi^{C_1}_1} & & \uparrow{\scriptstyle \pi^{C\times C}_\Delta} & & \uparrow{\scriptstyle \pi^C_1} \\ |G| \times G & \xrightarrow{\pi^\Delta_1} & (|G| \times G)/\Delta & \xrightarrow{\bar\mu} & G \end{array}$$

both squares are pull backs. Moreover, the equivariant transfer $\tau^{C_1}_1$ is G-homotopic to

$$\tau \wedge 1\colon \Sigma^\infty_+|G/C| \wedge \mathbf{\Sigma}^\infty_+ G \to \Sigma^\infty_+|G| \wedge \mathbf{\Sigma}^\infty_+ G,$$

cf. [LMS] IV 5.3. Now note that in (1.5.2) $\bar\mu \circ \pi^\Delta_1$ is the multiplication $\mu\colon |G| \times G \to G$. Therefore the claim follows from the commutativity of (1.5.2) upon taking G-fixed points. This concludes the proof. □

We shall need some fact about the multiplicative structure of topological Hochschild which we now recall, see also [HM], 1.7. There are G-spaces $\mathrm{THH}(A^{(n)}; S^n)$, $n \geqslant 0$, together with a transitive system of G-equivariant maps

$$(1.5.3)\qquad \mu_{m,n}\colon \mathrm{THH}(A^{(m)}; S^m) \wedge \mathrm{THH}(A^{(n)}; S^n) \to \mathrm{THH}(A^{(m+n)}; S^{m+n}),$$

where G acts diagonally on the left hand side. When $n \geqslant 1$ there is a natural chain of equivalences

$$T(A)(0) \leftarrow \mathrm{THH}(A) \to \Omega\,\mathrm{THH}(A^{(1)}; S^1) \to \underset{\overrightarrow{I}}{\mathrm{holim}}\, \Omega^n\,\mathrm{THH}(A^{(m)}; S^m) \leftarrow \Omega^n\,\mathrm{THH}(A^{(n)}; S^n),$$

where $T(A)(0)$ denotes the zero'th space of the G-spectrum $T(A)$. The last two maps are equivalences by the approximation theorem, [Bö], 1.6. More generally, the induced map of C-fixed sets is an equivalence for every finite subgroup $C \subset G$. In particular, we have a canonical isomorphism of groups

$$\pi_* T(A)^C \cong \pi_*(\Omega^n\,\mathrm{THH}(A^{(n)}; S^n)^C), \qquad n \geqslant 1.$$

and under this isomorphism the maps $\mu_{*,*}$ makes $\pi_* T(A)^C$ a graded commutative ring. When $n = 0$ we have

$$\mathrm{THH}(A^{(0)}; S^0) = |N^{\mathrm{cy}}_\wedge(A)_\bullet|.$$

The right hand side is the cyclic bar construction of A considered a pointed monoid under multiplication with $0 \in A$ as basepoint, *cf.* [HM], §6, and section 3.1 below. It is a commutative topological monoid under the product $\mu_{0,0}$. Moreover, there is a canonical G-equivariant map

$$\iota_n\colon \mathrm{THH}(A^{(0)}; S^0) \to \Omega^n\,\mathrm{THH}(A^{(n)}; S^n),$$

for every $n \geqslant 0$, and the diagram

$$\begin{array}{ccc} |N^{\mathrm{cy}}_\wedge(A)_\bullet| \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| & \xrightarrow{\iota_m \wedge \iota_m} & \Omega\,\mathrm{THH}(A^{(m)}; S^m) \wedge \Omega^n\,\mathrm{THH}(A^{(n)}; S^n) \\ \downarrow{\scriptstyle \mu_{0,0}} & & \downarrow{\scriptstyle \mu_{m,n}} \\ |N^{\mathrm{cy}}_\wedge(A)_\bullet| & \xrightarrow{\iota_{m+n}} & \Omega^{m+n}\,\mathrm{THH}(A^{(m+n)}; S^{m+n}) \end{array}$$

commutes. In particular, we obtain a multiplicative map on the level of homotopy groups

$$\iota\colon \pi_*(|N^{\mathrm{cy}}_\wedge(A)_\bullet|^C) \to \pi_* T(A)^C. \tag{1.5.4}$$

Next, recall from [HM], §6, that there are equivariant homeomorphisms

$$|N^{\mathrm{cy}}_\wedge(A)_\bullet| \xrightarrow{\Delta_C} |(\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet)^C| \xrightarrow{D} \rho_C^*|N^{\mathrm{cy}}_\wedge(A)_\bullet|^C,$$

where $\rho_C\colon G \to G/C$ is the root isomorphism. They are multiplicative by naturality and induce multiplicative isomorphisms on homotopy groups. Composing this with (1.5.4) we obtain a multiplicative map

$$\omega\colon \pi_*(|N^{\mathrm{cy}}_\wedge(A)_\bullet|) \to \pi_* T(A)^C. \tag{1.5.5}$$

When $* = 0$ this is the Teichmüller map $A \to \mathbf{W}_{<r>}(A)$, *cf.* [HM], addendum 2.3.

Lemma 1.5.6. *Suppose A is a ring. Then $F_r d\underline{x} = \underline{x}^{r-1} d\underline{x}$ for all $x \in A$.*

Proof. We again restrict ourselves to the case $s = 1$ leaving the general case to the reader. It suffices to prove that the diagram

$$\begin{array}{ccccc}
G_+ \wedge N^{\mathrm{cy}}_\wedge(A)_0 & \xrightarrow{D\circ\Delta_C} & G_+ \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet|^C & \xrightarrow{\rho_C \wedge \mathrm{id}} & G/C_+ \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet|^C \\
\downarrow{\scriptstyle \mathrm{incl}} & & & & \downarrow{\scriptstyle \mu} \\
G_+ \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| & & & & |N^{\mathrm{cy}}_\wedge(A)_\bullet|^C \\
\downarrow{\scriptstyle (P_{r-1},\mathrm{id})} & & & & \downarrow{\scriptstyle F} \\
G_+ \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| & \xrightarrow{\mu\wedge 1} & |N^{\mathrm{cy}}_\wedge(A)_\bullet| \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| & \xrightarrow{\mu_{0,0}} & |N^{\mathrm{cy}}_\wedge(A)_\bullet|
\end{array} \tag{1.5.7}$$

is homotopy commutative. Here μ is the action map, P_{r-1} is induced from the $r-1$'st power map on A and F is the obvious inclusion map. To this end note that the composition of the maps in the top row and the right hand column is equal to the composition

$$\begin{aligned}
G_+ \wedge N^{\mathrm{cy}}_\wedge(A)_0 &\xrightarrow{\mathrm{id}\wedge\mathrm{incl}} G_+ \wedge |N^{\mathrm{cy}}_\wedge(A)_\bullet| \xrightarrow{\mathrm{id}\wedge\Delta_C} G_+ \wedge |(\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet)^C| \\
&\xrightarrow{\mu} |(\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet)^C| \xrightarrow{F} |\,\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet| \xrightarrow{D} |N^{\mathrm{cy}}_\wedge(A)_\bullet|.
\end{aligned}$$

The (non-simplicial) homeomorphism $D\colon |\,\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet| \to |N^{\mathrm{cy}}_\wedge(A)_\bullet|$ is homotopic to the the realization of the simplicial map $\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet \to N^{\mathrm{cy}}_\wedge(A)_\bullet$ which in simplicial degree k is given by the iterated face map ${d_0}^{(r-1)(k+1)}$. In fact, this is true for any simplicial space $X_\bullet$ and follows from the proof of [BHM], 2.5. Now let $\mathbf{C}_\bullet$ be the standard cyclic model for the circle recalled in the proof of 1.4.5 and consider the diagram of simplicial sets (if there is more than one simplicial direction the diagonal simplicial set is understood)

$$\begin{array}{ccccc}
\mathbf{C}_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_0 & \longrightarrow & \mathbf{C}_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_\bullet & \xrightarrow{\mathrm{id}\wedge\Delta_C} & \mathbf{C}_\bullet \wedge (\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet)^C \\
\downarrow & & & & \downarrow{\scriptstyle \mu} \\
\mathbf{C}_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_\bullet & & & & (\mathrm{sd}_C\, N^{\mathrm{cy}}_\wedge(A)_\bullet)^C \\
\downarrow{\scriptstyle (P_{r-1},\mathrm{id})} & & & & \downarrow{\scriptstyle F\circ d_0^{(\bullet+1)(r-1)}} \\
\mathbf{C}_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_\bullet & \xrightarrow{\mu\wedge\mathrm{id}} & N^{\mathrm{cy}}_\wedge(A)_\bullet \wedge N^{\mathrm{cy}}_\wedge(A)_\bullet & \xrightarrow{\mu_{0,0}} & N^{\mathrm{cy}}_\wedge(A)_\bullet.
\end{array}$$

One verifies by inspection that it is commutative. Therefore (1.5.7) is homotopy commutative. Compare with the proof of 1.4.5. □

The universal property of the de Rham-Witt pro-complex immediately gives the

Proposition 1.5.8. *Suppose A is an $\mathbb{F}_p$-algebra. Then there is a natural map*

$$I\colon W_n\Omega_A^* \to \pi_* T(A)^{C_{p^{n-1}}},$$

such that $RI = IR$, $FI = IF$, $VI = IV$ and $dI = Id$, and such that for $* = 0$, I is the isomorphism of (1.3.6). □

The homotopy groups of $\mathrm{TR}(A;p)$, which we denote $\mathrm{TR}_*(A;p)$, are given by Milnor's short exact sequence

$$0 \to \varprojlim_R{}^{(1)} \pi_{*+1} T(A)^{C_{p^n}} \to \mathrm{TR}_*(A;p) \to \varprojlim_R \pi_* T(A)^{C_{p^n}} \to 0.$$

Therefore, if the derived limit on the left vanishes, 1.5.5 defines a natural map of complexes

$$I\colon W\Omega_A^* \to \mathrm{TR}_*(A;p). \tag{1.5.9}$$

The differential on $\mathrm{TR}_*(A;p)$ is given by 1.4.3.

2. Smooth algebras

2.1. Let k be any ring. We recall that a k-algebra A is said to be *smooth* if it is finitely presented and if every k-algebra map $A \to C/N$ into the quotient of a k-algebra C by a nilpotent ideal N has a lifting to a k-algebra map $A \to C$. If there is at most one such lifting, A is called *unramified*. A k-algebra is *étale* if it is smooth and unramified. Equivalently, A is smooth if there exists relatively prime elements $f_1, \dots, f_s \in A$ such that $A_{f_i} = A[1/f_i]$ is an étale extension of a polynomial algebra in finitely many variables over k,

$$k[X_1, \dots, X_n] \xrightarrow{\text{étale}} A_{f_i}.$$

See for example [L], the appendix. To prove theorem B we first calculate $\mathrm{TR}_*(A;p)$ when A is a polynomial algebra over $\mathbb{F}_p$. The proof in the general case is a covering argument based on the second characterization of smoothness. So let A denote $\mathbb{F}_p[X_1, \dots, X_n]$. We first recall the description of $W_\bullet\Omega_A^*$ from [I] I.2.

Consider the ring

$$C = \varinjlim_r \mathbb{Q}_p[X_1^{p^{-r}}, \dots, X_n^{p^{-r}}].$$

The formula $dX^k = kX^k d\log X$ shows that any $\omega \in \Omega^m_{C/\mathbb{Q}_p}$ may be written uniquely

$$\omega = \sum_{i_1<\dots<i_m} a_{i_1,\dots,i_m}(X) d\log X_{i_1} \dots d\log X_{i_m}, \tag{2.1.1}$$

such that each $a_{i_1,\dots,i_m}(X) \in C$ is divisible by $X_{i_1}^{p^{-s}} \dots X_{i_m}^{p^{-s}}$ for some $s \geqslant 0$. We say that ω is *integral* if the $a_{i_1,\dots,i_m}(X)$ have their coefficients in $\mathbb{Z}_p$ and let

$$E^* \subset \Omega^*_{C/\mathbb{Q}_p} \tag{2.1.2}$$

be the sub *DGA* of those forms ω such that ω and $d\omega$ are both integral. If we write ω as in (2.1.1), then the formula

$$F(x) = \sum_{i_1<\dots<i_m} a_{i_1,\dots,i_m}(X^p) d\log X_{i_1} \dots d\log X_{i_m}$$

defines an automorphism of $\Omega^*_{C/\mathbb{Q}_p}$ considered as a graded ring. We let $V = pF^{-1}$ and note that F and V restricts to endomorphisms of E^*. Moreover, the Teichmüller map $\omega\colon \mathbb{F}_p \to \mathbb{Z}_p$ extends to a multiplicative map $\omega\colon A \to E^0$ given by $\omega(X_i) = X_i$.

We filter E^* by the *DG* ideals

$$\mathrm{Fil}^r E^m = V^r E^m + dV^r E^{m-1}$$

and consider the quotient *DGA*'s

$$E_r^* = E^*/\mathrm{Fil}^r E^*.$$

The operations F and V restricts to $F\colon E_r^* \to E_{r-1}^*$ and $V\colon E_r^* \to E_{r+1}^*$ respectively and we let $R\colon E_r^* \to E_{r-1}^*$ be the projection. The formulas (1.2.1) are trivially verified, and moreover, one may prove that $E_r^0 \cong W_r(A)$ such that R, F and V correspond to restriction, Frobenius and Verschiebung respectively.

Theorem 2.1.3. (Deligne) *The canonical map $W_\bullet\Omega_A^* \to E_\bullet^*$ is an isomorphism.* □

Let $\mathbb{N}[1/p]$ be the monoid of non-negative rationals with denominator a power of p and let

$$K = \mathbb{N}[1/p]^n \tag{2.1.4}$$

be the n-fold product. We can grade E^* over K. For C is a K-graded ring in an obvious way and we call an m-form ω written as in (2.1.1) homogeneous of degree k if the $a_{i_1,\dots,i_m}(X)$ are so. We note that R and d preserves the grading after k whereas F (resp. V) multiplies (resp. divides) the degree by p. Let

$$_kE^* \subset E^*, \tag{2.1.5}$$

denote the subgroup of E^* of homogeneous elements of degree k and let $_kE_r^*$ be the image of $_kE^*$ in E_r^*. It is proved in [I] I.2.8 that $_kE^m$ is a f.g. free $\mathbb{Z}_p$-module and an explicit basis is given. We recall how this basis is defined.

We fix $k \in K$ and let

$$I_m = \{\underline{i} = (i_i, \dots, i_m) | 1 \leqslant i_s \leqslant n \text{ and } k_{i_s} \neq 0\}.$$

Reordering if necessary we may assume that $v(k_1) \leqslant \dots \leqslant v(k_n)$, where $v(-)$ denotes the p-adic valuation. For $\underline{i} \in I_m$ fixed we let

$$\begin{aligned} t_0 &= \begin{cases} 1, & \text{if } i_1 = 1, \\ p^{-v(k_{i_1})} X^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } v(k_1) < 0 \\ X^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } v(k_1) \geqslant 0 \end{cases} \\ t_s &= p^{-v(k_{i_s})} X^k_{[i_s,i_{s+1}[} \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) < 0 \\ u_s &= X^k_{[i_s,i_{s+1}[} \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) \geqslant 0, \end{aligned} \tag{2.1.6}$$

where $X_S^k = \prod_{i\in S} X_i^{k_i}$ for $S \subset [1,n]$ and $[i_m, i_{m+1}[= [i_m, n]$. We define

$$e_{\underline{i}}(k) \in {_kE^m} \tag{2.1.7}$$

by the formula

$$e_{\underline{i}}(k) = t_0 \prod_{\substack{1\leqslant s\leqslant m \\ v(k_{i_s})<0}} dt_s \prod_{\substack{1\leqslant s\leqslant m \\ v(k_{i_s})\geqslant 0}} u_s^{p^{v(k_{i_s})}-1} du_s.$$

Then by [I] I.2.8

$$_kE^m = \mathbb{Z}_p\langle e_{\underline{i}}(k) | \underline{i} \in I_m\rangle. \tag{2.1.8}$$

We also recall the description of $_kE_r^m$ from [I] I.2.12. Let $s = s(k) = -\min\{v(k_i)\}$ and set

$$v = v(r,k) = \begin{cases} r - s & \text{if } s > 0,\, r > s \\ 0 & \text{if } s > 0,\, r \leqslant s \\ r & \text{if } s \leqslant 0. \end{cases}$$

Then

$$_kE_r^m = \mathbb{Z}/p^v\langle e_{\underline{i}}(k) | \underline{i} \in I_m\rangle. \tag{2.1.9}$$

We note that $_kE_r^m$ is non-zero if and only if $v(r,k) > 0$ and $1 \leqslant m \leqslant n$. We shall also write $_kW_r\Omega_A^*$ for $_kE_r^*$.

2.2. In this section we evaluate $\pi_* T(A)^{C_{p^d}}$. The topological Hochschild spectrum $T(A)$ may be evaluated as the smash product

$$T(A) \simeq_G T(\mathbb{F}_p) \wedge |N^{\mathrm{cy}}_\bullet(\langle X_1, \dots, X_n\rangle)|_+, \tag{2.2.1}$$

where the second smash factor on the right is the realization of the cyclic nerve of the free abelian monoid on generators $X_1, \dots, X_n$, *cf.* [HM] §6. Moreover, we have a G-equivariant homeomorphism

$$|N^{\mathrm{cy}}_\bullet(\langle X_1, \dots, X_n\rangle)| \cong_G |N^{\mathrm{cy}}_\bullet(\langle X_1\rangle)| \times \dots \times |N^{\mathrm{cy}}_\bullet(\langle X_n\rangle)|,$$

since cyclic nerve and realization commutes with finite products. As a cyclic set

$$N^{\mathrm{cy}}_\bullet(\langle X\rangle) = \coprod_{l \geqslant 0} N^{\mathrm{cy}}_\bullet(\langle X\rangle; l), \tag{2.2.2}$$

where $N^{\mathrm{cy}}_\bullet(\langle X\rangle; l)$ is the cyclic subset whose k-simplices are the tuples $(X^{i_0}, \dots, X^{i_k})$ such that $i_0 + \dots + i_k = l$. Obviously, $N^{\mathrm{cy}}_\bullet(\langle X\rangle; 0) = *$. We let $S^1(l)$ denote the circle with G acting through the l'th power map for $l > 0$.

Lemma 2.2.3. *$S^1(l)$ is a strong G-homotopy retract of $|N^{\mathrm{cy}}_\bullet(\langle X\rangle; l)|$.*

Proof. As a cyclic set $N^{\mathrm{cy}}_\bullet(\langle X\rangle; l)$ is generated by the $(l-1)$-simplex $(X, \dots, X)$. Therefore the realization is a quotient of standard cyclic $(l-1)$-simplex Λ^{l-1}, *cf.* [G], [J]. In fact,

$$|N^{\mathrm{cy}}_\bullet(\langle X\rangle; l)| = \Lambda^{l-1}/C_l$$

where the generator of C_l acts as the cyclic operator τ_{l-1}. As a G-space $\Lambda^{l-1} \cong S^1 \times \Delta^{l-1}$ and the homeomorphism may be chosen such that the on the left G acts by multiplication in the first variable, see [HM], 6.2. Hence the inclusion of the barycenter in Δ^{l-1} induces a strong G-equivariant deformation retraction $S^1/C_l \to |N^{\mathrm{cy}}_\bullet(\langle X\rangle; l)|$. Finally, the l'th root provides a G-homeomorphism $S^1(l) \cong S^1/C_l$. □

Let $l = (l_1, \dots, l_n)$ be a tuple of non-negative integers and let $(l_{i_1}, \dots, l_{i_j})$ be the sub tuple of positive integers. We have shown that

$$T(A) \simeq_G \bigvee_{l \in \mathbb{N}^n} T(\mathbb{F}_p) \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+. \tag{2.2.4}$$

Since the splitting is G-equivariant it induces a similar splitting of C_{p^d}-fixed points and the maps F, V and d preserves this splitting. The restriction R divides the degree l by p in the sense that if l is not divisible by p then R is zero, compare [HM] §7. We write $e = \min\{v(l_i)\}$ and for $d \geqslant 1$ we let $w = w(d, l) = \min\{d, e\}$.

Proposition 2.2.5. *Let l and d be as above. Then*

$$\big(T(\mathbb{F}_p) \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+\big)^{C_{p^d}} \simeq T(\mathbb{F}_p)^{C_{p^w}} \wedge (S^1_{\alpha_{i_1}} \times \dots \times S^1_{\alpha_{i_j}})_+$$

where the α_{i_s} are dummies.

Proof. For any G-spectrum T and any C-trivial G-space X we have an equivalence of G/C-spectra

$$\big(T \wedge X\big)^C \simeq_{G/C} T^C \wedge X. \tag{2.2.6}$$

In the case at hand this shows that

$$\big(T(\mathbb{F}_p) \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+\big)^C \simeq_{G/C} T(\mathbb{F}_p)^C \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+$$

whenever $C \subset C_{p^e}$. The proposition follows when $d \leqslant e$. We consider the remaining case $d > e$ where we write $T = T(\mathbb{F}_p)$ and $C = C_{p^e}$. From (2.2.6) we have

$$\big(T \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+\big)^{C_{p^d}} \simeq \big(T^C \wedge (S^1(l_{i_1}) \times \dots \times S^1(l_{i_j}))_+\big)^{C_{p^d}/C}$$

and since C_{p^d}/C acts freely on $S^1(l_{i_1})\times\ldots\times S_1(l_{i_j})$ we have the transfer equivalence, also used in [HM],

$$\big(T^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+\big)^{C_{p^d}/C}\simeq T^C\wedge_{C_{p^d}/C}(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+.$$

This is in fact a G/C_{p^d}-equivariant equivalence. Now recall that if Γ is any group and X is any Γ-spectrum indexed on a trivial universe, then the composite

$$\xi\colon |X|\wedge\Gamma_+\xrightarrow{1\wedge\Delta}|X|\wedge\Gamma_+\wedge\Gamma_+\xrightarrow{\mu\wedge 1}X\wedge\Gamma_+, \tag{2.2.7}$$

where $|X|$ denotes the underlying non-equivariant spectrum of X, is an isomorphism of Γ-spectra. We assume that $l_i=p^e$ for some i. Then $S^1(l_i)\cong G/C$ through the p^e'th root and therefore ξ provides a G/C-equivariant isomorphism

$$T^C\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+\cong|T^C\wedge(S^1(l_{i_1})\times\ldots\times\widehat{S^1(l_i)}\times\ldots\times S^1(l_{i_j}))_+|\wedge S^1(l_i)_+. \tag{2.2.8}$$

Finally, $S^1(l_i)/(C_{p^d}/C)\cong S^1$ via the p^{d-e}'th power map.

We assumed above that $l_i=p^e$ for some i. In general we only have $l_i=p^em$ with $(m,p)=1$. To handle this we note that the m'th power map

$$P_m\colon S^1(p^e)\to S^1(p^em),\qquad P_m(z)=z^m,$$

is G-equivariant and that the cofiber is a Moore space for $\mathbb{Z}/m$. Since $T(\mathbb{F}_p)^C$ is a p-local spectrum P_m induces an equivalence

$$T^C\wedge_{C_{p^d}/C}(S^1(l_{i_1})\times\ldots S^1(l_i)\times\ldots\times S^1(l_{i_j}))_+\simeq\ T^C\wedge_{C_{p^d}/C}(S^1(l_{i_1})\times\ldots S^1(p^e)\times\ldots\times S^1(l_{i_j}))_+.$$

Now the scheme above applies. □

We compare the gradings in (2.1.5) and (2.2.4). Let $k\in K$ and $r\geqslant 1$ and suppose $v(r,k)>0$. Then $l=p^{r-1}k$ is an n-tuple of non-negative integers. On the other hand we obtain all n-tuples of non-negative integers this way. Therefore, if

$${}_kT(A)^{C_{p^{r-1}}}=\big(T(\mathbb{F}_p)\wedge(S^1(l_{i_1})\times\ldots\times S^1(l_{i_j}))_+\big)^{C_{p^{r-1}}}$$

then (2.2.4) rewrites

$$T(A)^{C_{p^{r-1}}}\simeq\bigvee_{\substack{k\in K\\ v(r,k)>0}}{}_kT(A)^{C_{p^{r-1}}}, \tag{2.2.9}$$

with $v=v(r,k)$ as in (2.1.9). From 2.2.5 we get

$${}_kT(A)^{C_{p^{r-1}}}\simeq T(\mathbb{F}_p)^{C_{p^{v-1}}}\wedge(S^1_{\alpha_{i_1}}\times\ldots\times S^1_{\alpha_{i_j}})_+ \tag{2.2.10}$$

and finally the homotopy groups become

$$\pi_*({}_kT(A)^{C_{p^{r-1}}})\cong S_{\mathbb{Z}/p^v}\{\sigma_r\}\otimes\Lambda_{\mathbb{Z}/p^v}\{\iota_{i_1},\ldots,\iota_{i_j}\}, \tag{2.2.11}$$

where $\deg(\sigma_r)=2$ and $\deg(\iota_{i_s})=1$. Let us note that in the grading after k the restriction R and the differential d preserve the grading whereas F (resp. V) multiplies (resp. divides) by p.

2.3. The isomorphism of 1.3.6 allows us to use the formulas (2.1.6) and (2.1.7) to define elements

$$g_{\underline{i}}(k) \in \pi_*({}_kT(A)^{C_{p^{r-1}}}).$$

Explicitly, we assume that $v(k_1) \leqslant \ldots \leqslant v(k_n)$ and let

(2.3.1)
$$a_0 = \begin{cases} 1, & \text{if } i_1 = 1, \\ p^{-v(k_{i_1})}\underline{X}^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } v(k_1) < 0 \\ \underline{X}^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } v(k_1) \geqslant 0 \end{cases}$$

$$a_s = p^{-v(k_{i_s})}\underline{X}^k_{[i_s,i_{s+1}[} \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) < 0$$

$$b_s = \underline{X}^k_{[i_s,i_{s+1}[} \quad \text{for } 1 \leqslant s \leqslant m \text{ with } v(k_{i_s}) \geqslant 0,$$

where $\underline{X}^k_S = \prod_{i\in S} \underline{X}_i^{k_i}$ for $S \subset [1,n]$ and $[i_m, i_{m+1}[= [i_j, n]$. Then

(2.3.2)
$$g_{\underline{i}}(k) \in \pi_m({}_kT(A)^{C_{p^{r-1}}})$$

is defined by the formula

$$g_{\underline{i}}(k) = a_0 \prod_{\substack{1\leqslant s\leqslant m \\ v(k_{i_s})<0}} da_s \prod_{\substack{1\leqslant s\leqslant m \\ v(k_{i_s})\geqslant 0}} b_s^{p^{v(k_{i_s})}-1}db_s.$$

Of course, $I(e_{\underline{i}}(k)) = g_{\underline{i}}(k)$. Our calculations in 2.2 above show that theorem B will follow in the case of a polynomial algebra over $\mathbb{F}_p$ from the

Proposition 2.3.3. *The $g_{\underline{i}}(k)$ generate the subgroup*

$$\pi_*^{(0)}({}_kT(A)^{C_{p^{r-1}}}) = \Lambda_{\mathbb{Z}/p^v}\{\iota_{i_1}, \ldots, \iota_{i_j}\}$$

of $\pi_*({}_kT(A)^{C_{p^{r-1}}})$.

Proof. We already know that this is true when $v(k_1) \geqslant 0$ and $r = 1$. For by 1.4.5 the composite

$$\Omega^*_A \to \pi_*T(A) \to \mathrm{HH}_*(A)$$

is the isomorphism of [HKR], see also [LQ]. Moreover, the case $v(k_1) \geqslant 0$ and $r > 1$ also follows because the iterated restriction

$$R^{r-1}\colon {}_kW_r\Omega^*_A \to {}_k\Omega^*_A$$

maps generators to generators. So we are left with the case $v(k_1) = -s < 0$. Again the iterated restriction $R^{r-s-1}\colon {}_kW_r\Omega^*_A \to {}_kW_{s+1}\Omega^*_A$ maps generators to generators so we may assume that $r = s+1$. This has the advantage that ${}_kW_{s+1}\Omega^*_A$ is an $\mathbb{F}_p$-vectorspace, *cf.* (2.1.9).

We first take $m = 1$ in (2.3.2). When $i_1 = 1$ we have

$$g_{i_1}(k) = da_1 = d(p^s\underline{X}^k)$$

and $p^s\underline{X}^k$ generates $\pi_0({}_kT(A)^{C_{p^s}}) \cong \mathbb{F}_p$. Recall that by definition

$$d(p^s\underline{X}^k) = (\mu_s)_*(\sigma \otimes p^s\underline{X}^k)$$

where μ_s is the composition

$${}_kT(A)^{C_{p^s}} \wedge S^1 \xrightarrow{\rho} {}_kT(A)^{C_{p^s}} \wedge (S^1/C_{p^s})_+ \xrightarrow{\mu} {}_kT(A)^{C_{p^s}}$$

and where $\sigma \in \pi_1(S^1_+)$ is as in (1.4.1). From (2.2.8) we have the G/C_{p^s}-equivariant equivalence

$$ {}_kT(A)^{C_{p^s}} \simeq |T(\mathbb{F}_p) \wedge (S^1(l_2) \times \ldots \times S^1(l_j))_+| \wedge S^1/C_{p^s+}. $$

It follows that $g_{i_1}(k) \in \pi_1({}_kT(A)^{C_{p^s}})$ is a generator.

When $i_1 > 1$ and $v(k_{i_1}) \geqslant 0$ we have

$$ g_{i_1}(k) = a_0 b_1^{p^{v(k_{i_1})}} db_1 $$

where $a_0 \in \pi_0({}_{k_{[1,i_1[}}T(A)^{C_{p^s}})$ and $b_1^{p^{v(k_{i_1})}} db_1 \in \pi_1({}_{k_{[i_1,n]}}T(A)^{C_{p^s}})$ both are generators. We proceed to show that $g_{i_1}(k) \in \pi_1({}_kT(A)^{C_{p^s}})$ is a generator. Recall that

$$ \begin{aligned} {}_{k_{[1,i_1[}}T(A)^{C_{p^s}} &= \big(T(\mathbb{F}_p) \wedge (S^1(l_1) \times \ldots \times \widehat{S^1(l_{i_1})})_+\big)^{C_{p^s}} \\ {}_{k_{[i_1,n]}}T(A)^{C_{p^s}} &= \big(T(\mathbb{F}_p) \wedge (S^1(l_{i_1}) \times \ldots \times S^1(l_j))_+\big)^{C_{p^s}} \end{aligned} $$

where $l_i = p^s k_i$. Hence $v(l_1) = 0$ and as demonstrated in the proof of (2.2.5) we may assume that $l_1 = 1$. For notational reasons we shall further assume that $i_1 = j = 2$. We shall also write $T = T(\mathbb{F}_p)$ and $C = C_{p^s}$. We must evaluate the multiplication map

$$ (T \wedge S^1(l_1)_+)^C \wedge (T \wedge S^1(l_2)_+)^C \to (T \wedge (S^1(l_1) \times S^1(l_2))_+)^C \tag{2.3.4} $$

under the equivalences of (2.2.5). Recall that the equivalence

$$ \tilde{\tau}\colon T \wedge_C S^1(l_1)_+ \xrightarrow{\simeq} (T \wedge S^1(l_1)_+)^C $$

is the adjoint of the C-equivariant transfer $\tau\colon T \wedge_C S^1(l_1)_+ \to T \wedge S^1(l_1)_+$ and consider the diagram

$$ \begin{array}{ccc} T \wedge S^1(l_1)_+ \wedge T \wedge S^1(l_2)_+ & \xrightarrow{m} & T \wedge (S^1(l_1) \times S^1(l_2))_+ \\ \uparrow \tau & & \uparrow \tau \\ (T \wedge S^1(l_1)_+ \wedge T \wedge S^1(l_2)_+)/C & \xrightarrow{m} & T \wedge_C (S^1(l_1) \times S^1(l_2))_+ \\ \uparrow 1\wedge F^s & & \| \\ T \wedge_C S^1(l_1)_+ \wedge T^C \wedge S^1(l_2)_+ & \xrightarrow{m'} & T \wedge_C (S^1(l_1) \times S^1(l_2))_+ \\ \cong \uparrow \xi_{12} & & \cong \uparrow \xi \\ |T| \wedge (S^1(l_1)/C)_+ \wedge |T^C \wedge S^1(l_2)_+| & \xrightarrow{m'} & |T| \wedge ((S^1(l_1)/C) \times |S^1(l_2)|)_+ \\ \cong \uparrow \xi_2^{-1} & & \\ |T| \wedge (S^1(l_1)/C)_+ \wedge T^C \wedge S^1(l_2)_+ & & \end{array} \tag{2.3.5} $$

The top square is homotopy commutative by naturality of transfer and the bottom square commutes because the multiplication $m\colon T \wedge T \to T$ is G-equivariant when G acts diagonally on $T \wedge T$. Moreover, the composition of the maps in the left hand column is equal to

$$ |T| \wedge (S^1(l_1)/C)_+ \wedge T^C \wedge S^1(l_2)_+ \xrightarrow{\xi_1 \wedge 1} T \wedge_C S^1(l_1)_+ \wedge T^C \wedge S^1(l_2)_+ \xrightarrow{\tau \wedge F^s} T \wedge S^1(l_1)_+ \wedge T \wedge S^1(l_2)_+. $$

Therefore, it follows from (2.3.5) that under the equivalences of (2.2.5) the multiplication (2.3.4) may be written

$$ \begin{aligned} |T| \wedge (S^1(l_1)/C)_+ \wedge T^C \wedge S^1(l_2)_+ &\xrightarrow{\xi_2^{-1}} |T| \wedge (S^1(l_1)/C)_+ \wedge |T^C \wedge S^1(l_2)_+| \\ &\xrightarrow{1 \wedge F^s} |T| \wedge (S^1(l_1)/C)_+ \wedge |T \wedge S^1(l_2)_+| \xrightarrow{m} |T| \wedge ((S^1(l_1)/C) \times |S^1(l_2))|)_+. \end{aligned} $$

The element

$$a_0 \otimes b_1^{p^{v(k_{i_1})}} db_1 \in \pi_1(T \wedge (S^1(l_1)/C)_+ \wedge T^C \wedge S^1(l_2)_+)$$

is fixed by the isomorphism $(\xi_2^{-1})_*$. Moreover,

$$F^s\colon \pi_1(T^C \wedge S^1(l_2)_+) \to \pi_1(T \wedge S^1(l_2)_+)$$

maps generators to generators. For $F^s\colon \pi_0 T^C \to \pi_0 T$ is the reduction $\mathbb{Z}/p^{s+1} \to \mathbb{F}_p$. Finally, the multiplication $m\colon T \wedge T \to T$ induces an isomorphism on $\pi_0(-)$. This proves that $g_{i_1}(k) \in \pi_1({}_kT(A)^{C_{p^s}})$ is a generator.

The case $i_1 > 1$ and $v(k_{i_1}) < 0$ requires some extra work. We have

$$g_{i_1}(k) = a_0 da_1,$$

and $a_0 \in \pi_0({}_{k_{[1,i_1[}}T(A)^{C_{p^s}})$ and $da_1 \in \pi_1({}_{k_{[i_1,n]}}T(A)^{C_{p^s}})$ are both generators. We prove that $g_{i_1}(k)$ is a generator under the same assumptions as above, *i.e.* $l_1 =$ and $l_2 = j = 2$. The general case is similar. We again write $T = T(\mathbb{F}_p)$ and $C = C_{p^s}$. We also let $C' = C_{p^{v(l_2)}}$ and $\bar{C} = C/C'$ and consider the diagram

$$\begin{array}{ccc}
T \wedge S^1(l_1)_+ \wedge T \wedge S^1(l_2)_+ & \xrightarrow{m} & T \wedge (S^1(l_1) \times S^1(l_2))_+ \\
\uparrow{\scriptstyle \tau_1^{C'}} & & \uparrow{\scriptstyle \tau_1^{C'}} \\
(T \wedge S^1(l_1)_+ \wedge T \wedge S^1(l_2)_+)/C' & \xrightarrow{m} & T \wedge_{C'} (S^1(l_1) \times S^1(l_2))_+ \\
\uparrow{\scriptstyle 1 \wedge F^s} & & \| \\
T \wedge_{C'} S^1(l_1)_+ \wedge T^{C'} \wedge S^1(l_2)_+ & \xrightarrow{m'} & T \wedge_{C'} (S^1(l_1) \times S^1(l_2))_+ \\
\uparrow{\scriptstyle \tau_{C'}^{C}} & & \uparrow{\scriptstyle \tau_{C'}^{C}} \\
(T \wedge_{C'} S^1(l_1)_+ \wedge T^{C'} \wedge S^1(l_2)_+)/\bar{C} & \xrightarrow{m'} & T \wedge_C (S^1(l_1) \times S^1(l_2))_+
\end{array}$$

It homotopy commutes by the naturality of transfers. The spectra in the left column are $G \times G$-spectra which we consider G-spectra via the diagonal $\Delta\colon G \to G \times G$. Let us write Δ (resp. Δ') for the image of C (resp. C'). Then the transfer $\tau_1^{C'}$ (resp. $\tau_{C'}^{C}$) of G-spectra is equal to the transfer $\tau_1^{\Delta'}$ (resp. $\tau_{\Delta'}^{\Delta}$) of $G \times G$-spectra. Moreover,

$$\tau_{C'}^{C} \wedge \tau_{C'}^{C} \simeq \tau_{\Delta}^{C \times C} \circ \tau_{\Delta'}^{\Delta}$$

as maps of $G \times G$-spectra

$$T \wedge_C S^1(l_1)_+ \wedge T^{C'} \wedge S^1(l_2)_+ \to T \wedge_{C'} S^1(l_1)_+ \wedge T^{C'} \wedge S^1(l_2)_+.$$

Finally, we consider the diagram

$$\begin{array}{ccc}
|T| \wedge (S^1(l_1)/C)_+ \wedge |T^{C'}| \wedge (S^1(l_2)/\bar{C})_+ & \xrightarrow{\xi \wedge \xi} & T \wedge_C S^1(l_1)_+ \wedge T^{C'} \wedge_{\bar{C}} S^1(l_2)_+ \\
\downarrow{\scriptstyle \tau_{\Delta}^{C \times C}} & & \downarrow{\scriptstyle \tau_{\Delta}^{C \times C}} \\
(|T| \wedge (S^1(l_1)/C')_+ \wedge |T^{C'}| \wedge S^1(l_2)_+)/\bar{C} & \xrightarrow{\xi \wedge \xi} & (T \wedge_{C'} S^1(l_1)_+ \wedge T^{C'} \wedge S^1(l_2)_+)/\bar{C} \\
\downarrow{\scriptstyle m''} & & \downarrow{\scriptstyle m'} \\
|T| \wedge ((S^1(l_1)/C) \times |S^1(l_2)|)_+ & \xrightarrow{\xi_{12}} & T \wedge_C (S^1(l_1) \times S^1(l_2))_+.
\end{array}$$

Again the top square homotopy commutes by naturality and m'' is defined to make the bottom square commute. Our analysis so far shows that under the equivalences of (2.2.5) the multiplication map is equal to $m'' \circ \tau_{\Delta}^{C \times C}$. One may argue as in the previous case that m'' maps generators to generators on $\pi_1(-)$. Therefore it will suffice to show that $\tau_{\Delta}^{C \times C}$ maps the generator $a_0 \otimes da_1$ non-trivially. To see this we note that $\tau_{\Delta}^{C\times C}$ is homotopic to the identity on $|T| \wedge |T^{C'}|$ smashed with the ordinary (non-equivariant) transfer associated to the p^s-fold covering

$$\mathrm{pr}_{\Delta}^{C\times C}\colon (S^1 \times (S^1/C'))/\Delta \to (S^1 \times S^1)/(C \times C).$$

The homology of the base is

$$H_*((S^1 \times S^1)/(C \times C); \mathbb{F}_p) \cong H_*(S^1/C; \mathbb{F}_p) \otimes H_*(S^1/C; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}\{\iota_1\} \otimes \Lambda_{\mathbb{F}_p}\{\iota_2\}$$

and $\tau_{\Delta}^{C\times C}(a_0 \otimes da_1)$ is non-trivial if and only if the homology transfer of $1 \otimes \iota_2$ is non-trivial. We have a pull back square

$$\begin{array}{ccc} (S^1/C) \times (S^1/C') & \xrightarrow{\xi} & (S^1 \times (S^1/C'))/\Delta \\ \big\downarrow{\scriptstyle 1\times \mathrm{pr}_{C'}^{C}} & & \big\downarrow{\scriptstyle \mathrm{pr}_{\Delta}^{C\times C}} \\ (S^1/C) \times (S^1/C) & \xrightarrow{\xi} & (S^1 \times S^1)/(C \times C), \end{array}$$

where ξ is the homeomorphism given by $\xi(z_1, z_2) = (z_1, z_1 z_2)$. On homology

$$\mathrm{trf}_{C'}^{C}(1) = p = 0, \qquad \mathrm{trf}_{C'}^{C}(\iota) = \iota$$

and therefore

$$(1 \otimes \mathrm{trf}_{C'}^{C}) \circ \xi^{-1}(1 \otimes \iota_2) = (1 \otimes \mathrm{trf}_{C'}^{C})(1 \otimes \iota_2) = 1 \otimes \iota_2$$

which is a generator. This shows that $g_{i_1}(k)$ is a generator.

We can now prove that the $g_{i_1}(k)$ generate

$$\pi_1^{(0)}({}_kT(A)^{C_{p^s}}) \cong \mathbb{F}_p\langle \iota_1, \dots, \iota_j \rangle.$$

We have seen that the $g_{i_1}(k)$, $i_1 = 1, \dots, j$, are non-trivial. However we have proved more, namely that

$$\iota_s \in \mathbb{F}_p\langle g_i(k), \dots, g_j(k) \rangle$$

if and only if $i \leqslant s$. This concludes the proof in the case where $m = 1$ in (2.3.2).

The general case is similar. Again one proves that the $g_{\underline{i}}(k)$, $\underline{i} \in I_m$, are non-trivial. The argument is only notationally more involved than the argument above. We have from (2.2.11)

$$\pi_m^{(0)}({}_kT(A)^{C_{p^s}}) \cong \Lambda^m(\mathbb{F}_p\langle \iota_1, \dots, \iota_j \rangle).$$

Finally, one orders I_m lexicographically to see that

$$\pi_m^{(0)}({}_kT(A)^{C_{p^s}}) \cong \mathbb{F}_p\langle g_{\underline{i}}(k) | \underline{i} \in I_m \rangle$$

as claimed. □

2.4. The proof of theorem B in the general case is a covering argument based on the fact that in a smooth k-algebra A one can find relatively prime elements $f_1, \dots, f_s$ such that $A_{f_i} = A[1/f_i]$ is an étale extension of a polynomial algebra in a finite number of variables,

$$k[X_1, \dots, X_n] \xrightarrow{\text{étale}} A_{f_i}. \tag{2.4.1}$$

We first study étale extensions.

Lemma 2.4.2. *If $f: A \to B$ is an étale extension of $\mathbb{F}_p$-algebras then $\pi_* T(B) \cong B \otimes_A \pi_* T(A)$.*

Proof. Recall the spectral sequence used in [Bö1],

$$E^2(B) = \mathrm{HH}_*(\mathcal{A}_B) \Rightarrow \mathcal{A} \otimes T_*(B), \tag{2.4.3}$$

where $\mathcal{A}_B = H_*^{\mathrm{spec}}(HB; \mathbb{F}_p)$ is the mod p spectrum homology of the Eilenberg-MacLane spectrum for B and $\mathcal{A} = \mathcal{A}_{\mathbb{F}_p}$. As an algebra $\mathcal{A}_B \cong B \otimes \mathcal{A}$. For the Hurewitz map induces a multiplicative homomorphism

$$B = \pi_* HB \to H_*^{\mathrm{spec}}(HB; \mathbb{Z}) \to H_*^{\mathrm{spec}}(HB; \mathbb{F}_p) = \mathcal{A}_B$$

and so does the unit map $\mathbb{F}_p \to B$. The ring homomorphism $B \otimes \mathcal{A} \to \mathcal{A}_B$ is an isomorphism because B as an abelian group is an $\mathbb{F}_p$-vectorspace and because spectrum homology commutes with sums. When B is an A-algebra we get

$$\mathcal{A}_B \cong B \otimes_A \mathcal{A}_A$$

as $\mathbb{F}_p$-algebras. We have a Künneth formula for Hochschild homology

$$\mathrm{HH}_*(\mathcal{A}_B) \cong \mathrm{HH}_*(B) \otimes \mathrm{HH}_*(\mathcal{A})$$

and similar with A in the place of B, *cf.* [CE] p. 204. The main result which proves the lemma is that when B is étale over A then $\mathrm{HH}_*(B) \cong B \otimes_A \mathrm{HH}_*(A)$. This result seems to have been proved in a varying generality by a number of people. A well written account is [WG]. In the case at hand we get

$$\mathrm{HH}_*(\mathcal{A}_B) \cong B \otimes_A \mathrm{HH}_*(\mathcal{A}_A).$$

The spectral sequence (2.4.3) is a spectral sequence of B-algebras. Therefore we also have

$$E^\infty(B) \cong B \otimes_A E^\infty(A).$$

Since the spectral sequence is concentrated in the first quadrant we may conclude that

$$\mathcal{A} \otimes T_*(B) \cong \mathcal{A} \otimes B \otimes_A T_*(A).$$

The lemma follows. □

We recall that for a ring A the topological Hochschild spectrum $T(A)$ is a ring spectrum and that the restriction map is multiplicative. This implies that (1.3.4) is a cofibration sequence of $T(A)^{C_{p^n}}$-module spectra,

$$T(A)_{hC_{p^n}} \xrightarrow{N} T(A)^{C_{p^n}} \xrightarrow{R} T(A)^{C_{p^{n-1}}}.$$

In particular, the associated homotopy long-exact sequence is a sequence of $W_{n+1}(A)$-modules. Moreover, there is a first quadrant spectral sequence of $W_{n+1}(A)$-modules

$$E^2 = H^*(C_{p^n}; (F^n)^* \pi_* T(A)) \Rightarrow \pi_* T(A)_{hC_{p^n}}, \tag{2.4.4}$$

where $F^n: W_{n+1}(A) \to A$ is the n-fold Frobenius. We refer to [HM] for a details. Our next result is also of interest in its own right, compare [WG].

Proposition 2.4.5. *If $f: A \to B$ is an étale map of $\mathbb{F}_p$-algebras then the canonical map*

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)^{C_{p^{r-1}}} \to \pi_* T(B)^{C_{p^{r-1}}}$$

is an isomorphism.

Proof. When $r = 1$ this is (2.4.2) and we proceed by induction on r. Consider the diagram

$$\begin{array}{ccccccc}
\ldots \to \pi_m T(A)_{hC_{p^{r-1}}} & \xrightarrow{N} & \pi_m T(A)^{C_{p^{r-1}}} & \xrightarrow{R} & \pi_m T(A)^{C_{p^{r-2}}} \to \ldots \\
\downarrow & & \downarrow & & \downarrow \\
\ldots \to \pi_m T(B)_{hC_{p^{r-1}}} & \xrightarrow{N} & \pi_m T(B)^{C_{p^{r-1}}} & \xrightarrow{R} & \pi_m T(B)^{C_{p^{r-2}}} \to \ldots
\end{array}$$

and recall that for an étale extension of $\mathbb{F}_p$-algebras one has

$$W_r(B) \otimes_{W_r(A)} W_s(A) \cong W_s(B)$$

whenever $s \leqslant r$, see [I] 1.5.8. Therefore,

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)^{C_{p^{r-2}}} \cong W_{r-1}(B) \otimes_{W_{r-1}(A)} \pi_* T(A)^{C_{p^{r-2}}}$$

so that we are done by induction if we can prove that

$$W_r(B) \otimes_{W_r(A)} \pi_* T(A)_{hC_{p^{r-1}}} \to \pi_* T(B)_{hC_{p^{r-1}}}$$

is an isomorphism. On the other hand this follows if the corresponding statement for the E^∞-terms of (2.4.4) holds. This in turns holds if it does on E^2. So it suffices to prove that

$$W_r(B) \otimes_{W_r(A)} (F^{r-1})^* \pi_* T(A) \to (F^{r-1})^* \pi_* T(B)$$

is an isomorphism. Recall from 1.1 that $F = R \circ F_A$ where $F_A \colon W_r(A) \to W_r(A)$ is the map induced by the Frobenius endomorphism of A. We have a series of identifications

$$\begin{aligned} W_r(B) \otimes_{W_r(A)} (F^{r-1})^* \pi_* T(A) &\cong W_r(B) \otimes_{W_r(A)} A \otimes_A (F_A^{r-1})^* \pi_* T(A) \\ &\cong (R^{r-1})^* B \otimes_A (F_A^{r-1})^* \pi_* T(A) \cong (R^{r-1})^* B \otimes_A (F_A^{r-1})^* A \otimes_A \pi_* T(A) \\ &\cong (R^{r-1})^* (F_B^{r-1})^* B \otimes_A \pi_* T(A) \cong (F^{r-1})^* B \otimes_A \pi_* T(A), \end{aligned}$$

where the second and the fourth identifications use that B is étale over A. Now the proposition follows from (2.4.2). □

Lemma 2.4.6. $\pi_* T(k[X_1, \dots, X_n])^{C_{p^{r-1}}} \cong W_r(k) \otimes \pi_* T(\mathbb{F}_p[X_1, \dots, X_n])^{C_{p^{r-1}}}$.

Proof. We recall that $\mathrm{HH}_*(k) = k$ concentrated in degree zero, see for example [HM], lemma 4.5. It follows from the proof of (2.4.2) that

$$\pi_* T(k[X_1, \dots, X_n]) \cong k \otimes \pi_* T(\mathbb{F}_p[X_1, \dots, X_n]).$$

Now recall that $W_r(k) \otimes W_s(\mathbb{F}_p) \cong W_s(k)$ whenever $s \leqslant r$, see [I] 1.3.22. Therefore, the claim follows by the same line of proof as (2.4.5). □

Lemma 2.4.7. *Let $f_1, \dots, f_s \in A$ be relatively prime elements. Then the complex*

$$\bigotimes\nolimits_{W_r(A)} (W_r(A) \to W_r(A_{f_i}))$$

where i runs through $1, \dots, s$ is acyclic.

Proof. We first consider the case $r = 1$. Since A_{f_i} is flat over A

$$H^*(\bigotimes\nolimits_A (A \to A_{f_i})) = H^s(\bigotimes\nolimits_A (A \to A_{f_i})) = \bigotimes\nolimits_A (A_{f_i}/A).$$

We write $1 = x_1 f_1 + \ldots + x_s f_s$ and note the formula

$$\frac{a_1}{f_1^{i_1}} \otimes_A \ldots \otimes_A \frac{a_s}{f_s^{i_s}} = \sum_{j=1}^{s} \frac{a_1}{f_1^{i_1}} \otimes_A \ldots \otimes_A \frac{x_j a_j}{f_j^{i_j - 1}} \otimes_A \ldots \otimes_A \frac{a_s}{f_s^{i_s}}.$$

Now the case $r = 1$ follows by induction. To handle the general case we note that $W_r(A_f) = W_r(A)_{\underline{f}}$. Therefore, it suffices to show that $\underline{f}_1, \dots, \underline{f}_s$ are relatively prime. Now

$$\underline{x}_1 \underline{f}_1 + \ldots + \underline{x}_s \underline{f}_s \in 1 + VW_r(A) \subset W_r(A)^\times.$$

Whence the claim. ∎

Proof of theorem B. Let us write the map in the statement as $X_r(A) \to Y_r(A)$ and let A_{f_i} be as in (2.4.1). We know from (2.3.3) and (2.4.6) that the theorem holds when $A = k[X_1, \dots, X_n]$. Moreover, if $f\colon A \to B$ is an étale extension then the horizontal maps in the square

$$\begin{array}{ccc} W_r(B) \otimes_{W_r(A)} X_r(A) & \xrightarrow{\cong} & X_r(B) \\ \downarrow & & \downarrow \\ W_r(B) \otimes_{W_r(A)} Y_r(A) & \xrightarrow{\cong} & Y_r(B). \end{array} \tag{2.4.10}$$

are isomorphisms by [I] I.1.14 and by (2.4.5). Therefore, the theorem holds for étale extensions of polynomial algebras over k and hence for A_{f_i}, $i = 1, \dots, s$. Now recall that $A_f \otimes_A A_g = A_{fg} = (A_f)_g$ which is étale over A_f. It follows that the chain map

$$\bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i})) \to \bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i}))$$

is an isomorphism in degrees $\geqslant 1$. We claim that both complexes are acyclic, whence the theorem. To prove the claim we write

$$\bigotimes_{W_r(A)} (X_r(A) \to X_r(A_{f_i})) \cong \big(\bigotimes_{W_r(A)} (W_r(A) \to W_r(A_{f_i}))\big) \otimes_{W_r(A)} X_r(A)$$

and use that by (2.4.7)

$$\bigotimes_{W_r(A)} (W_r(A) \to W_r(A_{f_i}))$$

is an acyclic complex of flat $W_r(A)$-modules. In view of (2.4.5) the same argument works with Y in place of X and the claim follows. ∎

Corollary 2.4.8. *Let A be as above. Then the map of (1.5.9)*

$$I\colon W\Omega^*_A \to \mathrm{TR}_*(A)$$

is well-defined and a natural isomorphism of complexes.

Proof. We recall the Milnor short exact sequence

$$0 \to \varprojlim_R{}^{(1)} \pi_{*+1} T(A)^{C_{p^s}} \to \mathrm{TR}_*(A) \to \varprojlim_R \pi_* T(A)^{C_{p^s}} \to 0$$

where by theorem B

$$\pi_* T(A)^{C_{p^{r-1}}} \cong W_r\Omega^*_A \otimes S_{\mathbb{F}_p}\{\sigma_r\}$$

with $\deg \sigma_r = 2$. In addition, $R(\sigma_r) = p\sigma_{r-1}$ and therefore the summand corresponding to the augmentation ideal of $S_{\mathbb{F}_p}\{\sigma_r\}$ does neither contribute the limit nor to the derived limit. The remaining part is the de Rham-Witt procomplex $W_\bullet\Omega^*_A$. It is a Mittag-Leffler system whose limit is $W\Omega^*_A$ by definition, *cf.* (1.2.2.). ∎

3. Typical curves in K-theory

3.1. Let A be a ring and let $K(A)$ denote Quillen's algebraic K-theory spectrum of A. Here we use the term spectrum in the sense of topology. Let us write $\widetilde{K}(A[X]/(X^n))$ for the homotopy fiber of the map

$$K(A[X]/(X^n)) \to K(A); \qquad X \mapsto 0$$

We recall from the introduction that the *curves in* $K(A)$ is the spectrum

$$C(A) = \operatorname*{holim}_{\overleftarrow{n}} \Omega \widetilde{K}(A[X]/(X^n)). \tag{3.1.1}$$

In order to evaluate $C(A)$ we use the cyclotomic trace of [BHM]. This is a map of spectra

$$\text{trc}\colon K(A) \to \text{TC}(A), \tag{3.1.2}$$

which is natural in the ring A. The codomain is the topological cyclic homology spectrum of A, defined in [BHM]. We briefly recall the construction and refer to [HM] §3 for details. Let $\mathbb{I}$ denote the category with objects the natural numbers and with two morphisms $R_r, F_r\colon m \to n$, whenever $m = rn$, subject to the relations

$$R_1 = F_1 = 1, \quad R_{rs} = R_r R_s, \quad F_{rs} = F_r F_s, \quad R_r F_s = F_s R_r.$$

We also let $\mathbb{I}_p$ denote the full subcategory on objects $\{1, p, p^2, \dots\}$. As recalled in 1.3, we have two maps $R_r, F_r\colon T(A)^{C_m} \to T(A)^{C_n}$, whenever $m = rn$. These maps satisfies the relations above such that we have a functor from $\mathbb{I}$ to the category of spectra, and we define

$$\text{TC}(A) = \underset{\overleftarrow{\mathbb{I}}}{\text{holim}}\, T(A)^{C_n}, \quad \text{TC}(A;p) = \underset{\overleftarrow{\mathbb{I}_p}}{\text{holim}}\, T(A)^{C_{p^s}}. \tag{3.1.3}$$

We note that by [HM], theorem 3.1, the canonical projection induces an equivalence after p-completion,

$$\text{TC}(A)^{\wedge}_{p} \simeq \text{TC}(A;p)^{\wedge}_{p}. \tag{3.1.4}$$

Let us also note that if A is an $\mathbb{Z}/p^j$-algebra, then $\text{TC}(A)$ is already p-complete. Indeed, any spectrum whose homotopy groups are bounded p-groups is p-complete, and hence [HM], addendum 2.3, shows that $T(A)^{C_n}$ is p-complete. Finally, a homotopy limit of p-complete spectra is again p-complete.

Next, we recall from [HM], §6, that a pointed monoid is a monoid Π in the symmetric monoidal category of pointed spaces and smash product, that is, a (topological) monoid Π with a base point $0 \in \Pi$ such that the multiplication factors over the smash product

$$\mu\colon \Pi \wedge \Pi \to \Pi.$$

A pointed monoid has a cyclic bar construction $N^{\text{cy}}_{\wedge}(\Pi)$. It is a cyclic space in the sense of Connes, with k-simplices

$$N^{\text{cy}}_{\wedge,k}(\Pi) = \Pi^{\wedge(k+1)}$$

and structure maps

$$\begin{aligned}
d_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \pi_{i+1} \wedge \dots \wedge \pi_k && ,0 \leqslant i < k \\
&= \pi_k \pi_0 \wedge \pi_1 \wedge \dots \wedge \pi_{k-1} && ,i = k \\
s_i(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_0 \wedge \dots \wedge \pi_i \wedge 1 \wedge \pi_{i+1} \wedge \dots \wedge \pi_k && ,0 \leqslant i \leqslant k \\
\tau_k(\pi_0 \wedge \dots \wedge \pi_k) &= \pi_k \wedge \pi_0 \wedge \dots \wedge \pi_{k-1}.
\end{aligned}$$

Since $N^{\text{cy}}_{\wedge}(\Pi)$ a cyclic space its realization has a G-action, where we remember, G denotes the circle group.

We shall here only consider *commutative* pointed monoids. In this case $N^{\text{cy}}_{\wedge}(\Pi)$ is a commutative cyclic pointed monoid, that is, a cyclic object in the category of commutative pointed monoids. Moreover, we have the following conceptual characterization of the cyclic bar construction: it is the left adjoint of the functor which takes a commutative cyclic pointed monoid $\Pi_{\bullet}$ to the commutative pointed monoid which consists of its zero simplices Π_0.

We consider the pointed monoid $\Pi_n = \{0, 1, X, \dots, X^{n-1}\}$, where 0 is the base point and $X^n = 0$. We also allow $n = \infty$. It follows from [HM], theorem 6.1, that

$$T(A[X]/(X^n)) \simeq_G T(A) \wedge |N^{\text{cy}}_{\wedge}(\Pi_n)|, \tag{3.1.5}$$

where the smash product on the right takes place in the category of G-spectra. Moreover, the cyclic bar construction splits as a wedge

$$N^{\text{cy}}_{\wedge}(\Pi_n) \cong \bigvee_{m=0}^{\infty} N^{\text{cy}}_{\wedge}(\Pi_n; m),$$

where the summand $N^{\text{cy}}_{\wedge}(\Pi_n; m)$ consists of 0 and the k-simplices $X^{i_0} \wedge \dots \wedge X^{i_k}$ with $i_0 + \dots + i_k = m$. The realization splits accordingly, and this splitting is G-equivariant.

Lemma 3.1.6. i) *Let $l(m)$ be the least integer greater than $m/(n-1)$. Then $|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|$ is at least $l(m)-3$ connected.*
ii) *Whenever $m<n$, $N^{\mathrm{cy}}_\wedge(\Pi_n;m)=N^{\mathrm{cy}}_\wedge(\Pi_\infty;m)$.*
iii) *If $m>0$, then S^1/C_{m+} is a strong G-deformation retract of $|N^{\mathrm{cy}}_\wedge(\Pi_\infty;m)|$, and $|N^{\mathrm{cy}}_\wedge(\Pi_\infty;0)|=S^0$.*
iv) *Let $P_n\colon\Pi_\infty\to\Pi_\infty$ be the map of pointed monoids given by $P_n(X)=X^n$. Then the diagram*

$$\begin{array}{ccc} S^1/C_{m+} & \longrightarrow & |N^{\mathrm{cy}}_\wedge(\Pi_\infty;m)| \\ \big\downarrow \mathrm{pr} & & \big\downarrow P_{n*} \\ S^1/C_{mn+} & \longrightarrow & |N^{\mathrm{cy}}_\wedge(\Pi_\infty;mn)|, \end{array}$$

is G-homotopy commutative.

Proof. i) Let us recall that the k-simplices in $N^{\mathrm{cy}}_\wedge(\Pi_n;m)$ has the form $X^{i_0}\wedge\dots\wedge X^{i_k}$, with $i_0+\dots+i_k=m$, and that $X^n=0$. Thus, if $(k+1)(n-1)<m$, there is only one k-simplex 0, and hence the k-skeleton of the realization is just a point.
iii) As a pointed monoid $\Pi_\infty=\langle X\rangle_+$, where $\langle X\rangle$ is the free abelian monoid on one generator X. Hence the claim follows from (2.2.3). □

Under the equivalence of (3.1.5) the restriction map R_r annihilates the summands where m is not divisible by r and induces maps

(3.1.7) $$R_{r,m}\colon (T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{rs}}\to(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m/r)|)^{C_s}$$

of the remaining summands. This follows from the fact that (3.1.5) is an equivalence of cyclotomic spectra, compare [HM], 6.1 and 7.2. The Frobenius map preserves the splitting and induces maps

$$F_{r,m}\colon (T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{rs}}\to(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_s}.$$

Theorem 3.1.8. *If A is a $\mathbb{Z}/p^j$-algebra then $\widetilde{\mathrm{TC}}_i(A[X]/(X^n))$ is a bounded p-group, i.e. any element is annihilated by p^N for some $N\geqslant 0$ which may depend on i.*

Proof. We recalled in (3.1.4) that the group in question is isomorphic to the group $\widetilde{\mathrm{TC}}_i(A[X]/(X^n);p)$. The topological Hochschild spectrum is given by (3.1.5) as the wedge sum

$$\widetilde{T}(A[X]/(X^n))\ \simeq\ \bigvee_{m=1}^{\infty} T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|$$

and the splitting being equivariant implies a corresponding splitting of the C_{p^r}-fixed set,

$$\widetilde{T}(A[X]/(X^n))^{C_{p^r}}\simeq\bigvee_{m=1}^{\infty}(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{p^r}}.$$

The restriction map of (3.1.7) fits into a cofibration sequence

$$(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)_{hC_{p^r}}\xrightarrow{N_{p,m}}(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|)^{C_{p^r}}\xrightarrow{R_{p,m}}(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m/p)|)^{C_{p^{r-1}}}.$$

Indeed, this follows from [HM], proposition 1.1, and the equivalences

$$\rho^\#_{C_p}\Phi^{C_p}(T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|))\simeq_G\rho^\#_{C_p}\Phi^{C_p}T(A)\wedge\rho^*_{C_p}|N^{\mathrm{cy}}_\wedge(\Pi_n;m)|^{C_p}\simeq_G T(A)\wedge|N^{\mathrm{cy}}_\wedge(\Pi_n;m/p)|.$$

In particular, when p does not divide m, the map $N_{p,m}$ is an equivalence. Now recall that taking homotopy orbits preserves connectivity. Hence the connectivity statement of (3.1.7) implies that $R_{p,m}$ is an $l(m)-3$ connected map, and if we write $m=p^sk$, with $(k,p)=1$, then the obvious induction argument shows that

$(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_{p^r}}$ is an $l(p^{s-r}k)-3$ connected spectrum. Accordingly, we may replace the wedge sums above by the corresponding products,

$$\widetilde{T}(A[X]/(X^n))^{C_{p^r}} \simeq \prod_{m=1}^{\infty}(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_{p^r}}.$$

Let us write $\mathrm{TF}(A[X]/(X^n);m;p)$ for the homotopy limit over the Frobenius maps $F_{p,m}$ of the m'th factor in this product decomposition. We then have

$$\widetilde{\mathrm{TF}}(A[X]/(X^n);p)\simeq\prod_{m=1}^{\infty}\mathrm{TF}(A[X]/(X^n);m;p).$$

and hence

$$\widetilde{\mathrm{TC}}(A[X]/(X^n);p)\ \simeq\prod_{(k,p)=1}\operatorname*{holim}_{\overleftarrow{R}}\mathrm{TF}(A[X]/(X^n);p^sk;p).$$

Moreover, the restriction map

$$R_{p,m}\colon \mathrm{TF}(A[X]/(X^n);m;p)\to\mathrm{TF}(A[X]/(X^n);m/p;p)$$

is $l(m)-3$ connected, and therefore, it suffices to show that the homotopy groups of $\mathrm{TF}(A[X]/(X^n);m;p)$ are bounded p-groups for every $m\geqslant 1$.

We fix $m=p^sk$ and consider $\pi_i(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_{p^r}}$ for $r\geqslant s$. We have the following tower of fibrations

$$\begin{array}{ccc}
(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)_{hC_{p^r}} & \xrightarrow{N_{p,m}} & (T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_{p^r}} \\
 & & \big\downarrow R_{p,m} \\
(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m/p)|)_{hC_{p^{r-1}}} & \xrightarrow{N_{p,m/p}} & (T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m/p)|)^{C_{p^{r-1}}} \\
 & & \big\downarrow R_{p,m/p} \\
 & & \vdots \\
 & & \big\downarrow R_{p,pk} \\
(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;k)|)_{hC_{p^{r-s}}} & \xrightarrow{N_{p,k}} & (T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;k)|)^{C_{p^{r-s}}} \\
 & & \big\downarrow R_{p,k} \\
 & & *
\end{array}$$

The homotopy groups of the homotopy orbit spectra which appear as fibers are approximated by homology type spectral sequences

$$E^2_{i,j}=H_i(C_{p^r};\pi_j(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|))\Rightarrow\pi_{i+j}(T(A)\wedge|N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|))_{hC_{p^r}}.$$

These spectral sequences are concentrated in the first quadrant above the line $y=l(m)-2$. Since A is a $\mathbb{Z}/p^j$-algebra the E^2-term, and hence also the E^∞-term, is a $\mathbb{Z}/p^j$-module. Hence $\pi_i(T(A)\wedge N^{\mathrm{cy}}_{\wedge}(\Pi_n;m))_{hC_{p^r}}$ is a p-group and every element has exponent less than or equal to $j(i-l(m)+3)$. Finally, the tower of fibrations above shows that $\pi_i(T(A)\wedge N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)|)^{C_{p^r}}$ is a p-group and the exponents of the elements are bounded by

$$\sum_{t=0}^{s}j(i-l(p^tk)+3)\leqslant j\big((s+1)(i+3)+\frac{k}{n-1}\frac{p^s-1}{p-1}\big).$$

Since this bound is independent of $r\geqslant s$ the proposition follows. □

We let A be a $\mathbb{Z}/p^j$-algebra and consider the arithmetic square for $\widetilde{K}(A[X]/(X^n))$, *i.e.* the homotopy cartesian square

$$\begin{array}{ccc} \widetilde{K}(A[X]/(X^n)) & \longrightarrow & \widetilde{K}(A[X]/(X^n))^{\wedge} \\ \downarrow & & \downarrow \\ \widetilde{K}(A[X]/(X^n))\mathbb{Q} & \longrightarrow & (\widetilde{K}(A[X]/(X^n))^{\wedge})\mathbb{Q}. \end{array}$$

Here the decoration $(-)^{\wedge}$ indicates profinite completion, *i.e.* the product of the p-completions, where p ranges over the primes. The rationalization is contractible by the main theorem of [G]. Indeed,

$$\widetilde{K}_*(A[X]/(X^n)) \otimes \mathbb{Q} \cong \widetilde{\mathrm{HC}}_{*-1}(A[X]/(X^n)) \otimes \mathbb{Q}$$

and it is easily seen that the cyclic homology of a $\mathbb{Z}/p^j$-algebra is rationally trivial. Similarly, a recent result of R. McCarthy, [Mc], recalled as theorem A in [HM], shows that the cyclotomic trace induces an equivalence

$$\widetilde{K}(A[X]/(X^n))^{\wedge} \simeq \widetilde{\mathrm{TC}}(A[X]/(X^n))^{\wedge}.$$

The right hand side is rationally trivial by (3.1.8) and therefore the spectrum in the lower right hand corner of the arithmetic square vanishes. Hence we have

Theorem 3.1.9. *If A is a $\mathbb{Z}/p^j$-algebra, then $\widetilde{K}(A[X]/(X^n)) \simeq \widetilde{\mathrm{TC}}(A[X]/(X^n))$.* □

As a corollary of (3.1.8) and (3.1.9) we get theorem E of the introduction. We note that, based on work of Jan Stienstra, Chuck Weibel has shown that $\widetilde{K}_i(A[X]/(X^n))$ is a p-group, [We]. However the fact that it is bounded is new. The bound may very well tend to infinity with i. Indeed, it is shown in [HM] that as i tends to infinity there are elements in $\widetilde{K}_i(\mathbb{F}_p[X]/(X^2))$ of arbitrarily large exponent.

Theorem 3.1.10. *Let A be a $\mathbb{Z}/p^j$-algebra. Then there is a natural equivalence*

$$C(A) \simeq \mathrm{TR}(A)$$

and the spectra are p-complete.

Proof. We have $C(A) \simeq \operatorname{holim}_{\longleftarrow} \Omega\widetilde{\mathrm{TC}}(A[X]/(X^n))$ by (3.1.9), and since homotopy limits commute

$$C(A) \simeq \underset{\overleftarrow{\mathbb{I}}}{\operatorname{holim}}(\underset{\overleftarrow{n}}{\operatorname{holim}}\, \Omega\widetilde{T}(A[X]/(X^n))^{C_r}),$$

compare (3.1.3). The connectivity estimate in the proof of (3.1.9) shows that

$$\widetilde{T}(A[X]/(X^n))^{C_r} \simeq \prod_{m=1}^{\infty} (T(A) \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n; m)|)^{C_r},$$

and since homotopy limits commute with products we may take the homotopy limit over n one factor at the time. But for a fixed m the limit system is constant equal to $(T(A) \wedge S^1/C_{m+})^{C_r}$ when $n > m$ by (3.1.6), and hence

$$\underset{\overleftarrow{n}}{\operatorname{holim}}\, \widetilde{T}(A[X]/(X^n))^{C_r} \simeq \prod_{m\geqslant 1} (T(A) \wedge S^1/C_{m+})^{C_r}.$$

Since the Frobenius maps F_s preserve the splitting after m, [HM], lemma 7.2, shows that

$$\underset{\overleftarrow{F}}{\operatorname{holim}}(\underset{\overleftarrow{n}}{\operatorname{holim}}\, \widetilde{T}(A[X]/(X^n))^{C_r}) \simeq \prod_{m\geqslant 1} (T(A) \wedge S^1/C_{m+})^{G} \simeq \prod_{m\geqslant 1} \Sigma T(A)^{C_m}. \tag{3.1.11}$$

The restriction map R_r induces maps $R_{r,m}$ from a factor m to the factor m/r and annihilates the factors where r does not divide m, compare (3.1.7). Moreover, it follows from [HM], theorem 6.1 that

$$R_{r,m} = \Sigma R_r \colon \Sigma T(A)^{C_m} \to \Sigma T(A)^{C_{m/r}},$$

where R_r is the restriction map associated with $T(A)$. Hence

$$\underset{\overleftarrow{n}}{\operatorname{holim}}\, \widetilde{\mathrm{TC}}(A[X]/(X^n)) \simeq \Sigma(\prod_{m=1}^{\infty} T(A)^{C_m})^{h\langle R_r | r\geqslant 1\rangle},$$

the homotopy fixed set of the multiplicative monoid of natural numbers acting through the restriction maps, see also [HM], 3.1. But this is homeomorphic to $\Sigma\,\mathrm{TR}(A)$, and so $C(A) \simeq \Omega\Sigma\,\mathrm{TR}(A) \simeq \mathrm{TR}(A)$. □

3.2. The spectrum $\mathrm{TR}(A)$ on the right hand side in (3.1.10) is a ring spectrum in a very strong sense: it is a commutative functor with smash product defined on spheres, *cf.* [HM], proposition 1.7. On the other hand Bloch defines a pairing on $C(A)$, which makes $C(A)$ a homotopy associative ring spectrum. We recall how this is defined.

For any A-algebra R we consider the exact category $\mathbf{Nil}(R)$ whose objects are pairs (M, α) where M is a f. g. projective R-module and α is a nilpotent endomorphism of M. It comes with an obvious ascending filtration given by the exponent of α. We also write $\mathbf{P}(R)$ for the category of f. g. projective R-modules. If $A_m = A[X]/(X^m)$ and $m > n$ we have a bi-exact functor

$$\mathrm{Fil}_n \mathbf{Nil}(R) \times \mathbf{P}(A_m) \to \mathbf{P}(R), \qquad ((M,\alpha), N) \mapsto M_\alpha \otimes_{A_m} N,$$

where M_α denotes M considered as A_m-module with X acting through α. It induces a natural (weak) map of K-theory spaces

$$K(\mathrm{Fil}_n \mathbf{Nil}(R)) \wedge K(A_m) \to K(R),$$

cf. [W], and since the bi-functors above are compatible as m and n varies we get in turn

$$K(\mathbf{Nil}(R)) \wedge \underset{\overleftarrow{m}}{\mathrm{holim}}\, K(A_m) \to K(R). \tag{3.2.1}$$

Here we have also used the natural equivalence $K(\mathbf{Nil}(R)) \simeq \underset{\longrightarrow}{\mathrm{holim}}\, K(\mathrm{Fil}_n \mathbf{Nil}(R))$. Next, we recall the localization sequence

$$K(\mathbf{H}) \to K(R[t]) \to K(R[t, t^{-1}]),$$

where $\mathbf{H}$ is the category of finitely generated t-torsion $R[t]$-modules of projective dimension $\leqslant 1$. It is isomorphic as a category to $\mathbf{Nil}(R)$. For any M in $\mathbf{H}$ is projective as an R-module. Hence the boundary map of the localization sequence provides a map

$$\Omega K(R[t, t^{-1}]) \to K(\mathbf{Nil}(R)). \tag{3.2.2}$$

Now let $R = A_k$ and let $A_k \to A_k[t, t^{-1}]$ be the map which takes X to Xt^{-1}. We may compose the induced map on K-theory with the maps (3.2.1) and (3.2.2) to get a map

$$\Omega K(A_k) \wedge \underset{\overleftarrow{m}}{\mathrm{holim}}\, K(A_m) \to K(A_k).$$

and hence $\Omega K(A_k) \wedge \mathrm{holim}\ \Omega K(A_m) \to \Omega K(A_k)$. Since these are strictly compatible as k varies we get

$$(\underset{\overleftarrow{k}}{\mathrm{holim}}\, \Omega K(A_k)) \wedge (\underset{\overleftarrow{m}}{\mathrm{holim}}\, \Omega K(A_m)) \to \underset{\overleftarrow{k}}{\mathrm{holim}}\, \Omega K(A_k),$$

and finally, this lifts to a pairing

$$C(A) \wedge C(A) \to C(A). \tag{3.2.3}$$

A somewhat lenght but nowadays standard argument shows that this pairing makes $C(A)$ a homotopy associative ring spectrum. Hence we get a ring homomorphism from $C_*(A)$, with the induced ring structure, to the ring $[C(A), C(A)]_*$ of cohomology operations in the spectrum $C(A)$. When A is a $\mathbb{Z}_{(p)}$-algebra, there is an idempotent splitting of the big Witt ring $\mathbf{W}(A)$ into a product indexed by the natural numbers prime to p of copies of the p-typical Witt ring $W(A)$, the ring homomorphism above gives a corresponding set of idempotents in the ring of degree zero cohomology operations in $C(A)$. Hence we get the splitting

$$C(A) \ \simeq \prod_{(k,p)=1} C(A; p) \tag{3.2.4}$$

of the curves in $K(A)$ as a product of copies of its p-typical part $C(A; p)$.

The homotopy groups $\mathrm{TR}_*(A) = \pi_* \mathrm{TR}(A)$ are given by Milnor's sequence

$$0 \to \underset{\overleftarrow{n}}{\lim}{}^{(1)} \pi_{i+1} T(A)^{C_n} \to \mathrm{TR}_i(A) \to \underset{\overleftarrow{n}}{\lim} \pi_i T(A)^{C_n} \to 0,$$

so in particular,

$$\mathrm{TR}_0(A) = \mathbf{W}(A). \tag{3.2.5}$$

Indeed, the limit on the right is equal to $\mathbf{W}(A)$ by [HM], addendum 2.3, and the proof of *loc. cit.* proposition 2.3 shows that the derived limit on the left vanishes.

Lemma 3.2.6. *The isomorphism $C_*(A) \cong \mathrm{TR}_*(A)$ given by (3.1.10) is $\mathbf{W}(A)$-linear.*

Proof. We recall that any element in $\mathbf{W}(A)$ may be written uniquely as an infinite sum

$$x = \sum_{n=1}^{\infty} V^n(\omega(a_n)),$$

where $\omega\colon A \to \mathbf{W}(A)$ is the Teichmüller character, [M]. Therefore, to show that a map is $\mathbf{W}(A)$-linear it suffices to show that it commutes with the Verschiebung maps and with multiplication by $\omega(a)$ for all $a \in A$. In the case at hand, the Verschiebung V^n on $C_*(A)$ is induced from the A-algebra map $v_n\colon A_m \to A_{mn}$, $v_n(X) = X^n$, and similarly, multiplication by $\omega(a)$ is induced from the A-algebra map $c_a\colon A_m \to A_m$, $c_a(X) = aX$. We claim that the same holds for $\mathrm{TR}_*(A)$. Given this the lemma follows from the naturality of the equivalence in (3.1.10). We recall from (3.1.11) the equivalence

$$\operatorname*{holim}_{\overleftarrow{F,k}} \widetilde{T}(A[X]/(X^k))^{C_r} \simeq \prod_{m\geqslant 1} (T(A) \wedge S^1/C_{m+})^G \simeq \prod_{m\geqslant 1} \Sigma T(A)^{C_m}$$

from which we get

$$\operatorname*{holim}_{\overleftarrow{k}} \widetilde{\mathrm{TC}}(A[X]/(X^k)) \simeq \Sigma\, \mathrm{TR}(A)$$

upon taking homotopy fixed sets for the action of the restriction maps. It follows from (3.1.6) that v_n maps the factor m to the factor mn by the map

$$v_{n,m}\colon (T(A) \wedge S^1/C_{m+})^G \to (T(A) \wedge S^1/C_{mn+})^G$$

induced from the projection $\pi_n^{mn}\colon S^1/C_m \to S^1/C_{mn}$. Therefore, the claim for V_n follows from [HM], lemma 7.1, which shows that the diagram

$$\begin{array}{ccc} \Sigma T(A)^{C_m} & \xrightarrow{\ \simeq\ } & (T(A) \wedge S^1/C_{m+})^G \\ \big\downarrow{\scriptstyle \Sigma V_n} & & \big\downarrow{\scriptstyle (\pi_n^{mn})_*} \\ \Sigma T(A)^{C_{mn}} & \xrightarrow{\ \simeq\ } & (T(A) \wedge S^/C_{mn+})^G \end{array}$$

is homotopy commutative.

The case of c_a is more involved. Consider the map

$$\tilde{c}\colon \operatorname*{holim}_{\overleftarrow{F,k}} \widetilde{T}(A[X]/(X^k))^{C_r} \wedge A \to \operatorname*{holim}_{\overleftarrow{F,k}} \widetilde{T}(A[X]/(X^k))^{C_r}$$

which is adjoint to the map which takes $a \in A$ to the map induced from c_a. Obviously, $\tilde{c}$ preserves the splitting of (3.1.11) and we shall prove that on the factor m it is given by the composite

$$\Sigma T(A)^{C_m} \wedge A \xrightarrow{\tilde{\omega}} \Sigma T(A)^{C_m} \xrightarrow{\xi_{22}} \Sigma T(A)^{C_m},$$

where $\tilde{\omega}$ is adjoint to the map which takes $a \in A$ to multiplication by $\omega(a)$ and where ξ_{22} corresponds under the equivalence

$$\Sigma T(A)^{C_m} \simeq (T(A)^{C_m} \wedge S^1/C_{m+})^{G/C_m}$$

to the map induced from the map ξ of (1.4.5). Since ξ_{22} is homotopic to the identity by (1.4.5) it follows that c_a induces multiplication by $\omega(a)$.

To verify this description of c we will need to recall precisely how the equivalence (3.1.5) is obtained. For details we refer to [HM], §1 and 6. We shall use the same notation as used there. For any ring B and any G-space Y one has an G-equivariant associtive pairing

$$\nu_{1,0}\colon \mathrm{THH}(B;Y) \wedge |N^{\mathrm{cy}}_{\wedge}(B)_{\bullet}| \to \mathrm{THH}(B;Y),$$

where B is viewed as a pointed monoid in the multiplicative structure. Indeed, this can be defined in the same manner as the pairings of [HM], 1.7. Let $B = A[X]/(X^n)$, then $\Pi_n \subset B$ is sub pointed monoid and $A \subset B$ is subring. Hence the pairing induces a map

$$f(Y)\colon \mathrm{THH}(A;Y) \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n)_\bullet| \to \mathrm{THH}(B;Y).$$

Now let $V \subset \mathcal{U}$ be a G-representation and let S^V denote its one-point compactification. We get in particular

$$f(V)\colon t(A)(V) \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n)_\bullet| \to t(B)(V),$$

and as V varies these form a map of G-prespectra indexed on $\mathcal{U}$. The equivalence of (3.1.5) is the induced map of associated G-spectra.

Next, we note the pairing of cyclic sets

$$\theta_\bullet\colon A \wedge N^{\mathrm{cy}}_{\wedge}(\Pi_\infty)_\bullet \to N^{\mathrm{cy}}_{\wedge}(A)_\bullet,$$

which is adjoint to the map which takes $a \in A$ to the map induced from $i_a\colon \Pi_\infty \to A$, $X \mapsto a$. The wedge summand $N^{\mathrm{cy}}_{\wedge}(\Pi_\infty; m)$ is equal to $N^{\mathrm{cy}}_{\wedge}(\Pi_n; m)$, provided that $m < n$, which we henceforth assume. It follows immediately from the definitions that the diagram

$$\begin{array}{ccc}
\mathrm{THH}(A;Y) \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)_\bullet| \wedge A & \xrightarrow{f(Y)\wedge \mathrm{id}} & \mathrm{THH}(B;Y) \wedge A \\
\downarrow{\scriptstyle \mathrm{id}\wedge \mathrm{tw}} & & \\
\mathrm{THH}(A;Y) \wedge A \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)_\bullet| & & \downarrow{\scriptstyle \tilde{c}} \\
\downarrow{\scriptstyle \mathrm{id}\wedge(\theta,\mathrm{id})} & & \\
\mathrm{THH}(A;Y) \wedge |N^{\mathrm{cy}}_{\wedge}(A)_\bullet| \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)_\bullet| & & \\
\downarrow{\scriptstyle \nu_{1,0}\wedge \mathrm{id}} & & \\
\mathrm{THH}(A;Y) \wedge |N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)_\bullet| & \xrightarrow{f(Y)} & \mathrm{THH}(B;Y)
\end{array}$$

commutes, and since the maps are equivariant the corresponding diagram of C_m-fixed points is also commutative. Let us write $C = C_m$. We consider the simplicial (but not cyclic) map

$$\psi_\bullet\colon A \to N^{\mathrm{cy}}_{\wedge}(A)_\bullet \xrightarrow{\Delta_C} (\mathrm{sd}_C\, N^{\mathrm{cy}}_{\wedge}(A)_\bullet)^C$$

and note that $\tilde{\omega}$ is induced from

$$\mathrm{THH}(A;Y)^C \wedge A \xrightarrow{\mathrm{id}\wedge D\circ\psi} \mathrm{THH}(A;Y)^C \wedge |N^{\mathrm{cy}}_{\wedge}(A)_\bullet|^C \xrightarrow{\nu_{1,0}} \mathrm{THH}(A;Y)^C,$$

where D is the equivariant (non-simplicial) homeomorphism

$$D\colon |(\mathrm{sd}_C\, N^{\mathrm{cy}}_{\wedge}(A)_\bullet)^C| \to \rho_C^*|N^{\mathrm{cy}}(A)_\bullet|^C.$$

The simplical map $\psi_\bullet$ induces a cyclic map

$$F\psi_\bullet\colon FA_\bullet \to (\mathrm{sd}_C\, N^{\mathrm{cy}}_{\wedge}(A)_\bullet)^C,$$

where $FA_\bullet$ is the free cyclic set generated by the (constant) simplicial set A, *cf.* [H], §3. One verifies easily that

$$FA_\bullet \cong A \wedge (\mathrm{sd}_C\, N^{\mathrm{cy}}_{\wedge}(\Pi_n;m)_\bullet)^C,$$

such that $F\psi_\bullet = (\mathrm{sd}_C\,\theta_\bullet)^C$. Moreover, $|FA_\bullet| \cong A \wedge G_+$ and the realization of $F\psi_\bullet$ is equal to the composition

$$A \wedge G_+ \xrightarrow{\psi} |(\mathrm{sd}_C\, N^{\mathrm{cy}}_{\wedge}(A)_\bullet)^C| \wedge G_+ \xrightarrow{\mu} |(\mathrm{sd}_C\, N^{\mathrm{cy}}(A)_\bullet)^C|,$$

where μ is the action. Now the claimed description of $\tilde{c}$ follows since $\nu_{1,0}$ is G-equivariant. □

3.3. In this section we give a splitting of the spectrum $\mathrm{TR}(A)$ into copies of its 'p-typical' part $\mathrm{TR}(A;p)$ and prove theorem A of the introduction.

Proposition 3.3.1. *Let A be a $\mathbb{Z}_{(p)}$-algebra. Then there is a natural equivalence*

$$\mathrm{TR}(A) \simeq \prod_{(k,p)=1} \mathrm{TR}(A;p),$$

where the product ranges over the natural numbers prime to p.

Proof. Let $\mathcal{F}$ be the category of natural numbers ordered after division and let $\mathcal{F}_p$ and $\mathcal{F}_{p'}$ be the full subcategories of natural numbers which are a power of p and prime to p, respectively. Then $\mathcal{F} = \mathcal{F}_p \times \mathcal{F}_{p'}$, and hence

$$\mathrm{TR}(A) = \underset{\overleftarrow{\mathcal{F}}}{\mathrm{holim}}\, T(A)^{C_n} = \underset{\overleftarrow{\mathcal{F}_{p'}}}{\mathrm{holim}}(\underset{\overleftarrow{\mathcal{F}_p}}{\mathrm{holim}}\, T(A)^{C_{p^s l}}),$$

where, we remember, the limit runs over the restriction maps. We claim that when A is a $\mathbb{Z}_{(p)}$-algebra

$$\prod R_{l/k} F_k \colon T(A)^{C_n} \to \prod_{k|l} T(A)^{C_{p^s}} \tag{3.3.2}$$

is an equivalence. Given this we get

$$\underset{\overleftarrow{\mathcal{F}_p}}{\mathrm{holim}}\, T(A)^{C_{p^s l}} \simeq \prod_{k|l} \mathrm{TR}(A;p)$$

from which the proposition follows. To prove the claim suppose first that $l = q^t$ where q is a prime. By [HM], proposition 1.3, $\rho^{\#}_{C_{p^s}} T(A)^{C_{p^s}}$ is a q-cyclotomic spectrum such that we have a cofibration sequence

$$(\rho^{\#}_{C_{p^s}} T(A)^{C_{p^s}})_{hC_{q^t}} \xrightarrow{N} T(A)^{C_{p^s q^t}} \xrightarrow{R_q} T(A)^{C_{p^s q^{t-1}}}.$$

When $t = 1$ the right hand side is the trivial spectrum. Moreover, we have a homotopy commutative diagram

$$\begin{array}{ccccc}
(\rho^{\#}_{C_{p^s}} T(A)^{C_{p^s}})_{hC_{q^t}} & \xrightarrow{N} & T(A)^{C_{p^s q^t}} & \xrightarrow{R_q} & T(A)^{C_{p^s q^{t-1}}} \\
\downarrow \mathrm{trf} & & \downarrow F_q & & \downarrow F_q \\
(\rho^{\#}_{C_{p^s}} T(A)^{C_{p^s}})_{hC_{q^{t-1}}} & \xrightarrow{N} & T(A)^{C_{p^s q^{t-1}}} & \xrightarrow{R_q} & T(A)^{C_{p^s q^{t-2}}},
\end{array}$$

where trf is the transfer of the projection going in the opposite direction. It is an equivalence because $T(A)^{C_{p^s}}$ is a ring spectrum and $1/q \in W_{s+1}(A) = \pi_0 T(A)^{C_{p^s}}$, and hence the right hand square is homotopy cartesian. Now the obvious induction argument proves (3.3.2) in the case where $l = q^t$. The general case follows by a further induction over the prime divisors in l. □

Addendum 3.3.3. *On homotopy groups the map $V_d F_d$ multiplies by d on the factors k with $d|k$ and annihilates the remaining factors.*

Proof. Recall from (1.3.4) that $\mathrm{TR}(A;p)$ is a G-spectrum indexed on the complete universe $\mathcal{U}$,

$$\mathrm{TR}(A;p) = \underset{\overleftarrow{R_p}}{\mathrm{holim}}\, \rho^{\#}_{C_{p^s}} T(A)^{C_{p^s}}.$$

Moreover, we have from (3.3.2) the (non-equivariant) equivalence

$$\prod R_{l/k} F_k \colon \mathrm{TR}(A;p)^{C_l} \to \prod_{k|l} \mathrm{TR}(A;p). \tag{3.3.4}$$

Evidently, $F_d\colon \mathrm{TR}(A;p)^{C_l} \to \mathrm{TR}(A;p)^{C_{l/d}}$ corresponds to the projection which sends a factor k divisible by d to the factor k/d and annihilates the remaining factors. Next, we claim that for k not divisible by d the composite

$$\mathrm{TR}(A;p)^{C_{l/d}} \xrightarrow{V_d} \mathrm{TR}(A;p)^{C_l} \xrightarrow{R_{l/k}F_k} \mathrm{TR}(A;p)^{C_{l/k}}$$

is null homotopic. To see this we note that there exists a prime q which divides both d and l/k. The spectrum $\rho^{\#}_{C_{p^s l/d}} T(A)^{p^s l/d}$ is q-cyclotomic by [HM], proposition 1.3, and therefore we have a cofibration sequence

$$(T(A)^{C_{p^s l/d}})_{hC_{q^r}} \xrightarrow{N} T(A)^{C_{p^s q^r l/d}} \xrightarrow{R_q} T(A)^{C_{p^s q^{r-1} l/d}}.$$

We take the homotopy limit over R_p and obtain a new cofibration sequence

$$(\mathrm{TR}(A;p)^{C_{l/d}})_{hC_{q^r}} \xrightarrow{N} \mathrm{TR}(A;p)^{C_{q^r l/d}} \xrightarrow{R_q} \mathrm{TR}(A;p)^{C_{q^{r-1} l/d}}.$$

The point of this is that [HM], lemma 2.2, shows that V_{q^r} factors through N such that $R_qV_{q^r}$ is null homotopic. Since $R_{l/k}F_kV_d = F_kR_{l/k}V_d = R_{l/qk}V_{d/q^r}R_qV_{q^r}$ the claim follows. Now recall that F_dV_d induces multiplication by d on homotopy groups. It follows that on homotopy groups $V_d\colon \mathrm{TR}(A;p)^{C_{l/d}} \to \mathrm{TR}(A;p)^{C_l}$ maps the factor k/d to the factor k by multiplication by d. Finally,

$$\mathrm{TR}(A) = \operatorname*{holim}_{\overleftarrow{\mathcal{F}_{p'}}} \mathrm{TR}(A;p)^{C_l}$$

and the limit system on the right induces a Mittag-Leffler system on the level of homotopy groups. Therefore we get the same description of F_d and V_d on $\mathrm{TR}_*(A)$. □

Recall that idempotent decomposition of $\mathbf{W}(A)$, preceeding (3.2.4), the projection onto factors k divisible by d is given by $\frac{1}{d}V_dF_d$. Hence the projection onto the factor k is given by

$$\mathrm{pr}_k = \prod_{(d,p)=1} (\tfrac{1}{k}V_kF_k - \tfrac{1}{dk}V_{dk}F_{dk}).$$

It follows that the product decompositions of $\mathrm{TR}_*(A)$ induced from (3.3.1) and from the idempotents in $\mathbf{W}(A)$ are equal. This proves theorem A of the introduction.

3.4. The differential defined by Bloch, [B], on the symbolic part of $SC_*(A) \subset C_*(A)$ for certain rings was extended to a degree one operator on all of $C_*(A)$ and all rings by Stienstra, [St]. The basis of the construction is a map

$$K(\mathbb{Z}[t,t^{-1}]) \to K(\mathbf{Nil}(\mathbb{Z}[t]/(t^n))) \tag{3.4.1}$$

which we now descibe. In the localization sequence

$$K(\mathbf{H}) \to K(\mathbb{Z}[t,y]) \to K(\mathbb{Z}[t,y,(1-ty)^{-1}])$$

the right hand map is split by the map induced from the ring homomorphism mapping t and y to zero, and moreover, the resolution and devissage theorems show that $K(\mathbf{H}) \simeq K(\mathbb{Z}[t,t^{-1}])$. In particular, we get a map

$$K(\mathbb{Z}[t,t^{-1}]) \to \Omega K(\mathbb{Z}[t,y,(1-ty)^{-1}]).$$

We compose this with the map

$$\Omega K(\mathbb{Z}[t,y,(1-ty)^{-1}]) \to \Omega K(\mathbb{Z}[t,u,u^{-1}]/(t^n))$$

given by the ring homomorphism which maps t to t and y to u^{-1}, and finally, we compose with (3.2.2) to get the required map.

We may combine (3.4.1), the map on K-theory induced from $\mathbb{Z}[t]/(t^n) \to A[t]/(t^n)$ and the pairing (3.2.1) to get a pairing

$$K(\mathbb{Z}[t,t^{-1}]) \wedge \operatorname*{holim}_{\overleftarrow{m}} K(A_m) \to K(A_n).$$

For varying n these are compatible, so get a pairing into the homotopy limit of the spaces on the right. Finally, this factors to

$$K(\mathbb{Z}[t,t^{-1}]) \wedge C(A) \to C(A). \tag{3.4.2}$$

We have $K_1(\mathbb{Z}[t,t^{-1}]) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ generated by t and -1. Let us note that t and -1 corresponds to σ and η under the isomorphism $\pi_1^S(S^1_+) \cong K_1(\mathbb{Z}[t,t^{-1}])$ given by the unit of the spectrum $K(\mathbb{Z})$ and the assembly map, compare (1.4.1). Now Stienstra defines a degree one map

$$\delta\colon C_*(A) \to C_{*+1}(A) \tag{3.4.3}$$

as the pairing with t. However, little is known about this map except that it extends Bloch's differential on the symbolic part and Stienstra's differential on $\widetilde{K}_*(\mathbf{End}(A))$. We leave it as an open question whether $\delta = d$ under the isomorphism of (3.1.10).

3.5. In this section we evaluate the complex of p-typical curves for the ring $k[\epsilon]$ of dual numbers over the perfect field k and prove theorem D of the introduction.

Let $W_s \subset \mathbb{R}[C_s]$ denote the maximal complex subrepresentation and let S^{W_s} be the one-point compactification. We shall write $T(k)_{W_s}$ for the smash product G-spectrum $T(k) \wedge S^{W_s}$. Recall from [HM], 7.2, that there is a cofibration sequence of G-spectra

$$\bigvee_{r\geqslant 1} T(k)_{W_{2r}} \wedge S^1/C_{r+} \xrightarrow{\text{sq}} T(k) \vee \bigvee_{s\geqslant 1} T(k)_{W_s} \wedge S^1/C_{s+} \to T(k[\epsilon]), \tag{3.5.1}$$

where the map sq takes the summand r to the summand $s = 2r$ by the map which is the identity on the first smash factor and the projection on the second. In fact, the spectra in (3.5.1) are cyclotomic in the sense of [HM], 1.2, and the maps preserve the cyclotomic structure. It follows that we may apply the construction $\mathrm{TR}(-;p)$ and still have a cofibration sequence of G-spectra,

$$\mathrm{TR}(\bigvee_{r\geqslant 1} T(k)_{W_{2r}} \wedge S^1/C_{r+};p) \xrightarrow{\text{sq}} \mathrm{TR}(T(k) \vee \bigvee_{s\geqslant 1} T(k)_{W_s} \wedge S^1/C_{s+};p) \to \mathrm{TR}(k[\epsilon];p). \tag{3.5.2}$$

We evaluate the map sq on homotopy groups. As in the smooth case the homotopy groups turns out to be rather big, so we first introduce some notation.

Consider the differential graded algebra

$$E^*_\sigma = W\Omega^*_{k[X]} \otimes_{W(k)} S_{W(k)}\{\sigma\}, \tag{3.5.3}$$

which is the tensor product of the de Rham-Witt complex for $k[X]$ and a polynomial algebra over $W(k)$ in one generator σ of degree 2 (with zero differential). Based on (2.1.3) we have the following alternate description: every element $\omega \in E^m_\sigma$ may be written uniquely as either

$$\omega = \sum_{j\in\mathbb{N}[1/p]} a_{ij}\sigma^i X^j \quad \text{or} \quad \omega = \sum_{j\in\mathbb{N}[1/p]-0} b_{ij}\sigma^i X^j d\log X,$$

depending on whether m is even or odd. The coefficients $a_{ij}, b_{ij} \in W(k)$ and are subject to the requirement that $\mathrm{den}(j)|a_{ij}$ and that for every N, $v_p(a_{ij})$ and $v_p(\mathrm{den}(j)b_{ij})$ are $\geqslant N$ for all but finitely many j. Here $v_p(a)$ denotes the p-adic valuation of a.

We next define DG-ideals $I_\sigma, J_\sigma \subset E_\sigma$. For $i \in \mathbb{N}$ and $j \in \mathbb{N}[1/p] - 0$ let

$$\epsilon(j) = \begin{cases} 1, & \text{if } \mathrm{num}(j) \text{ is even or } p = 2 \\ 0, & \text{else,} \end{cases}$$

and let $m(i,j)$ be the unique integer such that $p^{m(i,j)-1}j \leqslant 2i+1+\epsilon(j) < p^{m(i,j)}j$. We also set $m(i,0)=1$ for $i>0$ and $m(0,0)=0$. We define

$$I^m_\sigma = \{\omega \in E^m_\sigma | v_p(a_{ij}) \geqslant m(i,j) \text{ resp. } v_p(\mathrm{den}(j)b_{ij}) \geqslant m(i,j), \text{ for all } j \in \mathbb{N}[1/p]\},$$

and

$$J^m_\sigma = \{\omega \in E^m_\sigma | v_p(a_{ih} \geqslant m(i,2h) \text{ resp. } v_p(\mathrm{den}(h)b_{ih}) \geqslant m(i,2h), \text{ for all } h \in \mathbb{N}[1/p]\}.$$

One may check that this defines *DG*-ideals of E^*_σ. However, it would be desirable to have a more conceptual description. Also let $\tilde{E}^*_\sigma \subset E^*_\sigma$ be the augmentation ideal, *i.e.* the series with $a_{i,0}=0$, and let $\tilde{J}^*_\sigma = \tilde{E}^*_\sigma \cap J^*_\sigma$.

Proposition 3.5.5. *Let k be a perfect field of positive characteristic p. Then the map* sq *in (3.5.2) is given on homotopy groups as*

$$\mathrm{sq}_*\colon \tilde{E}^*_\sigma/\tilde{J}^*_\sigma \to E^*_\sigma/I^*_\sigma,$$

where sq_* *is the DG-map given by* $\mathrm{sq}_*(X)=X^2$.

Proof. Recall that for any cyclotomic spectrum T,

$$\mathrm{TR}(T;p) = \underset{\overleftarrow{R}}{\mathrm{holim}}\, \rho^\#_{C_{p^n}} T^{C_{p^n}}.$$

So we must evaluate the C_{p^n}-fixed points of the spectra in (3.5.1). Let us write $v=v(n,s)=\min\{n,v_p(s)\}$. Then we have from [HM],§7, that

$$\rho^\#_{C_{p^n}}(T(k)_{W_s}\wedge S^1/C_{s+})^{C_{p^n}} \simeq_G \rho^\#_{C_{p^v}} T(k)^{C_{p^v}}_{W_s} \wedge S^1/C_{s/p^v+}. \tag{3.5.6}$$

The cyclotomic structure on the spectra in (3.5.1) is given by the remark preceeding [HM], theorem 6.1. We recall that the map R takes a summand with s (resp. r) divisible by p to the summand s/p by a map

$$R^{(s)}\colon \rho^\#_{C_{p^v}} T(k)^{C_{p^v}}_{W_s} \wedge S^1/C_{s/p^v_+} \to \rho^\#_{C_{p^v}} T(k)^{C_{p^{v-1}}}_{W_{s/p}} \wedge S^1/C_{s/p^v_+}$$

and annihilates the remaining summands. Moreover, $R^{(s)} = R_{W_s}\wedge \mathrm{id}$ and there is a cofibration sequence of non-equivariant spectra

$$(T(k)_{W_s})_{hC_{p^v}} \xrightarrow{N} T(k)^{C_{p^v}}_{W_s} \xrightarrow{R_{W_s}} T(k)^{C_{p^{v-1}}}_{W_{s/p}},$$

see [HM], sections 1.2. It is convenient to reindex the wedge sums in (3.5.1) such that R preserves the index. So let $j=s/p^n$ and $h=r/p^n$, respectively. We may then rewrite (3.5.6) as

$$\rho^\#_{C_{p^n}}(T(k)_{W_s}\wedge S^1/C_{s+})^{C_{p^n}} \simeq_G \rho^\#_{C_{p^n/\mathrm{den}(j)}} T(k)^{C_{p^n/\mathrm{den}(j)}}_{W_{p^n j}} \wedge S^1/C_{\mathrm{num}(j)+},$$

and similarly we have

$$\rho^\#_{C_{p^n}}(T(k)_{W_{2r}}\wedge S^1/C_{r+})^{C_{p^n}} \simeq_G \rho^\#_{C_{p^n/\mathrm{den}(h)}} T(k)^{C_{p^n/\mathrm{den}(h)}}_{W_{p^n 2h}} \wedge S^1/C_{\mathrm{num}(h)+}.$$

With this indexing the map sq in (3.5.2) takes the summand h to the summand $j=2h$ by a map

$$\mathrm{sq}^{(h)}\colon \rho^\#_{C_{p^n/\mathrm{den}(h)}} T(k)^{C_{p^n/\mathrm{den}(h)}}_{W_{p^n 2h}} \wedge S^1/C_{\mathrm{num}(h)+} \to \rho^\#_{C_{p^n/\mathrm{den}(j)}} T(k)^{C_{p^n/\mathrm{den}(j)}}_{W_{p^n j}} \wedge S^1/C_{\mathrm{num}(j)+},$$

and [HM], lemma 7.1, shows that

$$\mathrm{sq}^{(h)} = \begin{cases} \mathrm{id}\wedge\mathrm{pr}, & \text{if } p \text{ is odd, or } p=2 \text{ and } \mathrm{den}(h)=1, \\ V_2\wedge\mathrm{id}, & \text{if } p=2 \text{ and } \mathrm{den}(h)>1. \end{cases} \tag{3.5.7}$$

It remains to take wedges over j and h and then homotopy limit over R. However, we have already shown in the proof of (3.1.8) that the wedge sums in (3.5.1) and (3.5.2) may be replaced by the corresponding products. Therefore, we may instead first take the homotopy limit over R and then wedges over h and j. The canonical map

$$\pi_{2i} \operatorname*{holim}_{\overleftarrow{R}} T(k)^{C_{p^n/\operatorname{den}(j)}}_{W_{p^n j}} \to \pi_{2i} T(k)^{C_{p^{m-1}/\operatorname{den}(j)}}_{W_{p^{m-1}j}}$$

is an isomorphism provided that $2i < \dim W_{p^m j}$. Indeed, this follows from the cofibration sequence above. On the other hand, [HM], proposition 8.1, shows that

$$\pi_{2i} T(k)^{C_{p^{m-1}/\operatorname{den}(j)}}_{W_{p^{m-1}j}} \cong W(k)/(p^m/\operatorname{den}(j)) \cong \operatorname{den}(j) W(k)/p^m W(k),$$

when $\dim W_{p^{m-1}j} \leqslant 2i$. Similarly, we have an isomorphism

$$\pi_{2i} \operatorname*{holim}_{\overleftarrow{R}} T(k)^{C_{p^n/\operatorname{den}(h)}}_{W_{p^n 2h}} \to \pi_{2i} T(k)^{C_{p^{m-1}/\operatorname{den}(h)}}_{W_{p^{m-1}2h}},$$

whenenver $2i < \dim W_{p^m 2h}$, and

$$\pi_{2i} T(k)^{C_{p^{m-1}/\operatorname{den}(h)}}_{W_{p^{m-1}2h}} \cong W(k)/(p^m/\operatorname{den}(h)) \cong \operatorname{den}(h) W(k)/p^m W(k),$$

if $\dim W_{p^{m-1}2h} \leqslant 2i$. Finally, $\dim W_s = s-1$, if s is odd, and $\dim W_s = s-2$, if s is even. This shows that the homotopy groups of the middle and left hand term in (3.5.2) are as claimed in the statement of the proposition. The map sq is given by (3.5.7) since pr induces the identity in even dimensions and multiplication by 2 in odd dimensions, respectively, while V_2 corresponds to the map V_2 on Witt vectors, *cf.* [HM], proposition 8.1. □

When p is odd, sq_* is injective and so we have

Corollary 3.5.8. *If* $\operatorname{char} k$ *is odd, then there is a short exact sequence of complexes*

$$0 \to \tilde{E}^*_\sigma/\tilde{J}^*_\sigma \xrightarrow{\mathrm{sq}_*} E^*_\sigma/I^*_\sigma \to C_*(k[\epsilon];p) \to 0. \qquad \square$$

Proof of theorem D. First note that if $\operatorname{den}(j) > 1$ then the differential induces an isomorphism

$$\operatorname{den}(j) W(k)/p^{m(i,j)} W(k)\langle \sigma^i X^j \rangle \xrightarrow{d} W(k)/(p^{m(i,j)}/\operatorname{den}(j)) W(k) \langle \sigma^i X^j d\log X \rangle,$$

so we may disregard the summands in E^*_σ/I^*_σ and $\tilde{E}^*_\sigma/\tilde{J}^*_\sigma$ generated by $\sigma^i X^j$ or $\sigma^i X^j d\log X$ with $\operatorname{den}(j) > 1$. So we only have to consider the summands where j is an natural number.

Suppose now that p is odd. Then sq_* maps $\tilde{E}^*_\sigma/\tilde{J}^*_\sigma$ onto the summands in E^*_σ/I^*_σ with $j > 0$ and even. So we only get a contribution in the cohomology from the summands $j = 0$ and $j \geqslant 1$, odd. The differential on such a summand is given by

$$d : W(k)/p^{m(i,j)} W(k) \langle \sigma^i X^j \rangle \to W(k)/p^{m(i,j)} \langle \sigma^i X^j d\log X \rangle, \qquad d(\sigma^i X^j) = j \sigma^i X^j d\log X.$$

This finishes the proof for p odd. The proof for $p = 2$ is similar. □

References

[B] P. Berthelot, *Cohomologie Cristalline des Schemas de Caracteristique* $p > 0$, Lecture Notes in Mathematics, Springer-Verlag **407** (1974).

[Bl] S. Bloch, *Algebraic K-theory and crystalline cohomology*, Publ. Math. I. H. E. S. **47** (1977), 187-268.

[Bö] M. Bökstedt, *Topological Hochschild homology*, preprint, Bielefeld.

[Bö1] M. Bökstedt, *Topological Hochschild homology of* $\mathbb{F}_p$ *and* $\mathbb{Z}$, preprint, Bielefeld.

[BHM] M.Bökstedt, W.C.Hsiang, I.Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465-540.

[BM] M. Bökstedt, I. Madsen, *Topological cyclic homology of the integers*, Asterisque **226** (1994), 57-143.

[BK] A. K. Bousfield, D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Springer-Verlag **304** (1972).

[CE] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press (1956).
[G] T. Goodwillie, *Algebraic K-theory and cyclic homology*, Ann. Math. **24** (1986), 344-399.
[GM] J. P. C. Greenlees, J. P. May, *Generalized Tate Cohomology*, Memoirs AMS **543** (1995).
[H] L. Hesselholt, *Stable topological cyclic homology is topological Hochschild homology*, Asterisque **226** (1994), 175-192.
[HM] L. Hesselholt, I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Preprint series, Aarhus University **2** (1995).
[HM1] L. Hesselholt, I. Madsen, *The S^1-Tate spectrum for J*, Bol. Soc. Math. Mexicana **37** (1992), 215-240.
[HKR] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Trans. A.M.S. **102** (1962), 383-408.
[I] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Éc. Norm. Sup. (4) **12** (1979), 501-661.
[I1] L. Illusie, *Report on crystalline cohomology*, Proc. Symp. Pure Math. **29** (1975), 459-478.
[K] K. Kato, *A generalization of local class field theory by using K-groups II*, Jour. Fac. Sc. Tokyo **27** (1980), 603-683.
[LMS] L. G. Lewis, J. P. May, M. Steinberger, *Stable equivariant homotopy theory*, Lecture Notes in Mathematics, Springer-Verlag **1213** (1986).
[L] J.-L. Loday, *Cyclic homology*, Springer-Verlag 1994.
[LQ] J.-L. Loday, D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comm. Math. Helv. **59** (1984), 565-591.
[Mc] R. McCarthy, *On relative algebraic K-theory and topological cyclic homology*, (in preparation).
[M] D. Mumford, *Lectures on curves on an algebraic surface*, Princeton University Studies.
[S] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, Springer-Verlag **67**.
[St] J. Stienstra, *Operations in the higher K-theory of endomorphisms*, in Current trends in algebraic topology, CMS Conference Proc. **2** (1982), 59-115.
[W] F. Waldhausen, *Algebraic K-theory of spaces*, Lecture Notes in Mathematics, Springer-Verlag **1126**, 318-419.
[We] C. A. Weibel, *Mayer-Vietoris sequences and module structures on NK_**, in Algebraic K-theory, Evanston 1980, Lecture Notes in Mathematics **854** (1981), 466-493.
[WG] C. A. Weibel, S. C. Geller, *Étale descent for Hochschild and cyclic homology*, Comment. Math. Helv. **66** (1991), 368-388.

Dept. of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
USA
email: larsh@math.mit.edu