On $K_4^{(3)}$ of curves over number fields.

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Abstract. In this paper we consider the group $K_4^{(3)}(F)$ of a field, in case F is the function field of a smooth geometrically irreducible curve over a number field. We do this using the complexes constructed in [4], together with an auxiliary complex. On the image in $K_4^{(3)}(F)$ of those complexes, we derive a formula for the Beilinson regulator, and compute an approximation of the boundary map at the closed points of the curve in the localization sequence. We give a way of finding examples of elliptic curves E with elements in $K_4^{(3)}(E)$, and in some cases use computer calculations to check numerically the relation between the regulator and the L–function, as conjectured by Beilinson.

1 Introduction

Let F be a field. The explicit representation of $K_2(F)$ in terms of generators and relations has given rise to a lot of interest in finding similar representations of higher K–groups. For this, it turns out to be better to decompose the K–theory according to the action of the adams operations. In this paper, for an abelian group A, let $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$, and write $K_n^{(j)}$ for the subspace of $K_n \otimes \mathbb{Q}$ on which all adams operations ψ^k act as multiplication by k^j. Suslin found a representation of $K_3^{(2)}(F)$, see [11]. (Actually, Suslin's result is much more precise, but we ignore this here.) It arises as the first cohomology group of a complex in degrees one and two, $B_2 \xrightarrow{\mathrm{d}} \bigwedge^2 F_{\mathbb{Q}}^*$ where B_2 is the free $\mathbb{Q}$–vector space on $\{x\}$, $x \in F \setminus \{0,1\}$, modulo certain explicit relations. $\mathrm{d}\{x\} = x \wedge (1-x)$, so the second cohomology group equals $K_2^M(F)_{\mathbb{Q}} = K_2^{(2)}(F)$. This complex computes the weight two part of the K–theory of F that is conjectured to be non–zero.

In [7], Goncharov defined explicit complexes in terms of generators and relations that should compute the weight n–part of the K–theory of fields. He also gives a double complex that should compute the weight 3 part of the K–theory of a regular scheme, which we reproduce here in case the scheme is a curve C defined over a number field k. Let C^1 be the set of closed points in C. The top row should compute $K_*^{(3)}(F)$, and the bottom row is known to compute $\coprod_{x \in C^1} K_*^{(2)}(k(x))$. ($\mathcal{B}_3(F)_{\mathbb{Q}}$ is placed in degree one, and both coboundaries have degree 1.)

$$\begin{tikzcd}
\mathcal{B}_3(F)_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] & F_{\mathbb{Q}}^* \otimes \mathcal{B}_2(F)_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & \bigwedge^3 F_{\mathbb{Q}}^* \arrow[d, "\partial_2"'] \\
 & \coprod_{x \in C^1} \mathcal{B}_2(k(x))_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] & \coprod_{x \in C^1} \bigwedge^2 k(x)_{\mathbb{Q}}^*
\end{tikzcd}$$

Here $\mathcal{B}_n(F)_{\mathbb{Q}}$ is a $\mathbb{Q}$–vector space generated by symbols $\{u\}_n$ (modulo certain relations), $u \in F \setminus \{0,1\}$, and similarly for the $k(x)$. The maps are given by

$$\begin{aligned}
&\mathrm{d}\{u\}_3 = u \otimes \{u\}_2 \\
&\mathrm{d}(v \otimes \{u\}_2) = v \wedge u \wedge (1-u) \\
&\partial_{1,x}(v \otimes \{u\}_2) = \mathrm{ord}_x(v)\{u(x)\}_2 \\
&\partial_{2,x}(f \wedge g \wedge h) = \mathrm{ord}_x(f)\,\overline{g_x} \wedge \overline{h_x} - \mathrm{ord}_x(g)\,\overline{f_x} \wedge \overline{h_x} + \mathrm{ord}_x(h)\,\overline{f_x} \wedge \overline{g_x}
\end{aligned}$$

with the convention that $\{0\}_2 = \{1\}_2 = \{\infty\}_2 = 0$. Here $\overline{f_x}$ is defined as follows. Let π_x be a uniformizer at x. Then $\overline{f_x} = (f\pi_x^{-\mathrm{ord}_x(f)})_{|x}$. ($\partial_{2,x}$ does not depend on the choice of π_x.)

For the case of a field, various complexes of the "right" shape have been constructed by various authors. In [4] the author constructed a candidate $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$ in degrees 1 to n for the complexes for F a field of characteristic zero, assuming the Soulé–Beilinson conjecure on weights. There is a natural map $H^p(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \to K^{(n)}_{2n-p}(F)$. In this paper we shall also obtain results related to the double complex, see Proposition 5.1. We also investigate the relation between ∂_1 and the boundary map $K_4^{(3)}(F) \to \coprod_{x \in C^1} K_3^{(2)}(k(x))$, see Corollary 5.4 and Remark 5.5.

Let us consider the situation for low weights in more detail. For weight two and three, we have the complexes

$$\widetilde{\mathcal{M}}^\bullet_{(2)}(F)\colon \quad \widetilde{M}_{(2)}(F) \to \bigwedge^2 F^*_{\mathbb{Q}}$$

and

$$\widetilde{\mathcal{M}}^\bullet_{(3)}(F)\colon \quad \widetilde{M}_{(3)}(F) \to F^*_{\mathbb{Q}} \otimes \widetilde{M}_{(2)}(F) \to \bigwedge^3 F^*_{\mathbb{Q}}.$$

Here $\widetilde{M}_{(2)}(F)$ (resp. $\widetilde{M}_{(3)}(F)$) is generated by elements $[u]_2$ (resp. $[u]_3$) with $u \in F^* \setminus \{1\}$. Differentials are as follows. In $\widetilde{\mathcal{M}}^\bullet_{(2)}(F)$, $\mathrm{d}[u]_2 = u \wedge (1-u)$, and in $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$ we have $\mathrm{d}[u]_3 = u \otimes [u]_2$, $\mathrm{d}(v \otimes [u]_2) = v \wedge u \wedge (1-u)$. We recall that if k is a number field, it follows from Suslin's work that the map $H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \to K_3^{(2)}(k)$ is an isomorphism ([4, Theorem 5.3]). For general fields F, we have $H^2(\widetilde{\mathcal{M}}^\bullet_{(2)}(F)) \cong K_2^{(2)}(F) \cong K_2^M(F)_{\mathbb{Q}}$.

For the complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$ the situation is as follows. If k is a number field, then $H^1(\widetilde{\mathcal{M}}^\bullet_{(3)}(k))$ is isomorphic with $K_5^{(3)}(k)$, see [4, Theorem 5.3]. As for the third cohomology group, it is isomorphic to $K_3^{(3)}(F) = K_3^M(F)_{\mathbb{Q}}$, which is explicit in generators and relations, and the boundary map in localizations can be computed using the analogue of the tame symbol here. That leaves $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$, mapping to $K_4^{(3)}(F)$ as the most interesting group to study. We do this in case F is the function field of a curve C over a number field k. In this case $K_4^{(3)}(C) \subset K_4^{(3)}(F)$

and the Beilinson regulator map from $K_4^{(3)}(C)$ lands in $H^2_{\mathcal{D}}(C \otimes_{\mathbb{Q}} \mathbb{C}; \mathbb{R}(3)) \cong H^1_{\mathrm{dR}}(C(\mathbb{C})_{\mathrm{an}}; \mathbb{R}(2))$, i.e., in the middle cohomology group. We give a formula for the Beilinson regulator on the intersection in $K_4^{(3)}(F)$ of $K_4^{(3)}(C)$ and the image of $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$, see Theorem 4.2 and Remark 4.5. We also compute the boundary map $K_4^{(3)}(F) \to \coprod_{x \in C^1} K_3^{(2)}(k(x))$ on the image of $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$, up to some indeterminacy coming from $K_3^{(2)}(k)$. See Corollary 5.4 and Remark 5.5 for the precise result. Because the Beilinson–Borel regulator is injective on $K_3^{(2)}(k(x))$, the computation in this case can be carried out at the level of Deligne cohomology. It should be pointed out that in this paper there are no assumptions about weights, see [4, Remark 3.23].

The starting point for this paper was [3], where the auxiliary complex $\mathcal{C}^\bullet$ to be defined in section 3 below was constructed, and some computations of the regulator map were carried out. This was written in reaction to [6], and it shows in particular that the formula for the regulator (see Theorem 4.2) coincides with the formulas found in [6]. As we shall see in the final section of the paper that the regulator can be non–zero, it follows that there exists non–trivial examples of the theory in [6].

The organization of this paper is as follows. Section 2 contains review of material needed from [4]. In the third section another complex $\mathcal{C}^\bullet$ is defined, which is much more useful for computations relating to $K_4^{(3)}(F)$ than the complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$. Then sections 4 and 5 contain the computation of the regulator map and the boundary. Finally, in section 6 we give ways of finding explicit elements in the groups considered, and give computer calculations corroborating Beilinson's conjectures on special values of L–functions. These computations also show that the theory does produce non–zero elements.

2 Review and preliminaries

We begin by introducing some notation. For any abelian group A, we write $A_{\mathbb{Q}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$. If S is a subset of a $\mathbb{Q}$–vector space, we denote by $<S>$ the subspace spanned by the elements of S. Finally, let $\mathbb{Q}(n) = (2\pi i)^n \mathbb{Q} \subset \mathbb{C}$, and similarly for $\mathbb{R}$. As $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n)$, we let π_{n-1} be the projection onto the first component, $\mathbb{R}(n-1)$.

We briefly recall the construction of the complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$ in [4]. Let t be the standard affine coordinate on $\mathbb{P}^1$, and let $X = \mathbb{P}^1_F \setminus \{t = 1\}$. Let $U \subset F^* \setminus \{1\}$ be finite. Let t_i be the coordinate on the i–th copy of X in X^n. In [4] a formalism of "multi–relative" K–theory with weights is developed. The relativity is taken step by step. As a shorthand we write $\square^n$ for $\{t_1 = 0, \infty\}, \ldots, \{t_n = 0, \infty\}$. The idea of this is that it follows from the long exact sequence in relative K–theory (cf. [4, (19)])

$$\cdots \to K_{m+1}^{(j)}(\{t_n = 0, \infty\}; \square^{n-1}) \to K_m^{(j)}(X^n_Y; \square^n) \to K_m^{(j)}(X^n_Y; \square^{n-1}) \to \cdots$$

together with the homotopy property for K–theory for some reasonable regular scheme Y that $K_m^{(j)}(X_Y^n;\square^n) \cong K_{m+1}^{(j)}(X_Y^{n-1};\square^{n-1})$ for $m \geq 0$. (We shall apply this isomorphism only in case Y is a Zariski open part of a smooth curve over a number field, or the Spec of its function field, in which case all conditions are satisfied.) Repeating this, we get $K_m^{(j)}(X_Y^n;\square^n) \cong K_{m+n}^{(j)}(Y)$. Note that there is no obvious choice of this isomorphism, which will result in statements up to sign below.

With $X^n_{\mathrm{loc}} = X^n \setminus \{t_i = u_j,\ u_j \in U,\ i = 1,\dots,n\}$, $K_n^{(n)}(X^{n-1}_{\mathrm{loc}};\square^{n-1})$ contains an element $[u]_n$ for $u \in U$. For the construction of $\widetilde{\mathcal{M}}^\bullet_{(2)}(F))$ and $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$, one starts with rows in spectral sequences:

(1)

$$\begin{aligned} C^\bullet_{(2)}\colon \quad & K_2^{(2)}(X_{\mathrm{loc}};\square) \to \coprod_{x\in U} F^*_{\mathbb{Q}} \\ C^\bullet_{(3)}\colon \quad & K_3^{(3)}(X^2_{\mathrm{loc}};\square^2) \to \left(\coprod_{x\in U} K_2^{(2)}(X_{\mathrm{loc}};\square)_{|t=u}\right)^{\oplus 2} \to \coprod_{x,y\in U} F^*_{\mathbb{Q}|t_1=x,t_2=y}. \end{aligned}$$

(The $\oplus 2$ here corresponds to the two directions $t_1 = u$ and $t_2 = u$.) Viewing those as cohomological complexes starting in degree one, there are natural maps $H^p(C^\bullet_{(n)}) \to K^{(n)}_{2n-p}(F)$ if one assumes the Beilinson-Soulé conjecture about weights. However, for the two complexes above the maps do exist without assumptions except for $n = 3$, $p = 1$, which is a case we shall not use below. See [4, p. 222] for details. The element $[u]_2$ has boundary $(1-u)^{-1}_{|t=u}$, and $[u]_3$ has boundary $-[u]_{2|t_1=u} + [u]_{2|t_2=u}$. Moreover, $C^\bullet_{(3)}$ carries an action of S_2 by interchanging the two coordinates, and $[u]_3$ is in the alternating part for the S_2–action. Let

$$(1+I)^* = K_1^{(1)}(X_{\mathrm{loc}};\square) = \left\{ F(t) = \prod_i (t-x_i)^{n_i}(t-1)^{-n_i} | x_i \in U,\ \prod_i x_i^{n_i} = 1 \right\}_{\mathbb{Q}}.$$

Note that there are two cup products

$$(1+I)^* \cup K_2^{(2)}(X_{\mathrm{loc}};\square) \to K_3^{(3)}(X^2_{\mathrm{loc}};\square^2)$$

depending on which of the two coordinates on X^2_{loc} we use for which factor. We let $(1+I)^* \tilde{\cup} K_2^{(2)}(X_{\mathrm{loc}};\square)$ denote the span of the images of both possiblities. With

$$\begin{aligned} \mathrm{symb}_2 &= <[u]_2> + (1+I)^* \cup F^*_{\mathbb{Q}} \subset K_2^{(2)}(X_{\mathrm{loc}};\square) \\ \mathrm{symb}_3 &= <[u]_3> + (1+I)^* \tilde{\cup}\, \mathrm{symb}_2 \subset K_3^{(3)}(X^2_{\mathrm{loc}};\square^2) \end{aligned}$$

we get subcomplexes

$$\begin{aligned} C^\bullet_{(2),\log}\colon \quad & \mathrm{symb}_2 \to \coprod F^*_{\mathbb{Q}} \\ C^\bullet_{(3),\log}\colon \quad & \mathrm{symb}_3 \to \left(\coprod \mathrm{symb}_2\right)^{\oplus 2} \to \coprod F^*_{\mathbb{Q}} \end{aligned} \tag{2}$$

of (1). The subcomplexes of those given by

$$\begin{aligned} J^\bullet_{(2)}\colon &\quad (1+I)^*\cup F^*_{\mathbb{Q}} \longrightarrow \mathrm{d}(\dots)\\ J^\bullet_{(3)}\colon &\quad (1+I)^*\tilde{\cup}\,\mathrm{symb}_2 \longrightarrow \mathrm{d}(\dots) + \Big(\coprod (1+I)^*\cup F^*_{\mathbb{Q}}\Big)^{\oplus 2} \longrightarrow \mathrm{d}(\dots)\end{aligned}$$

are acyclic ([4, Lemma 3.7]). Taking the quotient complex $C^\bullet_{(2),\log}/J^\bullet_{(2)}$ and the part of $C^\bullet_{(3),\log}/J^\bullet_{(3)}$ where interchanging the two coordinates acts as multiplication by -1 we find complexes ($<U> \subset F^*_{\mathbb{Q}}$)

$$\begin{aligned} \mathcal{M}^\bullet_{(2)}\colon &\quad M_{(2)} \longrightarrow <U>\otimes F^*_{\mathbb{Q}}\\ &\qquad [x]_2 \mapsto x\otimes(1-x)\\ \mathcal{M}^\bullet_{(3)}\colon &\quad M_{(3)} \longrightarrow <U>\otimes M_{(2)} \longrightarrow \bigwedge^2 <U>\otimes F^*_{\mathbb{Q}}\\ &\qquad\qquad\qquad\qquad y\otimes[x]_2 \mapsto y\wedge x\otimes(1-x)\end{aligned}$$

Here we let $y\otimes[x]_2$ correspond to the class of $-[x]_{2|t_1=y}+[x]_{2|t_2=y}$. One then takes limits over larger and larger U in order to obtain $\mathcal{M}^\bullet_{(2)}(F)$ and $\mathcal{M}^\bullet_{(3)}(F)$. $\widetilde{\mathcal{M}}^\bullet_{(2)}(F)$ is the quotient of $\mathcal{M}^\bullet_{(2)}(F)$ by the acyclic subcomplex $<[u]_2+[u^{-1}]_2> \longrightarrow \mathrm{Sym}^2(F^*_{\mathbb{Q}})$. $\widetilde{\mathcal{M}}^\bullet_{(3)}(F)$ is obtained as the quotient of $\mathcal{M}^\bullet_{(3)}(F)$ by the subcomplex $<[u]_3-[u^{-1}]_3> \longrightarrow F^*_{\mathbb{Q}}\otimes <[u]_2+[u^{-1}]_2> \longrightarrow \mathrm{d}(\dots)$. This last complex is acyclic in degrees two and three, and also in degree one if one assumes the Soulé–Beilinson conjecture on weights, see [4, Remark 3.23]. Note that

$$M_{(2)} = \frac{\mathrm{symb}_2}{(1+I)^*\cup F^*_{\mathbb{Q}}} \quad\text{and}\quad \widetilde{M}_{(2)}(F) = \frac{\widetilde{M}_{(2)}(F)}{<[u]_2+[u^{-1}]_2>}.$$

In [4] regulator maps $K_p^{(q)}(X^n_{Y,\mathrm{loc}};\square^n) \longrightarrow H^{2q-p}_{\mathcal{D}}(X^n_{Y,\mathrm{loc}};\square^n;\mathbb{R}(q))$ to relative Deligne cohomology were defined. We recall that

$$(3)\quad H^n_{\mathcal{D}}(X;E;\mathbb{R}(q)) \cong \left\{\begin{array}{l} (\omega_n,s_n)\text{ with }\omega_n\in F^q(D)^n,\\ s_n\in j_*S_X^{n-1}(q-1)\text{ such}\\ \text{that }\omega_{n|E}\equiv 0, s_{n|E}\equiv 0\text{ and}\\ \mathrm{d}s_n=\pi_{q-1}\omega_n \end{array}\right\} \Bigg/ \left\{\begin{array}{l} (\mathrm{d}\omega_{n-1},\pi_{q-1}\omega_{n-1}-\mathrm{d}s_{n-1})\\ \text{with }\omega_{n-1}\in F^q(D)^{n-1},\\ s_{n-1}\in j_*S_X^{n-2}(q-1)\text{ such}\\ \text{that }\omega_{n-1|E}\equiv 0, s_{n-1|E}\equiv 0 \end{array}\right\}.$$

(See [4, p. 218].) Here the notation means the following. We write X etc. for the underlying topological complex manifold consisting of the closed points of $X\times_{\mathrm{Spec}\,(\mathbb{Q})}\mathrm{Spec}\,(\mathbb{C})$. $\overline{X}$ is a compactification of X with complement D such that D and $D\cup E$ are a system of divisors with normal crossings. j is the imbedding of X into $\overline{X}$. $S^\bullet_X(q)$ is the complex of $\mathbb{R}(q)$–valued C^∞–forms on X, $F^q(D)^\bullet$ the complex of $\mathbb{C}$–valued C^∞–forms on $\overline{X}$ of type (p,r) with $p\geq q$ and with logarithmic poles along D. (So locally on $\overline{U}\subset\overline{X}$ an element in $F^q(D)^n$ is of the form $\phi\wedge\psi$

with $\phi \in \Omega^\bullet_{\overline{U}}(D \cap \overline{U})$ of degree $p \geq q$, and $\psi \in C^{0,n-p}(\overline{U})$.) Note that if $q > \dim X$, we get a natural isomorphism $H^n_{\mathcal{D}}(\mathbb{R}(q)) \cong H^{n-1}_{\mathrm{dR}}(\mathbb{R}(q-1))$.

The cup product in K–theory corresponds to the products of regulators (see [4, (22) and (40)]) given by

$$(\omega_p, s_p) \cup (\omega_q, s_q) = (\omega_p \wedge \omega_q, s_p \wedge \pi_q \omega_q + (-1)^{\deg \omega_p} (\pi_p \omega_p) \wedge s_q). \tag{4}$$

For computational purposes later on, we want to say a little more about the element $[S]_2$ which was constructed in [4] in $K_2^{(2)}(X_{\mathrm{loc}}; \square)$. Here $X_{\mathrm{loc}} = X_{\mathbb{G}_m} \setminus \{t = S\}$, S the parameter on $\mathbb{G}_m = \mathrm{Spec}\,(\mathbb{Q}[S, S^{-1}])$. It is determined by the fact that its boundary at $t = S$ is given by $(1-S)^{-1}$. Here is a different construction of the same element restricted to $X_{G,\mathrm{loc}}$, where $G = \mathbb{G}_m \setminus \{S = 1\}$. This will suffice for our purposes, and will give a more explicit form for its regulator. Consider the sequence in relative K–theory

$$\cdots \to K_3^{(2)}(G)^{\oplus 2} \to K_2^{(2)}(X_{G,\mathrm{loc}}; \square) \to K_2^{(2)}(X_{G,\mathrm{loc}}) \to K_2^{(2)}(G)^{\oplus 2} \to \cdots .$$

Now $K_3^{(2)}(G) = 0$, so it suffices to construct an element in $K_2^{(2)}(X_{G,\mathrm{loc}})$ with the required boundary, restricting to 0 for $t = 0, \infty$. $(1-S) \cup \dfrac{t-S}{t-1} = \dfrac{t-S}{t-1} \cup (1-S)^{-1}$ is the required element.

For the regulator, look at the corresponding localization sequence in Deligne cohomology:

$$\cdots \to H^1_{\mathcal{D}}(G, \mathbb{R}(2))^+ \to H^2_{\mathcal{D}}(X_{G,\mathrm{loc}}; \square; \mathbb{R}(2))^+ \to H^2_{\mathcal{D}}(X_{G,\mathrm{loc}}; \mathbb{R}(2))^+ \to \cdots .$$

(The regulator lands in the invariant (or plus) part of Deligne cohomology with respect to the combined action of complex conjugation on the underlying topological space and on the coefficients $\mathbb{R}(2)$.) As $H^1_{\mathcal{D}}(G; \mathbb{R}(2))^+ = H^0_{\mathrm{dR}}(G; \mathbb{R}(1))^+ = 0$, we only have to lift the image of $(1-S) \cup \dfrac{t-S}{t-1}$ in $H^2_{\mathcal{D}}(X_{G,\mathrm{loc}}; \mathbb{R}(2))^+$ back to $H^2_{\mathcal{D}}(X_{G,\mathrm{loc}}; \square; \mathbb{R}(2))^+$. The regulator of $(1-S) \cup \dfrac{t-S}{t-1}$ is given by (see (4))

$$\begin{aligned}&(\mathrm{d}\log(1-S), \log|1-S|) \cup (\mathrm{d}\log\frac{t-S}{t-1}, \log\left|\frac{t-S}{t-1}\right|) = \\ &(\mathrm{d}\log(1-S) \wedge \mathrm{d}\log\frac{t-S}{t-1}, \log|1-S| \mathrm{d}i\arg\frac{t-S}{t-1} - \log\left|\frac{t-S}{t-1}\right| \mathrm{d}i\arg(1-S)).\end{aligned}$$

If ρ is a bump function around $t = 0$ (so $\rho(0) \equiv 1$), symmetric with respect to complex conjugation, then

$$\begin{aligned}(\omega_2, \varepsilon_2) \overset{\mathrm{def}}{=} \Big(&\mathrm{d}\log(1-S) \wedge \mathrm{d}\log\frac{t-S}{t-1}, \\ &\log|1-S| \mathrm{d}i\arg\frac{t-S}{t-1} - \log\left|\frac{t-S}{t-1}\right| \mathrm{d}i\arg(1-S) + \mathrm{d}\big(\rho(t) P_{2,\mathrm{Zag}}(S)\big)\Big)\end{aligned} \tag{5}$$

is the required lift. Here the function $P_{2,\mathrm{Zag}}$ (also known as the Bloch–Wigner dilogarithm) is determined by $P_{2,\mathrm{Zag}}(1/2) = 0$ and $\mathrm{d}P_{2,\mathrm{Zag}}(z) = \log|z|\mathrm{d}\,i\arg(1-z) - \log|1-z|\mathrm{d}\,i\arg z$. $P_{2,\mathrm{Zag}}$ is a single valued on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$, and it can be extended continuously by setting $P_{2,\mathrm{Zag}}(0) = P_{2,\mathrm{Zag}}(1) = P_{2,\mathrm{Zag}}(\infty) = 0$.

The function $P_{2,\mathrm{Zag}}$ occurs in the theory in a natural way. Let k be a number field, $\sigma_1, \ldots, \sigma_r$ be all embeddings of k into $\mathbb{C}$. Then the Borel–Beilinson regulator on $H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \cong K_3^{(2)}(k)$ can be described as follows. Consider the commutative diagram (where we identify $H^2_{\mathcal{D}}$ with H^1_{dR}, $H^1_{\mathcal{D}}$ with H^0_{dR})

$$\begin{tikzcd}
H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k)) \arrow[r, "\mathrm{reg}"] \arrow[d, equal] & H^1_{\mathrm{dR}}(X_{k\otimes_{\mathbb{Q}}\mathbb{C}}; \Box; \mathbb{R}(1))^+ \arrow[d, "\sim"'] \\
K_3^{(2)}(k) \arrow[r, "\mathrm{reg}"] & H^0_{\mathrm{dR}}(\mathrm{Spec}\,(k\otimes_{\mathbb{Q}}\mathbb{C}); \mathbb{R}(1))^+ = \left(\oplus_\sigma \mathbb{R}(1)_\sigma\right)^+ .
\end{tikzcd} \tag{6}$$

The vertical isomorphsim on the right can be obtained as follows. $\varepsilon_2(\sigma(x))$ is an element of H^1_{dR}, and the vertical map is given by integration: $\int_{X_{\mathbb{C}}} \varepsilon_2(\sigma(x)) \wedge \mathrm{d}i\arg t = -P_{2,\mathrm{Zag}}(\sigma(x))$ if X has orientation given by $-\frac{1}{2i}\mathrm{d}t \wedge \mathrm{d}\bar{t}$. Then the map from upper left to lower right hand corner (up to sign) is given by $[x]_2 \mapsto (P_{2,\mathrm{Zag}}(\sigma_1(x)), \ldots, P_{2,\mathrm{Zag}}(\sigma_r(x)))$, (see [4, Proposition 4.1]). By Borel's theorem, this is an injection.

3 An auxiliary complex

Let F be a field of characteristic zero, $X = X_F$. Consider the commutative diagram with exact rows (which one obtains by taking direct limits over finitely many elements in F^*)

$$\begin{tikzcd}
\coprod_{F^*\setminus\{1\}} K_3^{(2)}(F) \arrow[r] & K_3^{(3)}(X;\Box) \arrow[r] & K_3^{(3)}(X_{\mathrm{loc}};\Box) \arrow[r] & \coprod_{F^*\setminus\{1\}} K_2^{(2)}(F) \arrow[r] & K_2^{(3)}(X;\Box) \\
 & 0 \arrow[u] \arrow[r] & (1+I)^* \cup K_2^{(2)}(F) \arrow[u] \arrow[r] & \coprod_{F^*\setminus\{1\}} K_2^{(2)}(F) \arrow[u, equal] \arrow[r] & F^*_{\mathbb{Q}} \otimes K_2^{(2)}(F) \arrow[u]
\end{tikzcd}$$

where the top row is part of the localization sequence. Because $K_3^{(3)}(X;\Box) \cong K_4^{(3)}(F)$, we get an exact sequence (cf. [2] and [3])

$$0 \to K_4^{(3)}(F)/F^*_{\mathbb{Q}} \cup K_3^{(2)}(F) \to K_3^{(3)}(X_{\mathrm{loc}};\Box)/(1+I)^* \cup K_2^{(2)}(F) \xrightarrow{\mathrm{d}} F^*_{\mathbb{Q}} \otimes K_2^{(2)}(F).$$

Let $\mathcal{C}^\bullet$ be the cohomological complex

$$K_3^{(3)}(X_{\mathrm{loc}};\square)/(1+I)^* \cup K_2^{(2)}(F) \xrightarrow{\mathrm{d}} F_{\mathbb{Q}}^* \otimes K_2^{(2)}(F)$$

in degrees 1 and 2. This complex is usually better for computations than the complexes $\mathcal{M}_{(3)}^\bullet$ or $\widetilde{\mathcal{M}}_{(3)}^\bullet$, and we begin with constructing a map from $\mathcal{M}_{(3)}^\bullet$ to $\mathcal{C}^\bullet$. Note that if $f, g \in F^*$, then $[g]_2 \cup f \in K_3^{(3)}(X_{\mathrm{loc}};\square)$, and $\mathrm{d}([g]_2 \cup f) = g \otimes \{f, 1-g\}$. (More precisely, $\mathrm{d}([g]_2 \cup f) = \{(1-g)^{-1}, f\}_{|t=g}$ which maps to $g \otimes \{f, 1-g\}$ in the above quotient.)

Lemma 3.1 *We have a commutative diagram*

$$\begin{tikzcd}
M_{(3)} \arrow[r] \arrow[d] & F_{\mathbb{Q}}^* \otimes M_{(2)} \arrow[r] \arrow[d] & \bigwedge^2 F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^* \arrow[d] \\
0 \arrow[r] & \mathcal{C}^1 \arrow[r] & \mathcal{C}^2
\end{tikzcd}$$

as follows. We map $f \otimes [g]_2$ to $[g]_2 \cup f$, and $f \wedge g \otimes h$ to $g \otimes \{f, h\} - f \otimes \{g, h\}$.

Proof It is easy to check that the right hand square commutes, so we need only check that this is well defined. For this we need only check that the product $[S]_2 \cup S$ is already zero in $K_3^{(3)}(X_G \setminus \{t = S\};\square)$ $(G = \mathbb{G}_m \setminus \{S = 1\})$. In fact, from the localization sequence

$$K_3^{(3)}(X_G;\square) \to K_3^{(3)}(X_G \setminus \{t = S\};\square) \xrightarrow{\mathrm{d}} K_2^{(2)}(G)_{|t=S}$$

and the fact that

$$K_3^{(3)}(X_G;\square) \cong K_4^{(3)}(G) \cong K_3^{(2)}(\mathbb{Q})^{\oplus 2} \cong 0$$

we only need to check that $[S]_2 \cup S$ satisfies $\mathrm{d}\,[S]_2 \cup S = (1-S)^{-1} \cup S = -\{1-S, S\} = 0$ in $K_2^{(2)}(G)_{|t=S}$.

Proposition 3.2 *The diagram*

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}^\bullet) \arrow[r] \arrow[d] & H^1(\mathcal{C}^\bullet) \arrow[d, equal] \\
K_4^{(3)}(F) \arrow[r] & K_4^{(3)}(F)/F_{\mathbb{Q}}^* \cup K_3^{(2)}(F)
\end{tikzcd}$$

where the top horizontal arrow is defined in Lemma 3.1 and the bottom arrow is the natural projection, is commutative (up to sign).

Proof In this proof we shall refer freely to the notations used in [4]. The map $H^2(\mathcal{M}^\bullet_{(3)}(F)) \longrightarrow K_4^{(3)}(F)$ arises as follows. Let W_p be the union of the p–fold intersections of $t_i = u_j$'s, $u_j \in U \overset{\text{def}}{=} F \setminus \{0,1\}$, $i = 1, 2$. (One should work with finite subsets of U, but one can take direct limits afterwards. For simplicity, we use notation as if one can work with U directly.) There exists a spectral sequence

$$E_1^{p,q} = H^{p+q}_{W_p \setminus W_{p+1}}(C \setminus W_{p+1}, K)^{(3)} \Rightarrow H^{p+q}(C,K)^{(3)}$$

for some suitable simplicial scheme C such that $H^{-p}(C,K)^{(3)} \cong K_p^{(3)}(X^2;\square^2)$. There are identifications

$$H^{p+q}_{W_p \setminus W_{p+1}}(C \setminus W_{p+1}, K)^{(3)} = K^{(3-p)}_{-p-q}(W_p \setminus W_{p+1};\square^2) \tag{7}$$

and using those, the spectral sequence becomes

$$E_1^{p,q} = K^{(3-p)}_{-p-q}(W_p \setminus W_{p+1};\square^2) \Rightarrow K^{(3)}_{-p-q}(X^2;\square^2).$$

The complex $C^\bullet_{(3)}$ (see (1)) is nothing but the row where $q = -3$, and $H^2(C^\bullet_{(3)}) = E_2^{1,-3}$. All higher differentials leaving $E_2^{1,-3}$ are zero, as are the incoming ones. Writing H for $H^{-2}(C,K)^{(3)}$, we therefore have maps

$$\begin{aligned} H^2(\mathcal{M}^\bullet_{(3)}(F)) \xleftarrow{\sim} H^2(C^\bullet_{(3),\log}) \longrightarrow H^2(C^\bullet_{(3)}) = E_2^{1,-3} = E_\infty^{1,-3} \xleftarrow{\sim} \mathrm{Im}(H_{W_1})/\mathrm{Im}(H_{W_2}) \\ \longleftarrow \mathrm{Im}(H_{W_1}) \subset H^{-2}(C,K)^{(3)} = K_2^{(3)}(X^2;\square^2) \xrightarrow{\sim} K_3^{(3)}(X;\square) \xrightarrow{\sim} K_4^{(3)}(F) \end{aligned}$$

where the images are in $H^{-2}(C,K)^{(3)}$ with respect to the natural map

$$H^{-2}_{W_p}(C,K)^{(3)} \longrightarrow H^{-2}(C,K)^{(3)},$$

and the last two maps are the isomorphisms coming from using the relativity with respect to t_1 resp. t_2, cf. the beginning of this section. It turns out that $H^{-2}_{W_2}(C,K)^{(3)} = 0$, so we have a map from $H^2(\mathcal{M}^\bullet_{(3)}(F))$ to $K_4^{(3)}(F)$.

The construction of the spectral sequence commutes with respect to localizing at $t_2 \in U$. So we can create the following commutative diagram corresponding to the previous sequence of maps from the E_2–term onwards, together with the localizations, i.e., intersection with $Y = X^2 \setminus \bigcup_{u\in U}\{t_2 = u\}$.

$$\begin{tikzcd}
H^{-2}_{W_1}(C,K)^{(3)} \arrow[r] \arrow[d] & H^{-2}_{W_1\cap Y}(C\cap Y,K)^{(3)} \arrow[d] \\
H^{-2}(C,K)^{(3)} \arrow[r] \arrow[d, equal] & H^{-2}(C\cap Y,K)^{(3)} \arrow[d, equal] \\
K_2^{(3)}(X^2;\square^2) \arrow[r] \arrow[d, "\sim"'] & K_2^{(3)}(Y;\square^2) \arrow[d, "\sim"'] \\
K_3^{(3)}(X;\square) \arrow[r] \arrow[d, "\sim"'] & K_3^{(3)}(X_{\mathrm{loc}};\square) \\
K_4^{(3)}(F) &
\end{tikzcd}$$

An element $\sum_j a_j g_j \otimes [f_j]_2$ in $H^2(\mathcal{M}^\bullet_{(3)})$ can be lifted to an element in $H^2(C^\bullet_{(3),\log})$, which is given by $\sum_j a_j([f_j]_{2|t_2=g_j} - [f_j]_{2|t_1=g_j})$, up to terms involving factors $(1+I)^*$, see (2). Using the identification as in (7), it follows from the definition of the boundary maps in the spectral sequence that $\sum_j a_j g_j \otimes [f_j]_2$ in $H^2(\mathcal{M}^\bullet_{(3)}(F))$ maps to $i_\sharp\left(\sum_j -a_j[f_j]_{2|t_1=g_j}\right)$ in $K_2^{(3)}(Y;\square^2)$ modulo $i_\sharp\left(\coprod_{u\in U}(1+I)^*\cup F^*_{\mathbb{Q}|t_1=u}\right)$. Here $i_\sharp$ is the composite map

$$\coprod_{t_1\in U} K_2^{(2)}(X_{\mathrm{loc}};\square)_{|t_1=u} = H^{-2}(C',K)^{(2)} \to H^{-2}_{W_1\cap Y}(C\cap Y,K)^{(3)} \to H^{-2}(C\cap Y,K)^{(3)} \to K_2^{(3)}(Y;\square^2),$$

see [4, Proposition 2.3]. Here C' is some other simplicial scheme such that there are identifications $\coprod_{u\in U} K_p^{(2)}(X_{\mathrm{loc}};\square)_{|t_1=u} = H^{-p}(C',K)^{(2)}$. $i_\sharp\left([f]_{2|t_1=g}\right) = [f]_2 \cup i_\sharp(1_{|t_1=g})$ where this last $i_\sharp$ is the map $K_0^{(0)}(F)_{|t=g} \to K_1^{(0)}(X;\square)$. By [4, Lemma 3.14], if we combine this with the isomorphism $K_0^{(1)}(X;\square) \cong K_1^{(1)}(F) = F^*_{\mathbb{Q}}$, $i_\sharp(1_{|t_1=g})$ maps to $g^{\pm 1}$ in $F^*_{\mathbb{Q}}$. As this last isomorphism is compatible with the isomorphism $K_2^{(3)}(Y;\square^2) \xrightarrow{\sim} K_3^{(3)}(X_{\mathrm{loc}};\square)$ under the cup product, we see that the element maps to $\pm\sum_j a_j[f_j]_2 \cup g_j$ in $K_3^{(3)}(X_{\mathrm{loc}};\square)$, modulo $(1+I)^* \cup K_2(F)$. Therefore it will map to $\pm\sum_j a_j[f_j]_2 \cup g_j$ in $H^1(\mathcal{C}^\bullet(F))$. The statement of the proposition now follows from the definition of the isomorphism

$$H^1(\mathcal{C}^\bullet(F)) \xrightarrow{\sim} K_4^{(3)}(F)/K_3^{(2)}(F)\cup F^*_{\mathbb{Q}},$$

at the beginning of this section.

Remark 3.3 Suppose that $\mathrm{d}\sum_j a_j[f_j]_2 \cup g_j = -\sum_j a_j f_j \otimes \{1-f_j, g_j\} = 0$ in $F_{\mathbb{Q}}^* \otimes K_2^{(2)}(F)$, with $a_j \in \mathbb{Q}$. Let $\{A_1, \dots, A_k\} \subset \{f_j\}$ be a basis of the subspace of $F_{\mathbb{Q}}^*$ spanned by the f_j. Then if we write $f_j = \prod_{l=1}^k A_l^{b_{j,l}}$, $b_{j,l} \in \mathbb{Q}$, expressing everything in terms of the independent A_j's, we see that, for each l, there exist $f_{l,m}$ and $c_{l,m} \in \mathbb{Q}$, such that $\sum_j a_j b_{j,l}(1-f_j) \otimes g_j = \sum_m c_{l,m}(1-f_{l,m}) \otimes f_{l,m}$. Then

$$\sum_j a_j g_j \otimes [f_j]_2 + \sum_{l,m} c_{l,m} A_l \otimes [f_{l,m}]_2$$

defines an element in $H^2(\mathcal{M}_{(3)}^\bullet)$. This provides some kind of map from $H^1(\mathcal{C}^\bullet)$ to $H^2(\mathcal{M}_{(3)}^\bullet)$.

Remark 3.4 This is in fact related to the following. In [6] C. Deninger constructs linearized Massey products for complexes $C(i)$, bounded below, that have associative cup–products $C(i)\hat{\otimes}C(j) \to C(i+j)$. ($C(i)\hat{\otimes}C(j)$ is the total complex associated to the tensor product of the two complexes.) We recall the case for the triple products that is of interest here, cf. [6, (2.4)]. We will assume for simplicity that the complexes consist of vector spaces over a field k. If λ_{rst} are elements in k, and A_{1r}, A_{2s} and A_{3t} are elements in $H^1(C(1))$ such that

1. for all r, $\displaystyle\sum_{s,t} \lambda_{rst}(A_{2s} \cup A_{3t}) = 0$ in $H^2(C(2))$
2. for all t, $\displaystyle\sum_{r,s} \lambda_{rst}(A_{1r} \cup A_{2s}) = 0$ in $H^2(C(2))$.

Then there exist b_{2r} and b_{1t} in $C(2)^1$ such that $\mathrm{d}\, b_{2r} = \sum_{s,t} \lambda_{rst}(A_{2s} \cup A_{3t})$ and $\mathrm{d}\, b_{1t} = \sum_{r,s} \lambda_{rst}(A_{1r} \cup A_{2s})$. Then $\sum_r A_{1r} \cup b_{2r} + \sum_t b_{1t} \cup A_{3t}$ represents an element in $H^2(C(3))$, well determined up to $H^1(C(1)) \cup H^1(C(2)) + H^1(C(2)) \cup H^1(C(1))$.

We can apply this to our complexes $\widetilde{\mathcal{M}}_{(p)}^\bullet$. Take $C(1)$ to be $F_{\mathbb{Q}}^*$ concentrated in degree 1, $C(2) = \widetilde{\mathcal{M}}_{(2)}^\bullet$ and $C(3) = \widetilde{\mathcal{M}}_{(3)}^\bullet$ with maps given by $a \cup b = a \wedge b$ for $a, b \in F_{\mathbb{Q}}^*$, and for the cup products $C(1) \otimes C(2)$ and $C(2) \otimes C(1)$ to $C(3)$ we take

$$\begin{array}{ll} a \cup [f]_2 = -a \otimes [f]_2 & [f]_2 \cup a = a \otimes [f]_2 \\ a \cup (b \wedge c) = a \wedge b \wedge c & (b \wedge c) \cup a = b \wedge c \wedge a = a \wedge b \wedge c \end{array}$$

So those complexes provide an analogue of Deninger's triple Massey products in Deligne cohomology in K–theory. In order to lift those products to the complexes $\mathcal{M}_{(p)}^\bullet$ it is better for the associativity of $C(1) \otimes C(1) \otimes C(1)$ to let the middle factor, call it $C(1)'$, play a special role. So we define $C(1) \otimes C(1)' \to C(2)$ via $a \cup b = a \otimes b$, $C(1)' \otimes C(1) \to C(2)$ via $b \cup c = -c \otimes b$. For the products $C(1) \otimes C(2)$ and $C(2) \otimes C(1)$ to $C(3)$ we take

$$\begin{array}{ll} a \cup [f]_2 = -a \otimes [f]_2 & [f]_2 \cup a = a \otimes [f]_2 \\ a \cup (b \otimes c) = a \wedge b \otimes c & (b \otimes c) \cup a = a \wedge b \otimes c \end{array}$$

4 The computation of the regulator map

Let C be a complete, geometrically irreducible, smooth curve over a number field k. Let F be its function field. Let $C(\mathbb{C})_{\mathrm{an}}$ be the complex manifold whose set of points is the set of closed points of $C \times_{\mathrm{Spec}\,(\mathbb{Q})} \mathrm{Spec}\,(\mathbb{C})$. Note that this is a disjoint union of $[k:\mathbb{Q}]$ complex curves of genus the genus of C, one for each embedding of k into $\mathbb{C}$. Let σ be the complex conjugation on $C(\mathbb{C})_{\mathrm{an}}$. If k has r_1 real embeddings, and $2r_2$ complex embeddings, so $[k:\mathbb{Q}] = r_1 + 2r_2$, then $C(\mathbb{C})_{\mathrm{an}}$ consists of r_1 curves that σ transforms into itself, and r_2 pairs of curves where σ interchanges the two curves. Keeping this in mind, it is easy to verify that $\dim_{\mathbb{C}} H^0(C(\mathbb{C})_{\mathrm{an}}, \Omega^1) = \mathrm{genus}(C) \cdot [k:\mathbb{Q}] \overset{\mathrm{def}}{=} r$, and that this space is spanned by $\omega_1, \ldots, \omega_r$ such that $\overline{\omega}_j = \omega_j \circ \sigma$. We view $H^*(C(\mathbb{C})_{\mathrm{an}}; \mathbb{Q}) \subset H^*_{\mathrm{dR}}(C(\mathbb{C})_{\mathrm{an}}; \mathbb{R})$ in the usual way. In order to avoid cumbersome notation, we will simply write C etc. from now on for the associated complex manifolds, it being understood that all algebraic varieties occurring in (co)homology groups and integrals in this section are to be interpreted as their associated complex manifolds.

There is an involution on (co)homology groups, given by the action of σ on the underlying topological space, followed by complex conjugation of the coefficients. If this involution acts on a group H, we let $H^{\pm}$ be the subspace where the involution acts as multiplication by ± 1.

Recall the definition of the Beilinson regulator c_3 for $K_4^{(3)}(C)$ (see, e.g., [10, p. 30]). Beilinson conjectures that the regulator map induces an isomorphism

$$K_4^{(3)}(C) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\mathrm{reg}} H^2_{\mathcal{D}}(C; \mathbb{R}(3))^+ \cong H^1_{\mathrm{dR}}(C; \mathbb{R}(2))^+,$$

so that conjecturally $\dim_{\mathbb{Q}} K_4^{(3)}(C) = r$. (The extra "integrality" condition is vacuous in this case, so we can work with $K_4^{(3)}(C)$ directly.) The regulator c_3 as an element in $\mathbb{R}^*/\mathbb{Q}^*$ is then defined as the determinant of the image of a basis $\alpha_1, \ldots, \alpha_r$ of $K_4^{(3)}(C)$ in $H^1_{\mathrm{dR}}(C; \mathbb{R}(2))^+$ with respect to the $\mathbb{Q}$–basis $H^1(C; \mathbb{Q}(2))^+$. Under the additional assumption that the L–series for $H^1(C)$, $L(C,s)$, can be extended meromorphically to the complex plane, Beilinson further conjectures that there is relation between the first non–vanising coefficient of $L(C,s)$ at $s=-1$, $L(C,-1)^*$, and c_3:

$$L(C,-1)^*/c_3 \in \mathbb{Q}^*.$$

We shall return to this in the final section of the paper.

Below, we refer to such a determinant as the regulator for any r elements of $K_4^{(3)}(C)$ (which may then well be zero).

We need some more notation in order to state the formula for c_3. Fix an orientation of C such that σ reverses the orientation. All integrals $\int_C$ below will be taken with respect to this orientation. We have a non–degenerate pairing $H^1_{\mathrm{dR}}(C; \mathbb{Q}) \times H^1_{\mathrm{dR}}(C; \mathbb{Q}) \longrightarrow \mathbb{Q}$, given by $(\psi_1, \psi_2) \mapsto \int_C \psi_1 \wedge \psi_2$. Under this pairing $H^1_{\mathrm{dR}}(C; \mathbb{Q})^+ \perp H^1_{\mathrm{dR}}(C; \mathbb{Q})^+$, and similarly for the $-$ space. From this it follows that $\dim_{\mathbb{Q}} H^1(C; \mathbb{Q})^+ = \dim_{\mathbb{Q}} H^1(C; \mathbb{Q})^- = r$.

The involution acts also on $H_1(C;\mathbb{Q})$, so this space splits into a $+$– and a $-$–part as well. From the pairing with $H^1(C;\mathbb{Q})$ one deduces that both pieces have $\mathbb{Q}$–dimension r. Let $\{s_{1,\pm},\dots,s_{r,\pm}\}$ be a basis of $H_1(C;\mathbb{Q})^{\pm}$, and let $\{s^*_{1,\pm},\dots,s^*_{r,\pm}\}$ in $H^1(C;\mathbb{Q})$ be its dual base, so that $\int_{s_{m,\pm}} s^*_{n,\pm} = \delta_{mn}$. Let $T^{\pm}_{n,l} = \left(\int_{s_{n,\pm}} \omega_l\right)$. As $\overline{\omega_l} = \omega_l \circ \sigma$, we also have $T^+ = \left(\int_{s_{n,+}} \pi_0\omega_l\right)$ and $T^- = \left(\int_{s_{n,-}} \pi_1\omega_l\right)$. In particular, $\pi_1\omega_l = \sum_n T^-_{n,l} s^*_{n,-}$ as $\int_{s_{n,+}} \pi_1\omega_l = 0$.

Proposition 4.1 *Suppose the Beilinson regulator maps* $\alpha_1,\dots,\alpha_r \in K_4^{(3)}(C)$ *to* $\psi_1,\dots,\psi_r \in H^1_{\mathrm{dR}}(C;\mathbb{R}(2))^+$. *Let* $\omega_1,\dots,\omega_r$ *and* T^- *be as above, and let* $R_{k,l} = \int_C \psi_k \wedge \overline{\omega_l}$. *Then the Beilinson regulator of* $\alpha_1,\dots,\alpha_r$ *is given by*

$$c_3 = \frac{\det(R)}{(2\pi i)^{2r}\det(T^-)}.$$

Proof Let M be the matrix defined by $\psi_k = \sum_m M_{k,m} s^*_{m+}$. Then by definition, $c_3 = (2\pi i)^{-2r}\det(M)$. Note that as σ reverses the orientation, $\omega_l \circ \sigma = \overline{\omega_l}$ and $\psi_k \circ \sigma = \psi_k$,

$$R_{k,l} = \int_C \psi_k \wedge \overline{\omega_l} = -\int_C \psi_k\circ\sigma \wedge \overline{\omega_l \circ \sigma} = -\int_C \psi_k \wedge \omega_l = -\overline{R_{k,l}}.$$

Therefore $R_{k,l}$ is purely imaginary, and

$$\begin{aligned} R_{k,l} &= -\int_C \psi_k \wedge \pi_1\omega_l \\ &= -\sum_n T^-_{n,l}\int_C \psi_k \wedge s^*_{n,-} \\ &= -\sum_{m,n} M_{k,m}T^-_{n,l}\int_C s^*_{m,+}\wedge s^*_{n,-} \\ &= -\sum_{m,n} M_{k,m}A_{m,n}T^-_{n,l} \end{aligned}$$

with $A_{m,n} = \int_C s^*_{m,+}\wedge s^*_{n,-}$. Hence $\det(R) = \pm\det(M)\det(A)\det(T^-)$. As $\det(A)$ expresses the non–degeneracy of the pairing $H^1(C;\mathbb{Q})\times H^1(C;\mathbb{Q}) \to H^2(C;\mathbb{Q})$, it is an element of $\mathbb{Q}^*$. So we get that the regulator c_3 of $\alpha_1,\dots,\alpha_r$ is given by $c_3 = \dfrac{\det(R)}{(2\pi i)^{2r}\det(T^-)}$.

The following Theorem allows us to compute the regulator explicitly on the intersection in $K_4^{(3)}(F)$ of the images of $H^2(\mathcal{M}^\bullet_{(3)}(F))$ and $K_4^{(3)}(C)$, or the intersection in $H^1(\mathcal{C}(F))$ of the image of $K_4^{(3)}(C)$ and elements of the form $\sum_j a_j[f_j]_2 \cup g_j$.

Theorem 4.2 *Fix $\omega \in H^0(C;\Omega^1)$ such that $\overline{\omega} = \omega \circ \sigma$. The map*

$$K_4^{(3)}(C) \xrightarrow{\text{reg}} H^1_{\text{dR}}(C;\mathbb{R}(2))^+ \xrightarrow{\psi \mapsto \int \psi \wedge \overline{\omega}} \mathbb{R}(1)$$

factors through

$$K_4^{(3)}(C) \to K_4^{(3)}(F) \to K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^* = H^1(\mathcal{C}^\bullet(F)).$$

If $\alpha \in K_4^{(3)}(C)$ maps to $\sum_j a_j[f_j]_2 \cup g_j$ in $H^1(\mathcal{C}^\bullet)$ and has image $\psi \in H^1_{\text{dR}}(C;\mathbb{R}(2))^+$ under the regulator map, then

$$R = \int_C \psi \wedge \overline{\omega} = -4 \sum_j a_j \int_C \log|g_j| \log|f_j| \mathrm{d}\log|1-f_j| \wedge \overline{\omega}.$$

Remark 4.3 By Proposition 3.2, this Theorem applies in particular to the intersecion in $K_4^{(3)}(F)$ of $K_4^{(3)}(C)$ and the image of $H^2(\mathcal{M}^\bullet_{(3)}(F))$. In Theorem 5.2 below, we shall give a formula for the boundary of elements in $H^2(\mathcal{M}^\bullet_{(3)}(F))$, up to some small indeterminacy. This is good enough to decide if the elements correspond to elements in $K_4^{(3)}(C)$. For elements in $H^1(\mathcal{C}^\bullet(F))$ the situation is less satisfactory. Computing the boundary amounts to lifting as in Remark 3.3 (see Theorem 5.6), which is not easy to do in practice. However, Theorem 5.6 is often good enough.

Remark 4.4 Combining this with the map

$$H^2(\mathcal{M}^\bullet_{(3)}) \to H^1(\mathcal{C}^\bullet)$$

defined in Lemma 3.1, this shows that the image of the regulator map for the elements of type $\sum_j a_j[f_j]_2 \cup g_j$ in $H^1(\mathcal{C}^\bullet)$ coincides with their "lifts" to $H^2(\mathcal{M}^\bullet_{(3)})$ as in Remark 3.3, up to a factor two, as the integral is multiplicatively linear in terms of f, $1-f$ and g. Something similar holds for the boundary map, see Theorem 5.6.

Remark 4.5 To get a regulator that factors through the isomorphism

$$H^2(\mathcal{M}^\bullet_{(3)}) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}),$$

i.e., vanishes on symbols $g \otimes ([f]_2 + [f^{-1}]_2)$, one can replace

$$\int_C \log|g| \log|f| \mathrm{d}\log|1-f| \wedge \overline{\omega}$$

with

$$\int_C \left(\log|g| \log|f| \mathrm{d}\log|1-f| + \frac{1}{3} \log|1-f| (\log|f| \mathrm{d}\log|g| - \log|g| \mathrm{d}\log|f|) \right) \wedge \overline{\omega}.$$

This follows from the fact that $\int_C \mathrm{d}(\log|g| \log^2|f|) \wedge \overline{\omega} = 0$ by Stokes' theorem. Because the second term vanishes on elements that lie in $H^2(\mathcal{M}^\bullet_{(3)})$, this does not change the value of the regulator map.

Proof of Theorem 4.2 Note that we have isomorphisms

$$H^3_{\mathcal{D}}(X_C;\square;\mathbb{R}(3)) \cong H^2_{\mathrm{dR}}(X_C;\square;\mathbb{R}(2)) \underset{\phi}{\xleftarrow{\sim}} H^1_{\mathrm{dR}}(C;\mathbb{R}(2)).$$

If we let h be a function on $X_{\mathrm{Spec}(\mathbb{C})}$ with $h(\infty)-h(0)=1$, then it is not hard to check that the isomorphism ϕ is given by taking the product with $\mathrm{d}h$, and that $\int_C \psi \wedge \overline{\omega} = -\frac{1}{2\pi i}\int_{X_C} \psi \wedge \mathrm{d}h \wedge \mathrm{d}\log \overline{t} \wedge \overline{\omega}$ if we give X_C the product orientation of C and the standard orientation on $\mathbb{P}^1(\mathbb{C})$.

Of course, we can restrict the integral to Zariski open parts of X_C, which we shall use in order to show that the regulator factors as stated in the theorem. This introduces the problem of convergence. (Indeed, problems of convergence seem to be an essential part of the fun in dealing with the Beilinson regulator, see, e.g., [5, (10.8)]). In order to deal with this, we need the following.

Let Y be an algebraic variety of dimension n with compactification $\overline{Y}$ such that the complement of Y is given by D, a divisor with normal crossings. Suppose moreover that there is another divisor D' on $\overline{Y}$ such that the union of D and D' is a divisor with normal crossings. We want to say something about the behaviour close to D of forms on Y that vanish on $Y \cap D'$. Suppose that locally in a compact subset of $\overline{Y}$, D is given by $\prod_{i=1}^k x_i = 0$. Let $r_i = |x_i|$, $\theta_i = \arg x_i$. We will consider differential forms β on Y that satisfy the following condition on the compact subset of the chosen neighbourhood of D.

(8) β vanishes on $D' \cap Y$ and can be written as sums of products of $\log r_i$, $\mathrm{d}\log r_i$, $\mathrm{d}\theta_i$, bounded functions on Y and the restriction of C^∞–forms on $\overline{Y}$ to Y.

Then if ω is a holomorphic or anti–holomorphic n–form on $\overline{Y}$, and β is a n–form as in (8), one checks that $\int_Y \beta \wedge \omega$ converges. In fact, the same is true if we allow ω to have logarithmic poles along D'.

Proposition 4.6 *Let Y, $\overline{Y}$, D and D' be as above. Suppose that β_1 and β_2 are two closed n–forms on Y as in (8) that represent the same class in relative de–Rham cohomology $H^n(Y;D';\mathbb{R}(j))$. Let ω be a holomorphic or anti–holomorphic n–form on $\overline{Y}$, possibly with logarithmic poles along D'. Then*

$$\int_Y \beta_1 \wedge \omega = \int_Y \beta_2 \wedge \omega. \tag{9}$$

In order not to interrupt the flow of the argument, we postpone the proof to the end of this section. Because of Proposition 4.6 we can compute our integral $\int_C \psi \wedge \overline{\omega} = -\dfrac{1}{2\pi i}\displaystyle\int_{X_C} \psi \wedge \mathrm{d}h \wedge \mathrm{d}\log \overline{t} \wedge \overline{\omega}$ on a Zariski open part, where we can replace $\psi \wedge \mathrm{d}h$ with a form as in (8) in the same class in H^2_{dR}. Let $\{f_j\}$ be a finite subset of F^*, and let $U \subset C$ be a Zariski open subset such that all f_j, $1-f_j$ and

$f_j - f_k$ are invertible if they do not vanish identically. Let $X_{U,\mathrm{loc}} = X_U \setminus t = f_j$'s. The complement of $X_{U,\mathrm{loc}}$ need not be a divisor with normal crossings. To achieve this one has to blow up some points. However, pulling the forms back to the blowup we see that all growth conditions are still satisfied, and that $\mathrm{d}\log\overline{t} \wedge \overline{\omega}$ has a pole only along the strict transform of $t = 0, \infty$, so that we can still apply Proposition 4.6. Consider the localization

$$\begin{tikzcd}
\coprod_j K_3^{(2)}(U)_{|t=f_j} \arrow[r] \arrow[d, "\mathrm{reg}"'] & K_3^{(3)}(X_U;\Box) \arrow[r] \arrow[d, "\mathrm{reg}"'] & K_3^{(3)}(X_{U,\mathrm{loc}};\Box) \arrow[d, "\mathrm{reg}"'] \\
\left(\coprod_j H^1_{\mathcal{D}}(U;\mathbb{R}(2))_{|t=f_j}\right)^+ \arrow[r] \arrow[d, "\sim"'] & H^3_{\mathcal{D}}(X_U;\Box;\mathbb{R}(3))^+ \arrow[r] \arrow[d, "\sim"'] & H^3_{\mathcal{D}}(X_{U,\mathrm{loc}};\Box;\mathbb{R}(3))^+ \arrow[d, "\sim"'] \\
\left(\coprod_j H^0_{\mathrm{dR}}(U;\mathbb{R}(1))_{|t=f_j}\right)^+ \arrow[r] & H^2_{\mathrm{dR}}(X_U;\Box;\mathbb{R}(2))^+ \arrow[r] & H^2_{\mathrm{dR}}(X_{U,\mathrm{loc}};\Box;\mathbb{R}(2))^+
\end{tikzcd}$$

The regulator of $K_3^{(2)}(U)_{|t=f_j}$ in $H^2_{\mathrm{dR}}(X_U;\Box;\mathbb{R}(2))^+$ is given by $\pm\gamma \mathrm{d}i\arg f_j \wedge \mathrm{d}h$ where γ in $H^1_{\mathcal{D}}(U;\mathbb{R}(2)) \cong H^0_{\mathrm{dR}}(U;\mathbb{R}(1))$. This obviously satisfies (8). By Proposition 4.6 the integral of those forms on $X_{U,\mathrm{loc}}$ vanishes. Because we can shrink U and localize at more f_j's, and any class in $H^2_{\mathrm{dR}}(X_{U,\mathrm{loc}};\Box;\mathbb{R}(2))$ has representatives as in (8), this shows that the regulator of $K_4^{(3)}(C)$ factors through $K_3^{(3)}(X_{F,\mathrm{loc}};\Box)$. As we shall see below, it vanishes on $(1+I)^* \cup K_2^{(2)}(F)$, hence factors through $H^1(\mathcal{C}^\bullet(F))$.

Now assume $\sum_j a_j[f_j]_2 \cup g_j$ is the image of an element of $K_4^{(3)}(C)$ in $H^1(\mathcal{C}^\bullet)$. Because the regulator factors over $H^1(\mathcal{C}^\bullet(F))$, the regulator can be found by first localizing suitably, then lifting $\sum_j a_j[f_j]_2 \cup g_j$ to an actual representative in $K_3^{(3)}(X_{U,\mathrm{loc}};\Box)$ with zero boundary, and finally applying the regulator map. So let U be a suitable Zariski open part of C. The result will involve the regulators of elements $[f]_2 \cup g$ and $F \cup \gamma$ where $\gamma \in K_2^{(2)}(U)$ and $F \in (1+I)^*$.

As for the terms $F \cup \gamma$, the regulator is given by (see (4))

$$(\mathrm{d}\log F, \log|F|) \cup (0, \varepsilon) = (0, -\mathrm{d}i\arg F \wedge \varepsilon)$$

for some form ε in $H^1_{\mathrm{dR}}(U;\mathbb{R}(1))$, which therefore can be taken to satisfy (8). By Fubini's theorem, $\int_{X_U} -\mathrm{d}i\arg F \wedge \varepsilon \wedge \mathrm{d}\log\overline{t} \wedge \overline{\omega} = 0$ as $F \in (1+I)^*$.

The regulator of $[f]_2 \cup g$ in $H^3_{\mathcal{D}}(X_{U,\mathrm{loc}};\Box;\mathbb{R}(3))$ is given by

$$(\omega_2(f), \varepsilon_2(f)) \cup (\mathrm{d}\log g, \log|g|) = (0, \varepsilon_2(f) \wedge \pi_1\,\mathrm{d}\log g + \log|g|\pi_2\omega_2(f)).$$

By (3) the class of $\varepsilon_2 \wedge \pi_1\,\mathrm{d}\log g + \log|g|\pi_2\omega_2$ is determined up to the boundary of a $\mathbb{R}(2)$–valued 1–form on $X_{U,\mathrm{loc}}$. As for the integrals of type

$$(10) \qquad -\frac{1}{2\pi i}\int_X\int_C (\varepsilon_2(f) \wedge \pi_1\mathrm{d}\log g + \log|g|\pi_2\omega_2(f)) \wedge \mathrm{d}\log\overline{t} \wedge \overline{\omega},$$

it is clear from the explicit shape of ε_2 (see (5)) that the conditions of (8) are satisfied. Hence the integrals in (10) compute the regulator after summing over j.

As for the actual computation, because of the presence of $\overline{\varpi}$, we can replace $\pi_1 \mathrm{d}\log g = \mathrm{d}\, i\arg g$ with d $\log|g|$. By Proposition 4.6 we can change the form with $\mathrm{d}(\log|g|\,\varepsilon_2)\wedge \mathrm{d}\log\bar{t}\wedge\overline{\varpi}$, so we have to compute

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int_X\int_C \log|g|\,\omega_2(f)\wedge \mathrm{d}\log\bar{t}\wedge\overline{\varpi}\\
=&-\frac{1}{2\pi i}\int_C \log|g|\left(\int_X \mathrm{d}\log\frac{t-f}{t-1}\wedge \mathrm{d}\log\bar{t}\right)\mathrm{d}\log(1-f)\wedge\overline{\varpi}\\
=&-\int_C \log|g|\log|f|^2 \mathrm{d}\log(1-f)\wedge\overline{\varpi} \qquad \text{as } \mathrm{d}\log\overline{\frac{t-f}{t-1}}\wedge\mathrm{d}\log\bar{t}\wedge\overline{\varpi}=0\\
=&-4\int_C \log|g|\log|f|\mathrm{d}\log|1-f|\wedge\overline{\varpi}.
\end{aligned}
$$

This finishes the proof of Theorem 4.2.

We now give the proof of Proposition 4.6.

Proof Because β_1 and β_2 represent the same class they differ by $\mathrm{d}\gamma$ where γ is an $(n-1)$–form vanishing on D'. If γ is as in (8) the statement of the Proposition is checked by removing a tubular neighbourhood of $D\cup D'$, so that we are left with a manifold, say M. Then the difference of the two integrals in (9) is given by making M larger and larger in

$$\int_M \mathrm{d}\gamma\wedge\omega=\int_{\partial M}\gamma\wedge\omega.$$

It is easy to check that the right hand side approaches zero as the radius of the tubes goes to zero, so we only have to prove that γ can be chosen with growth behaviour as in (8).

We prove this as follows. Let $\mathfrak{U}=\{U_i\}_{i\in I}$ be a good open cover of Y and let $\oplus_{p,q}C^{p,q}$ be the Čech-de Rham complex of p–forms on the $(q+1)$–fold intersections of this cover, with the condition on the forms that they vanish on D'. Let d be the boundary operator on forms, and let δ be the boundary for the intersections.

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
{} \arrow[r, "\mathrm{d}"] & C^{n-2,2} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & C^{n-1,2} \arrow[u, "-\delta"] \arrow[r, "\mathrm{d}"] & C^{n,2} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots \\
{} \arrow[r, "\mathrm{d}"] & C^{n-2,1} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & C^{n-1,1} \arrow[u, "-\delta"] \arrow[r, "\mathrm{d}"] & C^{n,1} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots \\
{} \arrow[r, "\mathrm{d}"] & C^{n-2,0} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & C^{n-1,0} \arrow[u, "-\delta"] \arrow[r, "\mathrm{d}"] & C^{n,0} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots
\end{tikzcd}$$

Observe that the conditions in (8) are stable under application of the Poincaré lemma (including the condition on vanishing on D'), so that we can get $\{\phi_i\} \in C^{n-1,0}$ with $\mathrm{d}\phi_i = (\beta_1 - \beta_2)_{|U_i}$, $\beta_1 - \beta_2 \in C^{n,0}$. We can now use the Poincaré lemma on $\delta\{\phi_i\}$, doing it repeatedly we can extend $\{\phi_i\}$ to an element A in $\oplus_{p+q=n-1} C^{p,q}$, with behaviour of each component $A^{p,q}$ as in (8), and boundary $\beta_1 - \beta_2 \in C^{n,0}$, constants in $C^{0,n}$, and zero elsewhere. In order to get rid of the constants we have to change the functions in $A^{0,n-1} = \{f_{i_0 \dots i_{n-1}}\}$ with constants that are zero on D'. This can be done as follows.

There exists γ', a global $(n-1)$–form, with $\mathrm{d}\gamma' = \beta_1 - \beta_2$. If $A^{n-1,0} = \{\phi_i\}$, then we can find $\psi_i \in C^{n-2,0}$ such that $\mathrm{d}\psi_i = \phi_i - \gamma'$. Changing $A^{n-2,1}$ with $\pm\delta\{\psi_i\}$ we get an element in $C^{n-2,1}$ with d–boundary 0 and δ–boundary equal to $\delta A^{n-2,1}$. We can now integrate this element to get $\{\psi_{ij}\}$ and use $\pm\delta\{\psi_{ij}\}$ to change $A^{n-3,2}$. That will give an element in $C^{n-3,2}$ with d–boundary 0 and δ–boundary equal to the δ–boundary of $A^{n-3,2}$. Continuing this way we arrive at functions $\{g_{i_0 \dots i_{n-2}}\}$ in $C^{0,n-2}$, such that $\{f_{i_0 \dots i_{n-1}}\} \pm \delta\{g_{i_0 \dots i_{n-2}}\}$ has the same δ–boundary as $\{f_{i_0 \dots i_{n-1}}\}$ and has d–boundary zero, so they differ by constants. This means that if we replace $A^{0,n-1}$ with $\mp\delta\{g_{i_0 \dots i_{n-2}}\}$ and keep the other components of A fixed we found an element $B \in \oplus_{p+q=n-1} C^{p,q}$ with boundary solely $\{(\beta_1 - \beta_2)_{|U_i}\}$ and behaviour at D and D' of all components as described in (8).

Finally, to get a global $(n-1)$–form out of B we use a partition of unity $\{\rho_i\}$ subordinate to the cover $\mathfrak{U}$. If $B^{p,q} = \{\phi_{i_0 \dots i_p}\}$ satisfies $\delta B^{p,q} = 0$, then we can change B with $\{\mathrm{d} \sum_i \rho_i \phi^{p,q}_{i i_0 \dots i_p}\} \pm \delta\{\sum_i \rho_i \phi^{p,q}_{i i_0 \dots i_p}\}$. It is well known that the last term equals $B^{p,q}$ so that this effectively kills $B^{p,q}$. Starting with $B^{0,n-1}$ and applying this repeatedly we can change B in such a way that $B^{n-p-1,p} = 0$ for $p > 0$. Then $B^{n-1,0}$ is a global $(n-1)$–form with $\mathrm{d}B^{n-1,0} = \beta_1 - \beta_2$. This form will be γ.

We have to check that the growth behaviour of γ is as in (8). By construction γ contains as summands wedge products of components of B and $\mathrm{d}\rho_i$'s. The latter can be assumed to behave well around D, e.g., by choosing the cover $\mathfrak{U}$ as follows. First take a good open cover of the complement of a small tubular neighbourhood of D in $\overline{Y}$. Then extend this to a good open cover for Y by shrinking the tubular neighbourhood to its boundary. This way we can get a partition of unity for which the ρ_i locally depend only on the θ_j, and hence $\mathrm{d}\rho_i$ will be of the form described in (8). Hence γ will satisfy the conditions in (8), and we are done.

5 The computation of the boundary under localization

As in the previous section, let C be a complete, geometrically irreducible, smooth curve over a number field k. Let F be its function field. Let C^1 be the points of codimension one in C.

We want to compute the boundary map ∂ in the localization sequence

$$\coprod_{x\in C^1} K_4^{(2)}(k(x)) \to K_4^{(3)}(C) \to K_4^{(3)}(F) \xrightarrow{\partial} \coprod_{x\in C^1} K_3^{(2)}(k(x)) \tag{11}$$

on the image of $H^2(\mathcal{M}^\bullet_{(3)}(F))$ in $K_4^{(3)}(F)$. One should compare the result (Corollary 5.4) with the formulas in [7, section 5.1]. In order to state the result conveniently, we want to have a map

$$\delta_1\colon H^2(\mathcal{M}^\bullet_{(3)}(F)) \cong H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \to \coprod_{x\in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))).$$

Proposition 5.1 *There exists a commutative diagram*

$$\begin{tikzcd}
\widetilde{M}_{(3)}(F) \arrow[r, "\mathrm{d}"] \arrow[d] & F^*_{\mathbb{Q}}\otimes \widetilde{M}_{(2)}(F) \arrow[r, "\mathrm{d}"] \arrow[d, "\delta_1"] & \bigwedge^3 F^*_{\mathbb{Q}} \arrow[d, "\delta_2"] \\
0 \arrow[r] & \coprod_{x\in C^1} \widetilde{M}_{(2)}(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_{x\in C^1} \bigwedge^2 k(x)^*_{\mathbb{Q}}
\end{tikzcd} \tag{12}$$

defined as follows. Let π_x be a uniformizer at x. For $f\in F^*$, let $\overline{f_x} = (f\pi_x^{-\mathrm{ord}_x(f)})_{|x}$.

$$f\wedge g\wedge h \overset{\delta_2}{\mapsto} \coprod \mathrm{ord}_x(f)\,\overline{g_x}\wedge\overline{h_x} - \mathrm{ord}_x(g)\,\overline{f_x}\wedge\overline{h_x} + \mathrm{ord}_x(h)\,\overline{f_x}\wedge\overline{g_x}$$

$$f\otimes[g]_2 \overset{\delta_1}{\mapsto} \coprod \mathrm{ord}_x(f)[g(x)]_2$$

with the convention that $[0]_2=[1]_2=[\infty]_2=0$.

Proof It easy to check that the diagram commutes, and that δ_2 does not depend on the choice of the π_x. So the only thing left is to check that δ_1 is well defined.

For this, fix $x\in C^1$, and define

$$\begin{aligned}\mathrm{sp}_{2,x}\colon \widetilde{M}_{(2)}(F) &\to \widetilde{M}_{(2)}(k(x))\\ [f]_2 &\mapsto [f(x)]_2\end{aligned}$$

with the convention that $[0]_2=[1]_2=[\infty]_2=0$. We want to show that this is well defined. For this, note that $\widetilde{M}_{(2)}(F)=\mathbb{Q}[F\setminus\{0,1\}]$ (free $\mathbb{Q}$–vector space on $F\setminus\{0,1\}$) modulo certain relations. With π a uniformizer at x, define $\wedge^2F^*_{\mathbb{Q}}\to\wedge^2k(x)^*_{\mathbb{Q}}$ by $f\wedge g\mapsto \overline{f}\wedge\overline{g}$ with $\overline{f}=f\pi^{-\mathrm{ord}_x(f)}{}_{|x}$. This depends on the choice of π, but always

$$\begin{tikzcd}
\mathbb{Q}[F\setminus\{0,1\}] \arrow[r, "\mathrm{d}"] \arrow[d, "\mathrm{sp}_{2,x}"] & \bigwedge^2 F^*_{\mathbb{Q}} \arrow[d] \\
\widetilde{M}_{(2)}(k(x)) \arrow[r, "\mathrm{d}"] & \bigwedge^2 k(x)^*_{\mathbb{Q}}
\end{tikzcd}$$

commutes. Therefore, if $\alpha \in \mathbb{Q}[F \setminus \{0,1\}]$ maps to zero in $\widetilde{M}_{(2)}(F)$, then $\mathrm{sp}_{2,x}(\alpha) \in H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k(x))) \cong K_3^{(2)}(k(x))$. As $k(x)$ is a number field, in order to show that $\mathrm{sp}_{2,x}(\alpha) = 0$, it suffices by Borel's theorem to show that the regulator vanishes on $\mathrm{sp}_{2,x}(\alpha)$. For this, we consider all complex embeddings of $k(x)$, which we can achieve by tensoring the curve with $\mathbb{C}$ over $\mathbb{Q}$, and considering all points lying above x. Then considering (6), it suffices to show that $P_{2,\mathrm{Zag}}(\mathrm{sp}_{2,x}(\alpha)) = 0$ in case x is a point of a curve over $\mathbb{C}$. Note that if y is a point in C such that α can be specialized to y directly, $\mathrm{sp}_{2,y}(\alpha)$ is nothing but the restriction to y of $\alpha = 0 \in H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(F))$, i.e., 0. Hence $P_{2,\mathrm{Zag}}(\mathrm{sp}_{2,y}(\alpha)) = 0$. $P_{2,\mathrm{Zag}}(\alpha)$ is constant because $\mathrm{d}P_{2,\mathrm{Zag}}(z) = \log|z| \mathrm{d}i \arg(1-z) - \log|1-z| \mathrm{d}i \arg z$ and $\mathrm{d}\alpha = 0$. Taking limits to x one finds $P_{2,\mathrm{Zag}}(\mathrm{sp}_{2,x}(\alpha)) = 0$ because $P_{2,\mathrm{Zag}}(0) = P_{2,\mathrm{Zag}}(1) = P_{2,\mathrm{Zag}}(\infty) = 0$ and $P_{2,\mathrm{Zag}}$ is continuous. This shows that δ_1 is well defined.

From (12) we get an induced map

$$\begin{aligned} \delta_1 : H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)) &\to \coprod_{x \in C^1} H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k(x))) \\ f \otimes [g]_2 &\mapsto \coprod_{x \in C^1} \mathrm{ord}_x(f)[g(x)]_2 . \end{aligned}$$

In the remainder of this section, we shall fix an identification of $K_3^{(2)}(k)$ with $H^1(\widetilde{\mathcal{M}}^{\bullet}_{(2)}(k))$, and take the same identification for all extensions of k, hence the same choice of sign in $K_2^{(2)}(X_{k(x)};\square) \cong K_3^{(2)}(k(x))$. Note also that because $k(x)$ is a number field, $K_4(k(x))_{\mathbb{Q}}$ is zero. Hence it follows from a localization sequence as in (11) that if U is a Zariski open part of C, then we can view $K_4^{(3)}(U)$ as a subset of $K_4^{(3)}(F)$, which we shall do from now on for the remainder of this section.

Theorem 5.2 *Let $\alpha = \sum_j a_j g_j \otimes [f_j]_2 \in H^2(\mathcal{M}^{\bullet}_{(3)}(F))$. Let U be the Zariski open part of C such that all f_j and g_j are invertible on U. Then under the map $H^2(\mathcal{M}^{\bullet}_{(3)}(F)) \longrightarrow K_4^{(3)}(F)$ in Proposition 3.2, α lands in $K_4^{(3)}(U) \subset K_4^{(3)}(F)$. The boundary of its image under the localization map*

$$K_4^{(3)}(U) \xrightarrow{\partial} \coprod_{x \in C \setminus U} K_3^{(2)}(k(x))$$

is given by $\pm 2\delta_1(\alpha)$, modulo the boundary of $K_3^{(2)}(k) \cup <f_j, g_j>$.

Remark 5.3 Suppose we have $\alpha \in H^2(\mathcal{M}^{\bullet}_{(3)}(F))$ with $\delta_1\alpha$ in the boundary of $K_3^{(2)}(k) \cup <f_j, g_j>$. As this last group injects into $\coprod_{x \in C^1} K_3^{(2)}(k(x))$ under ∂, we can change α with elements in $<f_j, g_j> \otimes H^1(\mathcal{M}^{\bullet}_{(2)}(k)) \cong <f_j, g_j> \otimes K_3^{(2)}(k)$ to obtain an element α' that maps to $K_4^{(3)}(C) \subset K_4^{(3)}(F)$. As the regulator does not care about this change by Theorem 4.2, computing the regulator of the global element α' amounts to computing the regulator of α.

Taking direct limits over U, we get, as $H^2(\mathcal{M}^\bullet_{(3)}(F)) \cong H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$

Corollary 5.4 *There is a commutative diagram (up to sign)*

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r] \arrow[d, "2\delta_1"'] & K_4^{(3)}(F) \arrow[r] & K_4^{(3)}(F)/K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \arrow[d, "\partial"'] \\
\coprod_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, equal] & \coprod_{x \in C^1} K_3^{(2)}(k(x)) \arrow[r] & \coprod_{x \in C^1} K_3^{(2)}(k(x)) \Big/ \partial\Big(K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}\Big)
\end{tikzcd}$$

Remark 5.5 If $K_3^{(2)}(k) = 0$, i.e., k is totally real, then of course we get the commutativity (up to sign) of

$$\begin{tikzcd}
H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \arrow[r] \arrow[d, "2\delta_1"'] & K_4^{(3)}(F) \arrow[d, "\partial"'] \\
\coprod_{x \in C^1} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \arrow[r, equal] & \coprod_{x \in C^1} K_3^{(2)}(k(x))
\end{tikzcd}$$

Proof of Theorem 5.2 Let U' be the Zariski open part of U where all $1 - f_j$ and all differences of distinct elements in $\{f_j, g_j\}$ are also invertible. Then the image α_1 of α in $K_4^{(3)}(F)$ is found as follows. Lift α to an element $\tilde{\alpha}$ of $H^2(C^\bullet_{(3),\log})$ (see (2)), which is given by $\sum_j -a_j([f_j]_{2|t_1=g_j} - [f_j]_{2|t_2=g_j})$, up to terms involving factors $(1+I)^*$. In fact, let $\{A_1, \dots, A_n\} \subset \{f_j, g_j\}$ be a basis for $<f_j, g_j> \subset F^*_{\mathbb{Q}}$. Then the only function of those terms involving $(1+I)^*$ is to move the boundary of this element, i.e., $1 - f_j$ at points $t_i = f_j$ or g_j $(i = 1, 2)$, to points $t_i = A_k$. Therefore they can be taken to be sums of $F(t_k) \cup (1-f_j)_{|t_{2-k}=f_j \text{ or } g_j}$ $(k = 1, 2)$ where $F(t)$ has only factors $t - f_j$, $t - g_j$ and $t - 1$. Hence the lift lives in $H^2(C_{(3),\log}(U')^\bullet)$. The proof of Proposition 3.2 works over U' as well, so that with $X_{U',\mathrm{loc}} = X_{U'} \setminus \{t = f_j \text{ or } g_j\}$ we have a commutative diagram (up to sign)

$$\begin{tikzcd}
\tilde{\alpha} \in H^2(C^\bullet_{(3),\log}(U')) \arrow[r] \arrow[d] & K_2^{(3)}(X^2_{U'} \setminus \{t_2 = f_j \text{ or } g_j\}; \square^2) \arrow[d, "\sim"'] \\
\alpha_3 \in K_3^{(3)}(X_{U'}; \square) \arrow[r] \arrow[d, "\sim"'] & K_3^{(3)}(X_{U',\mathrm{loc}}; \square) \ni \alpha_2 \\
\alpha_1 \in K_4^{(3)}(U') &
\end{tikzcd}$$

where all α's are mapped to each other, and $\alpha_2 = \pm \sum_j a_j [f_j]_2 \cup g_j$ plus some terms in $(1+I)^* \cup <f_j, g_j>$. In fact, with

$$f_j = \prod_k A_k^{b_{j,k}} \quad \text{and} \quad g_j = \prod_l A_l^{c_{j,l}}, \tag{13}$$

put $F_j(t) = \frac{t-f_j}{t-1} \prod_k \left(\frac{t-A_k}{t-1}\right)^{-b_{j,k}} \in (1+I)^*$. Then

$$\alpha_2 = \pm\left(\sum_j a_j[f_j]_2 \cup g_j + \sum_j a_j F_j(t) \cup \{1-f_j, g_j\}\right)$$

as it has zero boundary in $\coprod_{f_j,g_j} K_2^{(2)}(U')$ in the localization sequence

$$\coprod_{t=f_j \text{ or } g_j} K_3^{(2)}(U') \to K_3^{(3)}(X_{U'};\square) \to K_3^{(3)}(X_{U',\mathrm{loc}};\square) \to \coprod_{t=f_j \text{ or } g_j} K_2^{(2)}(U')$$

Note that α_2 determines α_1 up to $<f_j, g_j> \cup K_3^{(2)}(U')$, as $K_3^{(2)}(U')_{|t=f}$ maps to $K_3^{(2)}(U') \cup f$ in $K_4^{(3)}(U')$. As we also have a commutative diagram

$$\begin{tikzcd}
K_3^{(3)}(X_{U'};\square) \arrow[r, "\partial"] \arrow[d, "\sim"'] & \coprod_{x\in C\setminus U'} K_2^{(2)}(X_{k(x)};\square) \arrow[d, "\sim"'] \\
K_4^{(3)}(U') \arrow[r, "\partial"] & \coprod_{x\in C\setminus U'} K_3^{(2)}(k(x))
\end{tikzcd} \tag{14}$$

we can compute the boundary of α_1 by taking α_3 and computing its image in the lower right hand corner. Doing this with any lift α_3' of α_2 to $K_3^{(3)}(X_U;\square)$ gives us the boundary of α_1 modulo the boundary of $<f_j, g_j> \cup K_3^{(2)}(U')$. Note that $K_3^{(2)}(U') = K_3^{(2)}(C)$, so for $x \in U \setminus U'$ we do find $\partial_x \alpha_1$ using this procedure. As we shall find zero for such x, this proves that $\alpha_1 \in K_4^{(3)}(U) \subset K_4^{(3)}(U')$ as desired.

Because $k(x)$ is a number field, and the regulator is injective on its K–theory, we can perform the computation at the level of Deligne cohomology. For this we have to consider all embeddings of $k(x)$ into $\mathbb{C}$, which we can achieve by tensoring the curve C with $\mathbb{C}$ over $\mathbb{Q}$. Therefore we can assume from now on that the curve is defined over $\mathbb{C}$ (but it might not be connected).

The regulator map on all groups involved in (14) lands in Deligne–cohomology groups $H^m_{\mathcal{D}}(\mathbb{R}(n))$ which are isomorphic to $H^{m-1}_{\mathrm{dR}}(\mathbb{R}(n-1))$'s. We will identify the two, and work with the de–Rham cohomology. Under the regulator, the clockwise direction of the diagram (14) is transformed into

$$\begin{tikzcd}
H^2_{\mathrm{dR}}(X_{U'};\square;\mathbb{R}(2))^+ \arrow[r] & \left(\coprod_{x\in C\setminus U'} H^1_{\mathrm{dR}}(X_{k(x)};\square;\mathbb{R}(1))\right)^+ \arrow[d] \\
 & \left(\coprod_{x\in C\setminus U'} H^0_{\mathrm{dR}}(\mathrm{Spec}\,(k(x));\mathbb{R}(1))\right)^+
\end{tikzcd} \tag{15}$$

Consider the localization sequence

$$\left(\coprod_{f_j,g_j} H^0_{\mathrm{dR}}(U';\mathbb{R}(1))\right)^+ \longrightarrow H^2_{\mathrm{dR}}(X_{U'};\square;\mathbb{R}(2))^+ \longrightarrow H^2_{\mathrm{dR}}(X_{U',\mathrm{loc}};\square;\mathbb{R}(2))^+.$$

We will lift the regulator ψ_2 of $\pm\alpha_2 = \sum_j a_j[f_j]_2 \cup g_j + \sum_j a_j F_j(t) \cup \{1-f_j, g_j\}$ in $H^2_{\mathrm{dR}}(X_{U',\mathrm{loc}};\square;\mathbb{R}(2))^+$ to ψ_3 in $H^2_{\mathrm{dR}}(X_{U'};\square;\mathbb{R}(2))^+$. The sequence shows that ψ_3 differs from the regulator of α_3 by an element in the image of

$$\left(\coprod_{f_j,g_j} H^0_{\mathrm{dR}}(U';\mathbb{R}(1))\right)^+ = \left(\coprod_{f_j,g_j} H^0_{\mathrm{dR}}(\mathrm{Spec}\,(k\otimes_{\mathbb{Q}}\mathbb{C});\mathbb{R}(1))\right)^+ .$$

This is precisely the $\mathbb{R}$–vector space spanned by the image of $K_3^{(2)}(k)_{|t=f_j \text{ or } g_j}$, which injects into the $H^0_{\mathrm{dR}}(\mathrm{Spec}\,(k(x));\mathbb{R}(1))$'s in (15). Because we will give a ψ_3 such that its image in (15) at all $x \in C\setminus U'$ is in the image of the regulator of $K_3^{(2)}(k(x))$, and the same holds for the regulator of α_3, it follows that ψ_3 is actually the regulator of some α_3' lifting α_2. Therefore the image of ψ_3 in (15) determines the boundary of α_1 up to the boundary of $K_3^{(2)}(k)\cup\,<f_j,g_j>$, and the construction of ψ_3 together with the computation of its image in (15) will finish the proof.

All this having been said, we can concentrate on the regulators. The regulator of $[f]_2\cup g$ is given by $\varepsilon_2(f)\wedge \mathrm{d}i\arg g + \log|g|\pi_2\omega_2(f)$ according to (4) and (5). For $F\in(1+I)^*$, $\delta\in K_2^{(2)}(U')$, the regulator of $F\cup\delta$ is given by $-\mathrm{d}i\arg F\wedge\varepsilon$ for $\varepsilon\in H^1_{\mathrm{dR}}(U';\mathbb{R}(1))$.

We introduce the notation $\varepsilon(f,g) = \log|f|\mathrm{d}i\arg g - \log|g|\mathrm{d}i\arg f$, so $\varepsilon(f,g)$ is the regulator of $\{f,g\}$. Note that $\varepsilon(f,1-f) = \mathrm{d}P_{2,\mathrm{Zag}}(f)$. Then the regulator of $\sum_j a_j[f_j]_2\cup g_j + \sum_j F_j(t)\cup\{1-f_j,g_j\}$ is represented by

$$\psi_2 = \sum_j a_j\left(\varepsilon_2(f_j)\mathrm{d}i\arg g_j + \log|g_j|\pi_2\omega_2(t,f_j)\right) - \sum_j a_j \mathrm{d}i\arg F_j(t)\wedge\varepsilon(1-f_j,g_j).$$

We need a little more in order to proceed. Namely, from

$$\mathrm{d}\sum_j a_j g_j\otimes[f_j]_2 = \sum_j a_j g_j\wedge f_j\otimes(1-f_j) = 0$$

in $\wedge^2 F^*_{\mathbb{Q}}\otimes F^*_{\mathbb{Q}}$, it follows by writing all f_j and g_j in terms of the A_k's (see (13)) that for all k,l, we have an identity in $F^*_{\mathbb{Q}}$:

$$\prod_j (1-f_j)^{a_j b_{j,k} c_{j,l}} = \prod_j (1-f_j)^{a_j b_{j,l} c_{j,k}}. \tag{16}$$

We shall also use the identity for $\tilde\varepsilon_2(f) = \varepsilon_2(f) - \mathrm{d}\left(\rho(t)P_{2,\mathrm{Zag}}(f)\right)$ (see (5)):

$$\begin{aligned}\tilde\varepsilon_2(f)\wedge\mathrm{d}i\arg g + \log|g|\pi_2\omega_2(f) &= \varepsilon(g,1-f)\wedge\mathrm{d}i\arg\frac{t-f}{t-1}\\ &\quad + \log\left|\frac{t-f}{t-1}\right|\pi_0(\mathrm{d}\log g\wedge\mathrm{d}\log(1-f)) + \mathrm{d}\eta\end{aligned} \tag{17}$$

for $\eta = \eta(f,g) = -\log|g| \log\left|\dfrac{t-f}{t-1}\right| \mathrm{d}\log|1-f|$. Of course the term involving π_0 vanishes because we are on a curve.

As for lifting ψ_2 to a form on $X_{U'}$, vanishing at $t=0,\infty$, and satisfying (8), using (17) and letting $\eta_j = \eta(f_j, g_j)$, we can write for $\psi_2 - \sum_j a_j \mathrm{d}\rho(t) P_{2,\mathrm{Zag}}(f_j) \wedge \mathrm{d}i\arg g_j$

$$\begin{aligned}
&\sum_j a_j \left(\tilde{\varepsilon}_2(f_j)\mathrm{d}i\arg g_j + \log|g_j|\pi_2\omega_2(t,f_j)\right) - \sum_j a_j \mathrm{d}i\arg F_j(t)\wedge\varepsilon(1-f_j,g_j)\\
&= \sum_j a_j\varepsilon(g_j,1-f_j)\wedge\mathrm{d}i\arg\frac{t-f_j}{t-1} + \sum_j a_j\mathrm{d}\eta_j - \sum_j a_j\mathrm{d}i\arg F_j(t)\wedge\varepsilon(1-f_j,g_j)\\
&= \sum_{j,k} a_j b_{j,k}\varepsilon(g_j,1-f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} + \sum_j a_j\mathrm{d}\eta_j\\
&= \sum_{j,k,l} a_j b_{j,k}c_{j,l}\varepsilon(A_l,1-f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} + \sum_j a_j\mathrm{d}\eta_j\\
&= \sum_{j,k,l} a_j b_{j,l}c_{j,k}\varepsilon(A_l,1-f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} + \sum_j a_j\mathrm{d}\eta_j\\
&= \sum_{j,k} a_j c_{j,k}\varepsilon(f_j,1-f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} + \sum_j a_j\mathrm{d}\eta_j\\
&= \sum_{j,k} a_j c_{j,k}\mathrm{d}P_{2,\mathrm{Zag}}(f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} + \sum_j a_j\mathrm{d}\eta_j
\end{aligned}$$

by (16) and the identity $\varepsilon(f,1-f) = \mathrm{d}P_{2,\mathrm{Zag}}(f)$.

Let $\tilde{\rho}$ be a bump function such that $\tilde{\rho}\equiv 0$ for $t=0,\infty$, $\tilde{\rho}\equiv 1$ on small tubes around $t=f_j$, and symmetric with respect to complex conjugation. Then

$$\psi_3 = \psi_2 - \mathrm{d}\left(\tilde{\rho}\sum_{j,k} a_j c_{j,k} P_{2,\mathrm{Zag}}(f_j)\wedge\mathrm{d}i\arg\frac{t-A_k}{t-1} - \tilde{\rho}\sum_j a_j\eta_j\right)$$

is a lift of ψ_2 to $X_{U'}$, vanishing at $t=0,\infty$.

For $x\in C\setminus U'$, the map

$$H^2_{\mathrm{dR}}(X_{U'};\square;\mathbb{R}(2)) \longrightarrow H^1_{\mathrm{dR}}(X_x;\square;\mathbb{R}(1)) \xrightarrow{\sim} H^0_{\mathrm{dR}}(\mathrm{Spec}\,(\mathbb{C});\mathbb{R}(1))_x = \mathbb{R}(1)_x$$

can be computed as follows. If S^1_x is a little loop around x, it is given by

$$\frac{1}{2\pi i}\int_{X_{S^1_x}} \psi_3\wedge\mathrm{d}i\arg t. \tag{18}$$

(This does not depend on the representative of the class in $H^2_{\mathrm{dR}}(X_{U'};\square;\mathbb{R}(2))$, provided we consider representatives satisfying (8). It then follows from the fact that the class of ψ_3 has representatives of the form $\psi\wedge\mathrm{d}h$, $\psi\in H^1_{\mathrm{dR}}(U';\mathbb{R}(2))$.)

Here we use the following orientation. If θ is a parameter on S^1_x (oriented anti-clockwise), then we use $\frac{1}{2i}\mathrm{d}t \wedge \mathrm{d}\,\bar{t} \wedge \mathrm{d}\theta$.

One computes that

$$\begin{aligned}
&\int_{X_{S^1_x}} \varepsilon_2(f)\mathrm{d}i\arg g \wedge \mathrm{d}i\arg t = -2\pi i \int_{S^1_x} P_{2,\mathrm{Zag}}(f)\mathrm{d}i\arg g \\
&\int_{X_{S^1_x}} \log|g|\pi_2\omega_2(f) \wedge \mathrm{d}i\arg t = 2\pi i \int_{S^1_x} \log|g|\log|f|\mathrm{d}\log|1-f| \\
&\int_{X_{S^1_x}} \mathrm{d}i\arg F(t) \wedge \varepsilon(1-f,g) \wedge \mathrm{d}i\arg t = 0 \\
&\int_{X_{S^1_x}} \mathrm{d}\left(\tilde{\rho}P_{2,\mathrm{Zag}}(f) \wedge \mathrm{d}i\arg\frac{t-A}{t-1}\right) \wedge \mathrm{d}i\arg t = 2\pi i \int_{S^1_x} P_{2,\mathrm{Zag}}(f)\mathrm{d}i\arg A \\
&\int_{X_{S^1_x}} \mathrm{d}(\tilde{\rho}\eta) \wedge \mathrm{d}i\arg t = 0.
\end{aligned}$$

Hence (18) equals

$$\int_{S^1_x} -\sum_j a_j\left(2P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg g_j - \log|g_j|\log|f_j|\mathrm{d}\log|1-f_j|\right).$$

Note that the form we are integrating is closed by (16). Therefore, by shrinking the S^1_x's we obtain $2\sum_j a_j\mathrm{ord}_x(g_j)P_{2,\mathrm{Zag}}(f_j(x))$ as element of $H^0_{\mathrm{dR}}(\mathrm{Spec}\,\mathbb{C};\mathbb{R}(1))$. By (6), the image under the boundary of α at x is given by $\pm 2\sum_j a_j\mathrm{ord}_x(g_j)[f_j(x)]_2$ (with the convention $[0]_2 = [1]_2 = [\infty]_2 = 0$), as element in $H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) \cong K_3^{(2)}(k(x))$. This finishes the proof of Theorem 5.2.

It is hard in general to write down elements in $H^2(\mathcal{M}^\bullet_{(3)}(F))$ explicitly. In $H^1(\mathcal{C}^\bullet(F))$ this is easier. (We give explicit examples in Example 6.2 below.) In order to know if they are in the image of global elements we use the following modification of Theorem 5.2.

Theorem 5.6 *Suppose $\alpha = \sum_j a_j[f_j]_2 \cup g_j \in H^1(\mathcal{C}^\bullet)$. Let $U \subset C$ be the Zariski open set on which all f_j and g_j are invertible. Then α is the image of $\tilde{\alpha} \in K_4^{(3)}(U)$ under the map*

$$K_4^{(3)}(U) \longrightarrow K_4^{(3)}(F) \longrightarrow K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_{\mathbb{Q}} = H^1(\mathcal{C}^\bullet).$$

Moreover the boundary of $\tilde{\alpha}$ under the localization sequence

$$K_4^{(3)}(C) \longrightarrow K_4^{(3)}(U) \xrightarrow{\partial} \coprod_{x\in C\setminus U} K_3^{(2)}(k(x))$$

is given as follows, up to the boundary of $K_3^{(2)}(k) \cup <f_j>$, *and up to sign. Let* A_l, $f_{l,m}$ *and* $c_{l,m}$ *be as in Remark 3.3. Then the component of* $\partial\tilde{\alpha}$ *at* $x \in C \setminus U$ *is given by*

$$\sum_j a_j \mathrm{ord}_x(g_j)[f_j(x)]_2 + \sum_{l,m} c_{l,m} \mathrm{ord}_x(A_l)[f_{l,m}(x)]_2$$

in $H^1(\mathcal{M}^\bullet_{(2)}(k(x)) \cong K_3^{(2)}(k(x))$. *(So* $\partial\tilde{\alpha}$ *is half the boundary as in Theorem 5.2 of a lift of* α *to* $H^2(\mathcal{M}^\bullet_{(3)})$ *as in Remark 3.3.)*

Remark 5.7 The boundary map is not readily computable, because there does not seem to be a way to find the $f_{l,m}$ explicitly. However, the theorem puts restrictions on where the boundary map can be non–zero. In particular, it sometimes enables us to check if an element in $H^1(\mathcal{C}^\bullet(F))$ comes from a global element.

Example 5.8 The extra terms from the $f_{l,m}$'s can be necessary to get the correct boundary. In fact, consider the element $\alpha = [1+T^2]_2 \cup (1-T)$ for $\mathbb{Q}(T)$. Because $K_3^{(2)}(\mathbb{Q}) = 0$ we do get the boundary on the nose here. Then $\mathrm{d}\alpha = -\{-T^2, 1-T\} \otimes (1+T^2)$, and $(-T^2) \otimes (1-T) = 2\,(T \otimes (1-T))$. The contributions at the indicated points of $\mathbb{P}^1_{\mathbb{Q}}$ are:

$$\begin{array}{lll} T = 1 & : & [2]_2 = 0 \\ T = \infty & : & 0 \\ T^2+1=0: & & 2\,[1-T]_2 \end{array}$$

where in the last line we are working in the field $\mathbb{Q}[T]/(T^2+1)$. By using the function $P_{2,\mathrm{Zag}}$ one checks that the last is non–zero.

Proof [**of Theorem 5.6**] The proof is very similar to the proof of Theorem 5.2, so we shall be somewhat sketchy. Let U' be the open part of U on which all $1-f_j$ and $f_j - f_k$ are invertible if they are not identically zero. Consider the following localization sequence.

$$\coprod_j K_3^{(2)}(U')_{|t=f_j} \longrightarrow K_3^{(3)}(X_{U'};\square) \longrightarrow K_3^{(3)}(X_{U',\mathrm{loc}};\square) \longrightarrow \coprod_j K_2^{(2)}(U')_{|t=f_j}$$

We shall lift α to an element $\tilde{\alpha}$ of $K_3^{(3)}(X_{U'};\square) \cong K_4^{(3)}(U')$. Then a computation of the boundary of $\tilde{\alpha}$ will show that in fact $\tilde{\alpha}$ lies in $K_4^{(3)}(U) \subset K_4^{(3)}(U')$.

Recall that

$$\mathrm{d}\sum_j a_j[f_j]_2 \cup g_j = -\sum_j a_j f_j \otimes \{1-f_j, g_j\} = 0 \tag{19}$$

in $F^*_{\mathbb{Q}} \otimes K_2^{(2)}(F)$. Let $A = \{A_1, \ldots, A_k\} \subset \{f_j\}$ be a basis of the subspace of $F^*_{\mathbb{Q}}$ spanned (multiplicatively) by the f_j. So $f_j = \prod_{l=1}^k A_l^{b_{j,l}}$, $b_{j,l} \in \mathbb{Q}$. The functions

$$F_j(t) = \frac{t-f_j}{t-1} \prod_l \left(\frac{t-A_l}{t-1}\right)^{-b_{j,l}}$$

are elements of $(1+I)^*$. Then $\alpha_1 = \sum_j a_j[f_j]_2 \cup g_j + \sum_j a_j F_j(t) \cup \{1-f_j, g_j\}$ is an element in $K_3^{(3)}(X_{U',\mathrm{loc}};\square)$ which lifts α. Its boundary is concentrated in $\coprod_{t=A_j} K_2(U')$. In fact the boundary is zero because of (19) as the A_j are multiplicatively independent. Therefore it can be lifted (modulo $K_3^{(2)}(U') \cup {<}f_j{>} = K_3^{(2)}(C) \cup {<}f_j{>}$) to $\tilde{\alpha} \in K_3^{(3)}(X_{U'};\square)$.

For the regulator of $\tilde{\alpha}$ we will need that there exist $f_{l,m} \in F^*$ and $c_{l,m} \in \mathbb{Q}$ as in Remark 3.3, i.e., such that in $F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$,

$$\sum_j a_j b_{j,l}(1-f_j) \otimes g_j = \sum_m c_{l,m}(1-f_{l,m}) \otimes f_{l,m}. \tag{20}$$

The regulator of α_1 is represented by

$$\psi_1 = \sum_j a_j \left(\varepsilon_2(f_j)\mathrm{d}i\arg g_j + \log|g_j|\pi_2\omega_2(t,f_j)\right) - \sum_j a_j \mathrm{d}i\arg F_j(t) \wedge \varepsilon(1-f_j, g_j).$$

We want to lift ψ_1 to a form $\tilde{\psi}$ on $X_{U'}$, vanishing at $t=0,\infty$, and satisfying (8). As its residues at the closed points will turn out to be the regulators of elements in $K_3^{(2)}(k(x))$, $\tilde{\psi}$ is the regulator of $\tilde{\alpha}$ modulo the regulator of $K_3^{(2)}(k) \cup {<}f_j{>}$.

With $\eta_j = \eta(f_j, g_j)$, and using (17) we rewrite

$$\begin{aligned}
&\psi_1 - \sum_j a_j \mathrm{d}\left(\rho(t)P_{2,\mathrm{Zag}}(f_j)\right) \wedge \mathrm{d}i\arg g_j \\
=&\sum_j a_j \left(\tilde{\varepsilon}_2(f_j)\mathrm{d}i\arg g_j + \log|g_j|\pi_2\omega_2(t,f_j)\right) - \sum_j a_j \mathrm{d}i\arg F_j(t) \wedge \varepsilon(1-f_j, g_j) \\
=&\sum_j a_j \varepsilon(g_j, 1-f_j) \wedge \mathrm{d}i\arg\frac{t-f_j}{t-1} + \sum_j a_j \mathrm{d}\eta_j - \sum_j a_j \varepsilon(g_j, 1-f_j) \wedge \mathrm{d}i\arg F_j(t) \\
=&\sum_{j,l} a_j b_{l,j} \varepsilon(g_j, 1-f_j) \wedge \mathrm{d}i\arg\frac{t-A_l}{t-1} + \sum_j a_j \mathrm{d}\eta_j \\
=&\sum_{l,m} c_{l,m}\mathrm{d}P_{2,\mathrm{Zag}}(f_{l,m}) \wedge \mathrm{d}i\arg\frac{t-A_l}{t-1} + \sum_j a_j \mathrm{d}\eta_j
\end{aligned}$$

by (20) and the identity $\varepsilon(f, 1-f) = \mathrm{d}P_{2,\mathrm{Zag}}(f)$. One now proceeds as in the proof of Theorem 5.2 (using (20) instead of (16)) to find the boundary of $\tilde{\alpha}$ as given in the statement of the theorem:

$$\pm\left(\sum_j a_j \mathrm{ord}_x(g_j)[f_j(x)]_2 + \sum_{l,m} c_{l,m}\mathrm{ord}_x(A_l)[f_{l,m}(x)]_2\right).$$

Because there are no contributions at points in $U \setminus U'$, and this boundary equals the boundary of $\tilde{\alpha}$ up to the boundary of something in $K_3^{(2)}(k) \cup {<}f_j{>} \subset K_4^{(3)}(U)$, $\tilde{\alpha}$ actually came from $K_3^{(3)}(X_U;\square) \cong K_4^{(3)}(U)$.

6 Explicit examples

In this section we give construction methods of explicit examples in $H^1(\mathcal{C}^\bullet(F))$, where F is the function field of certain elliptic curves. Note that by Remark 3.3 this also gives rise to examples in $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F))$ and $H^2(\mathcal{M}^\bullet_{(3)}(F))$, although we cannot write them down explicitly. It will turn out that the regulators of the examples in $H^1(\mathcal{C}^\bullet)$ are non–zero, thus showing that the examples are non–trivial. It follows from Remark 4.4 that the corresponding examples in the H^2's are also non–zero, with regulators twice the regulators found here.

The methods are based on the same idea: essentially, we know the induced action of translation and multiplication by -1 on E on $\Gamma(E, \mathcal{K}_2)$, where $\mathcal{K}_2$ is the sheaf associated to the presheaf $U \mapsto K_2(\mathcal{O}_U)$.

Before going into the construction, recall how one computes $\Gamma(E, \mathcal{K}_2)$. Let E^1 be the set of points of codimension one. Then we have an exact sequence [9, Proposition 5.8, p. 48]

$$0 \to \Gamma(E, \mathcal{K}_2) \to K_2(k(E)) \xrightarrow{\partial} \coprod_{x \in E^1} k(x)^*. \tag{21}$$

Here ∂ is the inverse of the tame symbol, given for each $x \in E^1$ by

$$\mathrm{Tame}_x\{f, g\} = (-1)^{\mathrm{ord}_x(f)\mathrm{ord}_x(g)}(f^{\mathrm{ord}_x(g)} g^{-\mathrm{ord}_x(f)})_{|x},$$

and $k(x)$ is the residue field at x. Let $N_{k(x)/k}$ denote the norm from $k(x)$ to k. As E is complete, we have the product formula

$$\prod_{x \in E^1} N_{k(x)/k} \partial_x\{f, g\} = 1. \tag{22}$$

We now go on to the construction of the examples. Throughout, we shall be assuming that all points used are rational over the ground field k.

1. Assume $2P = O$, and assume $O, P, Q, -Q$ and $P+Q$ are all different. Let $d \in \mathbb{N}$ be such that $dQ = O$ and 2 divides d, and let $f_P, f_Q, f_{P+Q} \in k(E)^*$ be such that $(f_P) = 2(P) - 2(O)$, $(f_Q) = d(Q) - d(O)$, $(f_{P+Q}) = d(P+Q) - d(O)$, $f_P(Q) = 1$, $f_Q(P) = 1$ and $f_{P+Q}(P) = 1$. Then the tame symbol of $\{f_P, f_Q\}$ is trivial at P and Q, so it must be concentrated at O, where it must be trivial too by the product formula (22). So this is an element in $\Gamma(E, \mathcal{K}_2)$. It is well known that translation acts trivially on $\Gamma(E, \mathcal{K}_2)/K_2(k)$, where $K_2(k)$ comes from the base field via pull back. Translating over P we find that in $\Gamma(E, \mathcal{K}_2)$, up to $K_2(k)$,

$$\{f_P, f_Q\} = T_P\{f_P, f_Q\} = \{T_P f_P, T_P f_Q\} = \{c_1 f_P^{-1}, c_2 f_{P+Q} f_P^{-\frac{d}{2}}\}$$

for some $c_1, c_2 \in k^*$ by looking at the divisors of the functions involved; c_1, c_2 are determined up to torsion by the fact that the tame symbol is trivial. Using the relation $\{c_1 f_P, -c_1 f_P\} = 0$ this becomes

$$\{c_1 f_P^{-1}, c_3 f_{P+Q}\} = \{f_P^{-1} f_P(P+Q), f_{P+Q}\}$$

by computing the constants (up to d–torsion in k^*) using the tame symbol. By (21) we know that $\Gamma(E,\mathcal{K}_2)\subset K_2(k(E))$, so that we find that in $K_2(k(E))$ we have an identity (up to $K_2(k)$)

$$\{f_P, f_Q f_{P+Q}\}^d = \{f_P(P+Q), f_{P+Q}\}^d .$$

So if $f_P(P+Q)$ is an n–th root of unity, $\{f_P, f_Q f_{P+Q}\}^{\mathrm{lcm}(d,n)}$ lies in $K_2(k)$. Assuming this, we shall show it is zero by pulling it back to Q. For this, we write $\{f_P, f_Q f_{P+Q}\}^{\mathrm{lcm}(d,n)} = \{f_P, f_Q(1-f_P)^{-d} f_{P+Q}\}^{\mathrm{lcm}(d,n)}$, which can be pulled back to Q as all poles there have now been eliminated. Because $f_P(Q)=1$, it restricts to the trivial element at Q. Hence $\{f_P, f_Q f_{P+Q}\}^{\mathrm{lcm}(d,n)}$ is trivial in $K_2(k(E))$, and $[1-f_P]_2\cup(f_Q f_{P+Q})$ represents an element in $K_4^{(3)}(k(E))/k(E)^*\cup K_3^{(2)}(k(E))$. In fact, by Theorem 5.6 this is the image of an element with boundary at most at $Q, -Q, P+Q$ and O. As all those points are torsion and are defined over k, we can get a global element out of our element in the standard way: let $U = E\setminus\{Q,-Q,P+Q,O\}$, and lift the element to $\alpha\in K_4^{(3)}(U)$. If S is the finite subgroup generated by P and Q, then $\sum_{s\in S} T_s\alpha$ is a global element by the "product formula" for $K_4^{(3)}(k(E)) \xrightarrow{\partial} \coprod_{E^1} K_3^{(2)}(k(x))$.

2. Similar constructions can be used for $P,Q\in E(k)_{\mathrm{tor}}$, if $dP=dQ=O$, and $P, Q, P-Q$ and O are all different. Let f_P be such that $(f_P)=d(P)-d(O)$, $f_P(Q)=1$, and for other d–torsion points α let f_α be such that $(f_\alpha)=d(\alpha)-d(O)$, and $f_\alpha(P)=1$. Then the symbol $\{f_P,f_Q\}$ is an element of $\Gamma(E,\mathcal{K}_2)$ as before. Working up to $K_2(k)$, we find that in $\Gamma(E,\mathcal{K}_2)\subset K_2(k(E))$

$$\begin{aligned}\{f_P,f_Q\} &= T_{-P}\{f_P,f_Q\} = \{c_1 f_{-P}^{-1}, c_2 f_{Q-P} f_{-P}^{-1}\}\\ &= \{c_1 f_{-P}^{-1}, c_3 f_{Q-P}\} = \{c_4 f_P^{-1}, c_5 f_{P-Q}\}\end{aligned}$$

because $(-1)^*$ acts as multiplication by -1 on $\Gamma(E,\mathcal{K}_2)/K_2(k)$. c_4 and c_5 can be computed using the tame symbol, and one finds that this equals (up to d–torsion) $\{f_P^{-1} f_P(P-Q), f_{P-Q}\}$. As before we get an identity in $K_2(k(E))$ up to $K_2(k)$:

$$\{f_P, f_Q f_{P-Q}\}^d = \{f_P(P-Q), f_{P-Q}\}^d,$$

which leads to similar examples in $K_4^{(3)}(k(E))/k(E)^*\cup K_3^{(2)}(k(E))$. However, in general there might be non–torsion points where the corresponding element has a boundary, so that these elements might not globalize.

Remark 6.1 There are many variations on this theme to find examples. E.g., if P is a two torsion point, and Q is arbitrary, then one could try to work with the symbol $\{f_P, g\}$ where f_P is as before, and $(g)=2(Q)-(2Q)-O$, and $g(P)=1$. There is an obstruction (namely $f_P(2Q)\in k(2Q)^*_{\mathbb{Q}}$) for the element $\{f_P,g\}$ to be global, and if that is the case one finds an identity, again up to torsion and $K_2(k)$: $\{f_P, gT_Pg\} = \{f_P(2Q+P), T_Pg\}$. So there are two conditions on a two parameter family.

Example 6.2 To find explicit examples of the first construction, take the family of elliptic curves

$$Y^2 + (1-C)XY - (C^2+C)Y = X^3 - (C^2+C)X^2$$

for $C \neq 0, -\frac{1}{9}, -1$. The point $(0,0)$ is a 6–torsion point, see [8, p.91]. Then take $P = 3(0,0) = (C, C^2)$, and $Q = -2(0,0) = (C^2+C, 0)$. f with $(f) = 2(P) - 2(O)$ and $f_P(Q) = 1$ is $C^{-2}(X-C)$. So $f(P+Q) = -C^{-1}$. For $C = 1$, we find an example on $Y^2 - 2Y = X^3 - 2X^2$.

This curve is isomorphic to the one defined by $Y^2 = X^3 - 2X^2 + 1$, where now $P = (1,0)$, $Q = (2,-1)$ and $P+Q = (0,-1)$. Because there is only 2–torsion in $\mathbb{Q}^*$, the construction shows that

$$\{X-1, (Y+1)^3(Y+2X-3)\}^2$$

is in fact zero in $K_2(\mathbb{Q}(E))$. Therefore, letting $F(X,Y) = X-1$ and $G(X,Y) = (Y+1)^3(Y+2X-3)$, the element $[1-F]_2 \cup G$ in $H^1(\mathcal{C}^\bullet(\mathbb{Q}(E)))$ comes from $K_4^{(3)}(\mathbb{Q}(E))$. By Theorem 5.6, it comes from an element in $K_4^{(3)}(E \setminus \{O, P, Q, P+Q, -Q\})$. Because all the left–out points are $\mathbb{Q}$–rational, and $K_3^{(2)}(\mathbb{Q}) = 0$, it comes from $K_4^{(3)}(E)$. Note that the element $([F]_2 + [F^{-1}]_2) \cup G$ also has boundary zero, and also comes from an element in $K_4^{(3)}(E)$. One expects therefore a relation of the regulators of those elements with the leading coefficient of the L–function of E at -1, see [10, p. 31].

E is modular (in fact, it is curve 20A in [1]), and we have a functional equation $\Lambda(E,s) = \Lambda(E, 2-s)$, with $\Lambda(E,s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(E,s)$, $N = 20$ the conductor of E. From this, we get that the leading coeffcient of $L(E,s)$ at $s = -1$ is given by

$$L(E,-1)^* = -2\frac{20^2}{(2\pi)^4}L(E,3).$$

In order to compute the Beilinson regulator c_3, we have to compute the following integrals (see Proposition 4.1 and Theorem 4.2). Let ω be a non–zero invariant global holomorphic 1–form on E, with periods τ_1 (real) and τ_2 (purely imaginary in this case). Let

$$R_1 = \int_E \log|G| \log|1-F| \mathrm{d}\log|F| \wedge \overline{\omega}$$

for the first element, and for the second integral (R_2) we have to replace $1-F$ with F. One expects (see the beginning of section 4)

$$\frac{L(E,-1)^*}{c_3} = -\frac{2(20)^2}{(2\pi)^4}L(E,3)\frac{\tau_2\pi^2}{R_k}$$

to be a rational number. A (lengthy) computer calculation using PARI–GP gave the following values:

$$\begin{array}{ll} 1.99999999948 & (k=1) \\ -1.0000000000002433 & (k=2). \end{array}$$

Here the goal of the computation was 10 significant digits for R_1, and 15 for R_2. However, we lose some of this due to the large number of operations involved.

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