CODIMENSION 2 CYCLES
ON SEVERI-BRAUER VARIETIES

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ABSTRACT. For a given sequence of integers $(n_i)_{i=1}^{\infty}$ we consider all the central simple algebras A (over all fields) satisfying the condition $\operatorname{ind} A^{\otimes i} = n_i$ and find among them an algebra having the biggest torsion in the second Chow group CH^2 of the corresponding Severi-Brauer variety ("biggest" means that it can be mapped epimorphically onto each other).

We describe this biggest torsion in a way in general and more explicitly in some important special situations. As an application we prove indecomposability of certain algebras.

0. INTRODUCTION

We consider finite dimensional over fields central simple algebras. Let A be such an algebra and $X = \mathrm{SB}(A)$ the corresponding Severi-Brauer variety [2, §1]. We are interested to describe the torsion in the second Chow group $\mathrm{CH}^2(X)$ of 2-codimensional cycles on X modulo rational equivalence (the question seems more natural if one takes in account that the groups $\mathrm{CH}^0(X)$ and $\mathrm{CH}^1(X)$ never have a torsion). Here are some preliminary observations. The group $\operatorname{Tors} \mathrm{CH}^2(X)$ is finite and annihilated by $\operatorname{ind} A$. Further, if A' is another algebra Brauer equivalent to A and $X' = \mathrm{SB}(A')$ then [16, lemma (1.12)], [12, cor. 1.3.2]:

$$\operatorname{Tors} \mathrm{CH}^2(X) \simeq \operatorname{Tors} \mathrm{CH}^2(X') \ .$$

Finally, if

$$A = \bigotimes_p A_p$$

is the decomposition of an algebra A into the tensor product of its primary components and $X_p = \mathrm{SB}(A_p)$ for each prime p then

$$\operatorname{Tors} \mathrm{CH}^2(X) \simeq \bigoplus_p \operatorname{Tors} \mathrm{CH}^2(X_p)$$

or in other word, p-primary part of the group $\operatorname{Tors} \mathrm{CH}^2(X)$ is isomorphic to $\operatorname{Tors} \mathrm{CH}^2(X_p)$.

Summarizing we see that the problem to compute $\operatorname{Tors} \mathrm{CH}^2(X)$ for all algebras reduces itself to the case of primary division algebras.

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Now consider the Grothendieck group $K(X) = K_0(X)$ together with the gamma-filtration (1.5):

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots .$$

One has a canonical epimorphism

$$\Gamma^{2/3} K(X) \twoheadrightarrow \mathrm{CH}^2(X)$$

of the quotient

$$\Gamma^{2/3} K(X) = \Gamma^2 K(X) / \Gamma^3 K(X) .$$

We consider the group $\Gamma^{2/3} K(X)$ as an upper bound for $\mathrm{CH}^2(X)$ and will show that in the primary case this upper bound is in certain sense the least one.

To formulate it precisely let us call the sequence $(\operatorname{ind} A^{\otimes i})_{i=1}^{\infty}$ the behaviour of A. A sequence of integers $(n_i)_{i=1}^{\infty}$ will be called $\big((p\text{-})\text{primary}\big)$ behaviour if it is the behaviour of a $\big((p\text{-})\text{primary}\big)$ algebra.

Suppose that A is a division algebra. The Grothendieck group $K(X)$ depends only on the behaviour of A [18, §8, theorem 4.1]. Moreover, $K(X)$ together with the gamma-filtration (and the group $\Gamma^{2/3} K(X)$ in particular) depend only on the behaviour (2.1) and our main observation is (2.13):

For any primary behaviour (and any given field) there exists a division algebra $\widetilde{A}$ *(over an extension of the field) of the given behaviour for which the canonical epimorphism* $\Gamma^{2/3} K(\widetilde{X}) \twoheadrightarrow \mathrm{CH}^2(\widetilde{X})$ *with* $\widetilde{X} = \mathrm{SB}(\widetilde{A})$ *is bijective.*

The construction of the algebra $\widetilde{A}$ is rather simple (2.11). We take a division algebra (over a suitable extension of the field) of the index as in the given behaviour and of the exponent coinciding with the index. After that we pass to the function field of a product of certain generalized Severi-Brauer varieties in order to change the behaviour in the way prescribed.

Since the groups $\Gamma^{2/3} K(X)$ and $\mathrm{CH}^2(X)$ have the same rank (1.13) (rank 1 if X is a Severi-Brauer variety of dimension at least 2) we also have an epimorphism of the torsion subgroups

$$\operatorname{Tors} \Gamma^{2/3} K(X) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X)$$

which is moreover bijective iff $\Gamma^{2/3} K(X) \twoheadrightarrow \mathrm{CH}^2(X)$ is (1.14). So, formulating the main observation we may replace (and we do replace) both the groups $\Gamma^{2/3} K(X)$ and $\mathrm{CH}^2(X)$ by their torsion subgroups.

The gamma-filtration for a Severi-Brauer variety X and the group

$$\operatorname{Tors} \Gamma^{2/3} K(X)$$

in particular are from the so to say "algebra-geometrical" point of view very easy to compute (3.1): $K(X)$ is a subring of $K(\mathbb{P})$ where $\mathbb{P}$ is the $\dim X$-dimensional projective space and the Chern classes on $K(X)$ are simply the contractions of the Chern classes on $K(\mathbb{P})$. Although to get the answer in a final form (say, to find the canonical decomposition of the finite abelian group $\operatorname{Tors} \Gamma^{2/3} K(X)$ for any primary behaviour) further calculations are required which can be done e.g. by computer in any concrete situation (i.e. for a concrete behaviour) but seem to be not easy in the general case. Our main efforts in this direction are made in (3.3), (3.5), (3.8) and (3.9).

To the structure of the note.

In § 1 we recall and partially prove certain general facts on the Chern classes (with various values) and on the gamma-filtration. In § 2 we make the main observation. In § 3 we investigate the group $\Gamma^{2/3}K$ for various primary behaviours. In § 4 we consider algebras of prime exponent.

Some further notations concerning filtrations on $K(X)$ are introduced in § 1.

1. Chern classes and gamma-filtration

In this § we are working with the category of smooth projective irreducible algebraic varieties over a fixed field. The Grothendieck ring K is considered as a contravariant functor on this category.

Definition 1.1 (Chern classes with values in K)**.** Total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow K[[t]]^\times$$

(where the left-hand side is the additive group of the ring K while the right-hand side is the multiplicative group of series in one variable t over K) satisfying the following property: if $\xi \in K(X)$ is a class of an invertible sheaf on a variety X then

$$c_t(\xi) = 1 + (\xi - 1)t \; .$$

One defines the Chern classes $c^i : K \to K$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i \; .$$

Proposition 1.2. *Chern classes with values in K are unique.*

Proof. Follows from the

Lemma 1.3 (Splitting principle, [15, prop. 5.6])**.** *For any variety X and any $x \in K(X)$ there exists a morphism $f : Y \to X$ such that:*

1. *f is a composition of some projective bundle morphisms;*
2. *$f^*(x) \in K(Y)$ is a sum (with integer coefficients) of classes of some invertible sheaves.*

To obtain uniqueness of the Chern classes just note that the homomorphism $f^* : K(X) \to K(Y)$ from the lemma is injective. □

Proposition 1.4. *Chern classes with values in K exist.*

Proof. Here is the way of constructing due to Grothendieck with the original notations [15, theorem 3.10 and §8].

Take a variety X. First one constructs a homomorphism

$$\lambda_t : K^+ \longrightarrow K[[t]]^\times$$

by sending class of a locally free sheaf $\mathcal{E}$ to

$$\lambda_t([\mathcal{E}]) = \sum_{i=0}^{\infty} [\Lambda^i \mathcal{E}] \cdot t^i$$

where $\Lambda^i \mathcal{E}$ is the i-th exterior power of $\mathcal{E}$.

After that one considers another homomorphism

$$\gamma_t : K^+ \longrightarrow K[[t]]^\times$$

namely,

$$\gamma_t = \lambda_{\frac{t}{1-t}}$$

(this γ_t gave the name of the gamma-filtration).

Finally, one puts

$$c_t = \gamma_t \circ (\mathrm{id} - \mathrm{rk})$$

where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the rank homomorphism (followed by the inclusion $\mathbb{Z} \hookrightarrow K(X)$ more precisely). □

Definition 1.5 (Gamma-filtration)**.** The gamma-filtration

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots$$

is the smallest ring filtration on $K(X)$ such that

$$c^i\big(K(X)\big) \subset \Gamma^i K(X) \text{ for all } i \geq 1 .$$

In other words, $\Gamma^l K(X)$ is the subgroup of $K(X)$ generated by all the products

$$c^{i_1}(x_1) \dots c^{i_r}(x_r) \text{ with } x_j \in K(X) \text{ and } \sum_{i=1}^{r} i_j \geq l .$$

In particular, $\Gamma^1 K(X) = \mathrm{Ker}(\mathrm{rk} : K(X) \to \mathbb{Z})$.

We denote by $\mathrm{G}^*\Gamma K(X)$ the adjoint graded ring.

Definition 1.6 (Chern classes with values in $\mathrm{G}^*\Gamma K$)**.** For any variety X, we call the induced maps

$$c^i : K(X) \to \mathrm{G}^i\Gamma K(X)$$

the Chern classes with values in $\mathrm{G}^*\Gamma K$. The total Chern class c_t is the homomorphism

$$c_t : K(X)^+ \longrightarrow \left(\sum_{i=0}^{\infty} \mathrm{G}^i\Gamma K(X) \cdot t^i \right)^\times$$

It is a morphism of functors and

$$c_t(\xi) = 1 + (\xi - 1)t$$

for a class $\xi \in K(X)$ of an invertible sheaf on X ($(\xi - 1)$ is considered as an element of $\mathrm{G}^1\Gamma K(X)$ in the last formula).

Definition 1.7 (Chern classes with values in CH^*)**.** Total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow \left(\sum_{i=0}^{\infty} \mathrm{CH}^i \cdot t^i \right)^{\times}$$

satisfying the property

$$c_t(\xi) = 1 + (\xi - 1)t$$

where $\xi \in K(X)$ is a class of an invertible sheaf ($(\xi - 1)$ is considered as an element of $\mathrm{CH}^1(X)$ in the last formula).

One defines the Chern classes $c^i : K \to \mathrm{CH}^i$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i \ .$$

Proposition 1.8. *Chern classes with values in* CH^* *are unique.*

Proof. Follows from the splitting principle (1.3) since the homomorphism

$$f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

is injective. □

Proposition 1.9 ([4, §3.2])**.** *Chern classes with values in* CH^* *exist.*

Side by side with the gamma-filtration we consider the topological filtration on $K(X)$ (in fact defined on $K_0'(X)$) [18, §7]:

$$K(X) = \mathrm{T}^0 K(X) \supset \mathrm{T}^1 K(X) \supset \dots \ .$$

Note that

$$\mathrm{T}^1 K(X) = \mathrm{Ker}(\mathrm{rk} : K(X) \to \mathbb{Z}) = \Gamma^1 K(X) \ .$$

We will denote by $\mathrm{G}^*\mathrm{T}K(X)$ the adjoint graded ring.

Definition 1.10 (Chern classes with values in $\mathrm{G}^*\mathrm{T}K$)**.** The total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow \left(\sum_{i=0}^{\infty} \mathrm{G}^i\mathrm{T}K \cdot t^i \right)^{\times}$$

satisfying the property:

$$c_t(\xi) = 1 + (\xi - 1)t \ .$$

One defines the Chern classes $c^i : K \to \mathrm{G}^i\mathrm{T}K$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i \ .$$

Proposition 1.11. *Chern classes with values in* $\mathrm{G}^*\mathrm{T}K$ *are unique.*

Proof. Follows from the splitting principle (1.3) since the homomorphism

$$f^* : \mathrm{G}^*\mathrm{T}K(X) \to \mathrm{G}^*\mathrm{T}K(Y)$$

is injective. □

Proposition 1.12. *Chern classes with values in* $\mathrm{G}^*\mathrm{T}K$ *exist.*

Proof. Simply compose the Chern classes with values in CH^* with the canonical epimorphism $\mathrm{CH}^* \twoheadrightarrow \mathrm{G}^*\mathrm{T}K$. □

Now we establish certain connections between the gamma-filtration and the topological one.

Proposition 1.13. *For any variety* X*,*

1. $\Gamma^i K(X) \subset \mathrm{T}^i K(X)$ *for all* i*.*
2. $\Gamma^i K(X) = \mathrm{T}^i K(X)$ *for* $i \leq 2$*.*
3. $\Gamma^i K(X) \otimes \mathbb{Q} = \mathrm{T}^i K(X) \otimes \mathbb{Q}$ *for all* i*.*

Proof. 1. We only need to show that for any $x \in K(X)$ the coefficient at t^i in the series $c_t(x)$ lies in $\mathrm{T}^i K(X)$. Let us call a series $f \in K(X)[[t]]$ "good" if for any i the coefficient at t^i in f is from $\mathrm{T}^i K(X)$. If serieses f and g are "good" then the serieses fg and f^{-1} (if exists) are obviously "good" too (compatibility of the topological filtration with the multiplication is used). From the other hand the series $c_t(x)$ is evidently "good" in the case when x is a class of an invertible sheaf. Hence $c_t(x)$ is also "good" when x is a sum (with integer coefficients)of some invertible sheaves what implies according to the splitting principle (1.3) that $c_t(x)$ is "good" always.

2. We only need to manage the case $i = 2$.

Consider a homomorphism $K(X) \to \mathrm{Pic}(X)$ into the Piccard group of X given by the rule $[\mathcal{E}] \mapsto [\Lambda^{\mathrm{rk}\,\mathcal{E}}\mathcal{E}]$. It gives a surjection $\Gamma^1 K(X) \twoheadrightarrow \mathrm{Pic}(X)$ with the kernel $\Gamma^2 K(X)$ [15, §10]. So, we get an isomorphism

$$\mathrm{G}^1\Gamma K(X) \simeq \mathrm{Pic}(X)\ .$$

One has also an isomorphism [18, §7.5]

$$\mathrm{CH}^1(X) \simeq \mathrm{G}^1\mathrm{T}K(X)\ .$$

Since $\Gamma^2 K(X) \subset \mathrm{T}^2 K(X)$ we have a surjection $\mathrm{G}^1\Gamma K(X) \twoheadrightarrow \mathrm{G}^1\mathrm{T}K(X)$ which gives a homomorphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$. But the latter map is an isomorphism [5, cor. 6.16]. Thus $\Gamma^2 K(X) = \mathrm{T}^2 K(X)$.

3. We will give a proof in the end of this §. □

Corollary 1.14. *One has an exact sequence*

$$0 \to \mathrm{T}^3 K(X)/\Gamma^3 K(X) \to \operatorname{Tors} \mathrm{G}^2\Gamma K(X) \to \operatorname{Tors} \mathrm{CH}^2(X) \to 0\ .$$

Proof. The equality $\Gamma^2 K(X) = \mathrm{T}^2 K(X)$ and the inclusion $\Gamma^3 K(X) \subset \mathrm{T}^3 K(X)$ from the proposition give the exact sequence

$$0 \to \mathrm{T}^3 K(X)/\Gamma^3 K(X) \to \mathrm{G}^2\Gamma K(X) \to \mathrm{G}^2\mathrm{T}K(X) \to 0\ .$$

Consider the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{Tors} \mathrm{G}^2\Gamma K(X) \arrow[r, "(1)"] \arrow[d] & \operatorname{Tors} \mathrm{G}^2\mathrm{T}K(X) \arrow[d] \\
\mathrm{G}^2\Gamma K(X) \arrow[r, "(2)"] \arrow[d] & \mathrm{G}^2\mathrm{T}K(X) \arrow[d] \\
\mathrm{G}^2\Gamma K(X)/\operatorname{Tors} \arrow[r, "(3)"] \arrow[d] & \mathrm{G}^2\mathrm{T}K(X)/\operatorname{Tors} \arrow[d] \\
0 & 0
\end{tikzcd}$$

The map (2) is surjective. Hence the map (3) is surjective too. Since by the proposition the ranks of the groups $\mathrm{G}^2\Gamma K(X)$ and $\mathrm{G}^2\mathrm{T}K(X)$ coincide the map (3) is bijective. Thus (1) is surjective and the kernels of (1) and (2) coincide. So, we get the exact sequence

$$0 \to \mathrm{T}^3K(X)/\Gamma^3K(X) \to \operatorname{Tors} \mathrm{G}^2\Gamma K(X) \to \operatorname{Tors} \mathrm{G}^2\mathrm{T}K(X) \to 0 .$$

Finally, the canonical map $\mathrm{CH}^2(X) \to \mathrm{G}^2\mathrm{T}K(X)$ is an isomorphism (see e.g. [7, (3.1)]). □

Here is a connection between some Chern classes with distinguished values:

Lemma 1.15. *The following diagram of maps commutes:*

$$\begin{tikzcd}
K(X) \arrow[r, "c^i"] \arrow[d, "c^i"] & \mathrm{CH}^i(X) \arrow[d, "can."] \\
\mathrm{G}^i\Gamma K(X) \arrow[r] & \mathrm{G}^i\mathrm{T}K(X)
\end{tikzcd}$$

Proof. Both the compositions are Chern classes with values in $\mathrm{G}^*\mathrm{T}K$ (1.10) which are unique (1.11). □

Remark 1.16. One can formulate a criteria for when the gamma-filtration coincides with the topological one. It is clear from the very definition (1.5) that for any variety X the ring $\mathrm{G}^*\Gamma K(X)$ is generated by the Chern classes (with values in $\mathrm{G}^*\Gamma K$). So, if the gamma and the topological filtrations are the same the ring $\mathrm{G}^*\mathrm{T}K$ is generated by the Chern classes (with values in $\mathrm{G}^*\mathrm{T}K$ this time) too.

The other way round, if $\mathrm{G}^*\mathrm{T}K$ is generated by the Chern classes then the homomorphism $\mathrm{G}^*\Gamma K(X) \to \mathrm{G}^*\mathrm{T}K(X)$ is surjective whence the filtrations coincide.

Such a principle can be used to prove (3) from (1.13). We need

Lemma 1.17 ([4, example 15.3.6])**.** *For any $i > 0$, the restriction of the i-th Chern class*

$$c^i : K(X) \to \mathrm{CH}^i(X)$$

to $\mathrm{T}^i K(X)$ *is a homomorphism trivial on* $\mathrm{T}^{i+1}K(X)$. *The composition of the canonical epimorphism*

$$\mathrm{CH}^i(X) \twoheadrightarrow \mathrm{G}^i\mathrm{T}K(X)$$

with the induced map

$$\mathrm{G}^i\mathrm{T}K(X) \to \mathrm{CH}^i(X)$$

coincides with the multiplication by $(-1)^{i-1}(i-1)!$.

Corollary 1.18. *Changing the order of the composed maps one gets the multiplication by* $(-1)^{i-1}(i-1)!$ *too.*

Proof. The homomorphism $\mathrm{CH}^i(X) \twoheadrightarrow \mathrm{G}^i\mathrm{T}K(X)$ is surjective. □

Proof of (3) from (1.13). For any i the homomorphism

$$c^i : \mathrm{G}^i\mathrm{T}K(X) \to \mathrm{G}^i\mathrm{T}K(X)$$

coincides due to the corollary with multiplication by an integer. Whence

$$c^i : \mathrm{G}^i\mathrm{T}K(X) \otimes \mathbb{Q} \to \mathrm{G}^i\mathrm{T}K(X) \otimes \mathbb{Q}$$

is an isomorphism and in particular surjective what means that the quotients of the topological filtration on $K(X) \otimes \mathbb{Q}$ are generated by the Chern classes. According to our principle it implies that the gamma-filtration on $K(X) \otimes \mathbb{Q}$ coincides with the topological one. □

2. Main observation

From now on X denotes the Severi-Brauer variety corresponding to a central simple algebra A.

Lemma 2.1. *For a division algebra* A*, the group* $K(X)$ *together with the gamma-filtration depends only on the behaviour of* A.

Proof. Let $\mathbb{P}$ be the $\dim X$-dimensional projective space and $\xi \in K(\mathbb{P})$ the class of $\mathcal{O}_{\mathbb{P}}(-1)$. The ring $K(\mathbb{P})$ equals $\mathbb{Z}[\xi]/(1-\xi)^n$ where $n = \dim X + 1 = \deg A$. We identify $K(X)$ with a subring of $K(\mathbb{P})$. It is generated (as a subgroup) by all $(\operatorname{ind} A^{\otimes i}) \cdot \xi^i$ $(i \geq 0)$ [18, §8, theorem 4.1], so it is uniquely determined by the behaviour of A. The Chern classes on $K(X)$ (with values in K) are contractions of the Chern classes on $K(\mathbb{P})$, so they are uniquely determined too. □

Proposition 2.2 ([8, theorem 1]). *If* $\operatorname{ind} A = \exp A$ *for an algebra* A *then (for any* $l \geq 0$*) the* l*-th term of the topological filtration* $\mathrm{T}^l K(X)$ *is generated by all*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \quad \text{with} \quad l \leq i < \deg A$$

(see the the proof above for the definition of ξ*;* $(\cdot,\cdot)$ *denotes the greatest common divisor). In particular, the group* $\mathrm{G}^*\mathrm{T}K(X)$ *is torsion-free.*

Proposition 2.3. *If A is a primary algebra then*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \in \Gamma^i K(X) \ \textit{ for any } \ i \geq 0 \ .$$

Proof. Put $n = \operatorname{ind} A$. For $n\xi \in K(X)$ we have:

$$c_t(n\xi) = c_t(\xi)^n = \big(1 + (\xi - 1)t\big)^n$$

$\big($the last equality holds by (1.1)$\big)$. Whence

$$c^i(n\xi) = \binom{n}{i}(\xi - 1)^i \in \Gamma^i K(X) \ .$$

In particular,

$$(\xi - 1)^n \in \Gamma^n K(X)$$

thereby for the rest of the proof we may assume that $i \leq n$. Moreover,

$$n^i(\xi - 1)^i = c^1(\xi)^i \in \Gamma^i K(X) \ .$$

The last observation is

Lemma 2.4. *If n is a primary number and $n \geq i \geq 0$ then*

$$\left(n^i, \binom{n}{i}\right) = \frac{n}{(i, n)} \ .$$

Proof. Suppose that n is p-primary and denote by $v_p(j)$ the multiplicity of p in j. If $1 \leq j < n$ then $v_p(j) < v_p(n)$ and so $v_p(n - j) = v_p(j)$. Hence

$$v_p\left(\frac{(n-1)(n-2)\dots(n-(i-1))}{1 \cdot 2 \dots (i-1)}\right) = 0$$

and

$$v_p\binom{n}{i} = v_p\left(\frac{n}{i}\right) = v_p\left(\frac{n}{(i, n)}\right) \ .$$

$\square$

$\square$

Corollary 2.5. *If A is a primary algebra and* $\operatorname{ind} A = \exp A$ *then the gamma-filtration on $K(X)$ coincides with the topological one.*

$\square$

Theorem 2.6. *Let A be as in (2.5), $X = \mathrm{SB}(A)$. Let $Y_1, \dots, Y_m$ be some generalized Severi-Brauer varieties* [3, §4] *of some algebras which are (Brauer equivalent to) some tensor powers of A.*

The gamma-filtration on the Grothendieck group of the variety X over the function field $F(Y_1 \times \dots \times Y_m)$ coincides with the topological one. In particular, the epimorphism (1.14)

$$\operatorname{Tors} \mathrm{G}^2\Gamma K \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2$$

for this variety is bijective.

Proof. For every Y_i the product $X \times Y_i$ is a Grassman bundle over X (with respect to the first projection). Hence $\mathrm{CH}^*(X \times Y_i)$ is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes of a locally free sheaf (see e.g. [14, (3.2)]). Taking the product of all $X \times Y_i$ over X we obtain that

$$\mathrm{CH}^*(X \times Y_1 \times \cdots \times Y_m)$$

is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes (of some locally free sheaves).

The homomorphism of $\mathrm{CH}^*(X)$-algebras

$$\mathrm{CH}^*(X \times Y_1 \times \cdots \times Y_m) \to \mathrm{CH}^*(X_{F(Y_1 \times \cdots \times Y_m)})$$

(given by the pull-back) is surjective (see e.g. [13, theorem 3.1]). Whence the right-hand side is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes too.

Using the epimorphism $\mathrm{CH}^* \twoheadrightarrow \mathrm{G}^*\mathrm{T}K$ of the Chow ring onto the adjoint graded Grothendieck ring we obtain the same statement as in the previous paragraph but for $\mathrm{G}^*\mathrm{T}K$ instead of CH^* by meaning the Chern classes with values in $\mathrm{G}^*\mathrm{T}K$ this time.

The gamma-filtration on $K(X)$ coincides with the topological one (2.5) and therefore the ring $\mathrm{G}^*\mathrm{T}K(X)$ is generated by the Chern classes (1.16). Consequently, $\mathrm{G}^*\mathrm{T}K(X_{F(Y_1 \times \cdots \times Y_m)})$ is generated by the Chern classes as a ring, not only as a $\mathrm{G}^*\mathrm{T}K(X)$-algebra. It means that the gamma-filtration on $K(X_{F(Y_1 \times \cdots \times Y_m)})$ coincides with the topological one (1.16). □

Definition 2.7. Let A be a p-primary algebra. The sequence of integers

$$\left(\log_p \operatorname{ind} A^{\otimes p^i}\right)_{i=0}^{\log_p \exp A}$$

will be called the *reduced behaviour* of A.

Example 2.8. The reduced behaviour of a p-primary algebra A with $\operatorname{ind} A = \exp A = p^n$ is $n,\ n-1,\ n-2,\ \dots,\ 1,\ 0$.

Lemma 2.9. *The behaviour of a primary algebra is completely determined by its reduced behaviour. The reduced behaviour of an algebra is a finite strong decreasing sequence of integers with 0 in the end. Any finite strong decreasing sequence of integers with 0 in the end is for any prime p the reduced behaviour of a p-primary division algebra.*

Proof. Let A be a p-primary algebra. If i is an integer prime to p then $\operatorname{ind} A^{\otimes i} = \operatorname{ind} A$. It proves the first sentence of the lemma.

If in addition $\operatorname{ind} A \neq 1$ then $\operatorname{ind} A^{\otimes p} < \operatorname{ind} A$. It proves the second sentence.

Finally, fix a sequence $n_0 > n_1 > \cdots > n_m = 0$ and a prime p. A construction of a p-primary algebra having the reduced behaviour $(n_i)_{i=0}^m$ is given in [20, construction 2.8]. This construction involves function fields of usual Severi-Brauer varieties only. We describe another known construction which involves function fields of generalized Severi-Brauer varieties too and is more suitable for our purposes.

We start with a division algebra A (over a suitable field) for which

$$\operatorname{ind} A = \exp A = p^{n_0} .$$

For each $i = 1, 2, \dots, m$ consider the generalized Severi-Brauer variety

$$Y_i = \mathrm{SB}(p^{n_i}, A^{\otimes p^i})$$

(Y_i is by the definition [3, §2] the variety of rank p^{n_i} right ideals in $A^{\otimes p^i}$; its function field is a generic extension making the index of $A^{\otimes p^i}$ to be equal to p^{n_i}).

Finally, we denote the function field $F(Y_1 \times \cdots \times Y_m)$ by $\widetilde{F}$ and put $\widetilde{A} = A_{\widetilde{F}}$. Using the index reduction formula [3, theorem 5] or an improved version of this formula [17, formula I] one can easy show that the algebra $\widetilde{A}$ has the reduced behaviour $(n_i)_{i=0}^m$. □

Remark 2.10. In the construction described in the proof above it is not necessary to use all of the varieties Y_i: if $n_i = n_{i-1} - 1$ for some i then the variety Y_i can be omitted.

Definition 2.11. The algebra $\widetilde{A}$ from the proof above will be called a *"generic" p-primary division algebra of the reduced behaviour* $(n_i)_{i=0}^m$. Note that it can be constructed over an extension of any pregiven field.

Remark 2.12. The algebra $\widetilde{A}$ is really generic if A is (although for our purposes A can be chosen arbitrary). Also note that $\widetilde{A}$ is obtained from A by a generic scalar extension among the extensions changing the reduced behaviour of the algebra to $(n_i)_{i=0}^m$; it is another motivation for our terminology.

Theorem 2.13. *Fix a prime p and a reduced behaviour. If $\widetilde{A}$ is a "generic" p-primary division algebra of a given reduced behaviour (2.11) then the epimorphism (1.14)*

$$\operatorname{Tors} \mathrm{G}^2\Gamma K(\widetilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(\widetilde{X}) \quad \textit{(where } \widetilde{X} = \mathrm{SB}(\widetilde{A})\textit{)}$$

is bijective. If A is an arbitrary p-primary algebra of the same reduced behaviour as $\widetilde{A}$ then there exists an epimorphism

$$\operatorname{Tors} \mathrm{CH}^2(\widetilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X) .$$

Proof. The first part follows from (2.6) and from the definition of "generic" algebras (2.11). The second part follows from (1.14), (2.1) and from the first one. □

3. Computation of gamma-filtration

Remind that we have put forever $X = \mathrm{SB}(A)$; we also put $\widetilde{X} = \mathrm{SB}(\widetilde{A})$.

Proposition 3.1. *Let A be a p-primary algebra and $(n_i)_{i=0}^m$ its reduced behaviour. For any $l \geq 0$, the group $\Gamma^l K(X)$ is generated by all the products*

$$(*) \qquad \prod_{i=0}^{m} \frac{p^{n_i}}{(j_i, p^{n_i})} (\xi^{p^i} - 1)^{j_i} \quad \textit{with } j_i \geq 0 \textit{ and } \sum_{i=0}^{m} j_i \geq l$$

where $\xi = [\mathcal{O}(-1)] \in K(\mathbb{P})$.

Moreover, one can omit in $(*)$ *all the factors with numbers* i *such that* $n_i = n_{i-1} - 1$.

Proof. The formula

$$c^j(p^{n_i}\xi^{p^i}) = \binom{p^{n_i}}{j}(\xi^{p^i} - 1)^j$$

and (2.4) show that each product $(*)$ lies in $\Gamma^l K(X)$.

For the opposite inclusion we need

Lemma 3.2. *Consider the polynomials over* $\mathbb{Z}$ *in one variable* ζ. *For any integers* $j,\ r \geq 0$ *the polynomial* $(\zeta^r - 1)^j$ *is equal to a sum*

$$\sum_{s \geq j} a_s(\zeta - 1)^s$$

with integers a_s *such that* $s \cdot a_s$ *is a multiple of* $j \cdot r$.

Proof. It is clear that

$$(\zeta^r - 1)^j = \sum_{s=j}^{j \cdot r} a_s(\zeta - 1)^s$$

for some (uniquely determined) $a_s \in \mathbb{Z}$. Taking the derivative we obtain the statement on the coefficients. □

The additive group $K(X)$ is generated by all $p^{n_i}\xi^{rp^i}$ with $0 \leq i \leq m$ and $r \geq 0$. We have:

$$c^j(p^{n_i}\xi^{rp^i}) = \binom{p^{n_i}}{j}(\xi^{rp^i} - 1)^j = \sum_{s \geq j} \binom{p^{n_i}}{j} \cdot a_s \cdot (\xi^{p^i} - 1)^s\ .$$

Since by the lemma $j \mid s \cdot a_s$ the coefficient

$$\binom{p^{n_i}}{j} \cdot a_s \ \text{ is divisible by } \ \frac{p^{n_i}}{(s, p^{n_i})}\ .$$

Thus the Chern class $c^j(p^{n_i}\xi^{rp^i})$ is a sum (with integer coefficients) of

$$\frac{p^{n_i}}{(s, p^{n_i})}(\xi^{p^i} - 1)^s \ \text{ with } \ s \geq j$$

and we obtain the first part of the proposition.

Now suppose that $n_i = n_{i-1} - 1$ for some i. One has:

$$(\xi^{p^i} - 1)^j = \sum_{s \geq j} a_s \cdot (\xi^{p^{i-1}} - 1)^s$$

for some integer a_s with $j \cdot p \mid s \cdot a_s$. Consequently,

$$\begin{aligned}\frac{p^{n_i}}{(j,p^{n_i})}(\xi^{p^i}-1)^j &= \sum_{s\geq j} a_s \cdot \frac{p^{n_i}}{(j,p^{n_i})}(\xi^{p^{i-1}}-1)^s = \\ &= \sum_{s\geq j} b_s \cdot p \cdot \frac{p^{n_i}}{(s,p^{n_i})}(\xi^{p^{i-1}}-1)^s = \\ &= \sum_{s\geq j} c_s \cdot \frac{p^{n_{i-1}}}{(s,p^{n_{i-1}})}(\xi^{p^{i-1}}-1)^s\end{aligned}$$

for some integers b_s and c_s (the equality $n_i = n_{i-1} - 1$ is used in the last step). Thus the i-th factor in $(*)$ can be omitted. □

The proposition gives in particular a description of the group $\mathrm{G}^2\Gamma K(X)$. Now we want to find out when this group has a non-trivial torsion. We start with the case of an odd prime.

Proposition 3.3. *Let A be a p-primary algebra with an odd p. The group $\mathrm{G}^2\Gamma K(X)$ has a torsion iff $\operatorname{ind} A > \exp A$.*

Proof. See (2.2) with (2.5) for the "only if" part.

Suppose that $\operatorname{ind} A > \exp A$. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A one has:

$$n_s \leq n_{s-1} - 2 \text{ for some } s\ .$$

Consider the element

$$x = p^{n_{s-1}-2}(\xi^{p^s}-1)^2 - p^{n_{s-1}}(\xi^{p^{s-1}}-1)^2 \in \Gamma^2 K(X)\ .$$

Since in the polynomial ring $\mathbb{Z}[\xi]$ (if we imagine that ξ is just a variable) x is divisible by $(\xi-1)^3$ it is clear that a multiple of x lies in $\Gamma^3 K(X)$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3 K(X)$.

Let us act in the polynomial ring $\mathbb{Z}[\xi]$ modulo $p^{n_{s-1}-1}$. We have:

$$x \equiv p^{n_{s-1}-2}(\xi^{p^s}-1)^2\ .$$

Consider a generator of $\Gamma^3 K(X)$ from (3.1):

$$(*) \qquad \prod_{i=0}^{m} \frac{p^{n_i}}{(j_i,p^{n_i})}(\xi^{p^i}-1)^{j_i} \text{ where } j_i \geq 0 \text{ and } \sum_{i=0}^{m} j_i \geq 3\ .$$

We state that

$$(*) \equiv (\xi^{p^s}-1)^3 \cdot f(\xi^{p^s})$$

where f is a polynomial. If we would manage to show it we could proceed as follows. Suppose that $x \in \Gamma^3 K(X)$. Then in the polynomial ring $\mathbb{Z}[\xi]$ we obtain an equality:

$$p^{n_{s-1}-2}(\xi^{p^s}-1)^2 = (\xi^{p^s}-1)^3 \cdot f(\xi^{p^s}) + p^{n_{s-1}-1} \cdot g(\xi^{p^s})$$

for some polynomials f and g. Canceling by $p^{n_{s-1}-2}$ and $(\xi^{p^s}-1)^2$ and substituting $\zeta = \xi^{p^s} - 1$ we get:

$$1 = \zeta f_0(\zeta) + p g_0(\zeta) \in \mathbb{Z}[\zeta]$$

what is a contradiction because ζ and p do not generate the unit ideal in the polynomial ring $\mathbb{Z}[\zeta]$.

It remains to show that

$$(*) \equiv (\xi^{p^s} - 1)^3 \cdot f(\xi^{p^s}) \ .$$

If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \neq 0$ for some $i < s$. Write down this j_i as $j_i = p^r \cdot j$ with j prime to p. If $n_i - r \geq n_{s-1} - 1$ then

$$\frac{p^{n_i}}{(j_i, p^{n_i})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 1$. We have:

$$r > n_i - n_{s-1} + 1 \geq (s-1) - i + 1 = s - i \ .$$

In order to proceed we need

Lemma 3.4. *In the polynomial ring $\mathbb{Z}[\zeta]$ one has a congruence*

$$(\zeta - 1)^{p^k} \equiv (\zeta^p - 1)^{p^{k-1}} \mod p^k$$

for any prime p and any integer $k > 0$.

Proof. Induction on k starting from $k = 1$:

$$(\zeta - 1)^{p^{k+1}} = \Big((\zeta - 1)^{p^k} \Big)^p =$$
$$= \Big((\zeta^p - 1)^{p^{k-1}} + p^k \cdot f(\zeta) \Big)^p \equiv (\zeta^p - 1)^{p^k} \mod p^{k+1}$$

($f(\zeta)$ is a polynomial, it exists by the induction hypothesis). □

According to the lemma we have:

$$(\xi^{p^i} - 1)^{p^r} \equiv (\xi^{p^s} - 1)^{p^{r-s+i}} \mod p^{r-s+i+1} \ .$$

Hence

$$\frac{p^{n_i}}{(j_i, p^{n_i})} (\xi^{p^i} - 1)^{j_i} \equiv \frac{p^{n_i}}{(j_i, p^{n_i})} (\xi^{p^s} - 1)^{p^{r-s+i} \cdot j} \mod p^{n_i - s + i + 1} \ .$$

Since $p^{r-s+i} \cdot j \geq p \geq 3$ and $n_i - s + i + 1 \geq n_{s-1} - 1$ we are done. □

The analogous statement in the case $p = 2$ looks a little bit more complicated:

Proposition 3.5. *Let A be a 2-primary algebra. The group $\mathrm{G}^2\Gamma K(X)$ has a torsion iff $\operatorname{ind} A > \exp A$ and the reduced behaviour of A is not of the kind*

$$n, \ n-1, \ \ldots, \ 3, \ 2, \ 0 \ .$$

Proof. We start with the "only if" part. The case $\operatorname{ind} A = \exp A$ is covered by (2.2) with (2.5). Suppose that A has the reduced behaviour

$$n, \ n-1, \ \ldots, \ 3, \ 2, \ 0 \ .$$

Using the same method as in [8] one can show that the whole adjoint graded group is torsion-free. Namely, a formula like one from [8, prop. 2] states:

$$|\operatorname{Tors} \mathrm{G}^*\Gamma K(X)| = \frac{|\mathrm{G}^*\Gamma K(\mathbb{P})/\operatorname{Im} \mathrm{G}^*\Gamma K(X)|}{|K(\mathbb{P})/K(X)|}$$

where $|\,.\,|$ denotes the order of a group. Since we know the behaviour of A we can compute that

$$|K(\mathbb{P})/K(X)| = \frac{1}{2}\prod_{i=0}^{2^n-1} \frac{2^n}{(i, 2^n)}$$

(to avoid unnecessary complications we assume here that A is a division algebra). From the other hand, (2.3) shows that

$$|\mathrm{G}^i\Gamma K(\mathbb{P})/\operatorname{Im} \mathrm{G}^i\Gamma K(X)| \le \frac{2^n}{(i, 2^n)} \text{ for any } i\ .$$

Moreover,

$$|\mathrm{G}^1\Gamma K(\mathbb{P})/\operatorname{Im} \mathrm{G}^1\Gamma K(X)| \le 2^{n-1}$$

because $\xi^{2^{n-1}} - 1 \in \Gamma^1 K(X)$ (see also the computation of $\mathrm{CH}^1(X)$ [2, §2]) and therefore

$$|\mathrm{G}^*\Gamma K(\mathbb{P})/\operatorname{Im} \mathrm{G}^*\Gamma K(X)| \le \frac{1}{2}\prod_{i=0}^{2^n-1} \frac{2^n}{(i, 2^n)}\ .$$

Thus, $|\operatorname{Tors} \mathrm{G}^*\Gamma K(X)| = 1$.

Now we will "correct" the "if" proof of the previous proposition in order to match the current 2-primary situation. Suppose that we have an algebra A for which existence of torsion is stated. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A we have:

$$n_s \le n_{s-1} - 2 \text{ and } n_{s-1} \ge 3 \text{ for some } s\ .$$

Consider the element

$$x = 2^{n_{s-1}-3}(\xi^{2^s} - 1)^2 - 2^{n_{s-1}-1}(\xi^{2^{s-1}} - 1)^2 \in \Gamma^2 K(X)\ .$$

Since in the polynomial ring $\mathbb{Z}[\xi]$ the polynomial x is divisible by $(\xi - 1)^3$ it is clear that a multiple of x lies in $\Gamma^3 K(X)$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3 K(X)$.

Let us act in the polynomial ring $\mathbb{Z}[\xi]$ modulo $2^{n_{s-1}-2}$. We have:

$$x \equiv 2^{n_{s-1}-3}(\xi^{2^s} - 1)^2\ .$$

Consider a generator of $\Gamma^3 K(X)$ from (3.1):

$$(*) \qquad \prod_{i=0}^m \frac{2^{n_i}}{(j_i, 2^{n_i})}(\xi^{2^i} - 1)^{j_i} \text{ where } j_i \ge 0 \text{ and } \sum_{i=0}^m j_i \ge 3\ .$$

We state that

$$(*) \equiv (\xi^{2^s} - 1)^3 \cdot f(\xi^{2^s})$$

where f is a polynomial. If we would manage to show it we could proceed in the same manner as in the proof of the previous proposition.

If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \neq 0$ for some $i < s$. Write down this j_i as $j_i = 2^r \cdot j$ with j prime to 2. If $n_i - r \geq n_{s-1} - 2$ then

$$\frac{2^{n_i}}{(j_i, 2^{n_i})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 2$. We have:

$$r > n_i - n_{s-1} + 2 \geq (s-1) - i + 2 = s - i + 1 \; .$$

According to (3.4) we have:

$$(\xi^{2^i} - 1)^{2^r} \equiv (\xi^{2^s} - 1)^{2^{r-s+i}} \mod 2^{r-s+i+1} \; .$$

Hence

$$\frac{2^{n_i}}{(j_i, 2^{n_i})}(\xi^{2^i} - 1)^{j_i} \equiv \frac{2^{n_i}}{(j_i, 2^{n_i})}(\xi^{2^s} - 1)^{2^{r-s+i} \cdot j} \mod 2^{n_i - s + i + 1} \; .$$

Since $2^{r-s+i} \cdot j \geq 2^2 \geq 3$ and $n_i - s + i + 1 \geq n_{s-1} - 1$ we are done. □

Now we want to compute the group $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ in a special situation explicitly. The situation in mean is described in the following

Definition 3.6. We say that a reduced behaviour $(n_i)_{i=0}^m$ "makes (exactly) one jump" iff there exists exactly one s such that $n_s \leq n_{s-1} - 2$.

Example 3.7. Fix a prime p and integers $n > m \geq 1$. One can define a "generic" division algebra $\widetilde{A}$ of index p^n and exponent p^m in spirit of (2.11): take a division algebra A of index and exponent p^n, put $Y = \mathrm{SB}(A^{\otimes p^m})$ and $\widetilde{A} = A_{F(Y)}$.

The resulting algebra $\widetilde{A}$ can be also obtained as a "generic" p-primary division algebra of the reduced behaviour

$$n, \; n-1, \; \ldots, \; n-m+2, \; n-m+1, \; 0 \; .$$

In particular, it is an example of algebra which reduced behaviour "makes one jump".

Proposition 3.8. *Let A be a p-primary algebra with an odd p and suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A "makes one jump". Then* $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ *is a cyclic group of order p to the power*

$$\min\{s, \; n_0 - n_s - s\}$$

where s is the subscript for which $n_s \leq n_{s-1} - 2$.

Proof. According to (3.1) for any $l \geq 0$ the group $\Gamma^l K(X)$ is generated by all of the products:

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\xi - 1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\xi^{p^s} - 1)^{j_s} \;\; \text{with} \;\; j_i \geq 0 \;\; \text{and} \;\; j_0 + j_s \geq l \; .$$

In particular for the quotient $\mathrm{G}^2\Gamma K(X)$ we get three generators:

$$u = p^{n_0}(\xi - 1)^2 \ ;$$
$$v = p^{n_0}(\xi - 1) \cdot p^{n_s}(\xi^{p^s} - 1) \ ;$$
$$w = p^{n_s}(\xi^{p^s} - 1)^2 \ .$$

The second one can be excluded:

$$v = p^{n_s+s}u \in \mathrm{G}^2\Gamma K(X) \ .$$

Since u and w have the infinite order in the quotient, any torsion element x of the kind $x = u - kw$ or $x = ku - w$ with an integer k (if exists) generates the torsion subgroup. Consider two cases: if $n_0 \geq n_s + 2s$ then we put

$$x = u - p^{n_0-n_s-2s}w \ ;$$

otherwise we put

$$x = p^{n_s+2s-n_0}u - w \ .$$

The element $x \in \mathrm{G}^2\Gamma K(X)$ is evidently a torsion element. We finish the proof when we show that x has order p^s in the first case and order $p^{n_0-n_s-s}$ in the second. In both the cases it means the same:

$$p^{n_0+s}(\xi - 1)^2 - p^{n_0-s}(\xi^{p^s} - 1)^2 \in \Gamma^3 K(X) \tag{1}$$

and

$$p^{n_0+s-1}(\xi - 1)^2 - p^{n_0-s-1}(\xi^{p^s} - 1)^2 \notin \Gamma^3 K(X) \ . \tag{2}$$

In order to avoid repetition of some boredom computations we prove the inclusion (1) in the following way. Take an algebra B of the same degree and index as A and of the exponent coinciding with the index. We have an inclusion $K(\mathrm{SB}(B)) \subset K(X)$ (at this point it is better to consider both the Grothendieck rings in a formal way: just as rings defined by generators and relations supplied with the Chern classes). This inclusion is compatible with the gamma-filtrations and the element from (1) is obviously in $\Gamma^2 K(\mathrm{SB}(B))$. Since the group $\mathrm{G}^*\Gamma K(\mathrm{SB}(B))$ is torsion-free our element lies in $\Gamma^3 K(\mathrm{SB}))$, hence (1).

The proof of (2) goes parallel to the proof of (3.3) and does not contain any new idea. Let us act in the polynomial ring $\mathbb{Z}[\xi]$ modulo p^{n_0-s}. The element we are interested in is congruent to

$$p^{n_0-s-1}(\xi^{p^s} - 1)^2 \ .$$

Consider a generator of $\Gamma^3 K(X)$:

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\xi - 1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\xi^{p^s} - 1)^{j_s} \text{ with } j_i \geq 0 \text{ and } j_0 + j_s \geq 3 \ . \tag{$*$}$$

The proof is complete when we show that

$$(*) \equiv (\xi^{p^s} - 1)^3 \cdot f(\xi^{p^s})$$

where f is a polynomial (compare with the proof of (3.3)).

If $j_0 = 0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0 \neq 0$. Write down j_0 as $j_0 = p^r \cdot j$ with j prime to p. If $n_0 - r \geq n_0 - s$ then

$$\frac{p^{n_0}}{(j_0, p^{n_0})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_0 - r < n_0 - s$, i.e. that $r > s$. According to (3.4) we have:

$$(\xi - 1)^{p^r} \equiv (\xi^{p^s} - 1)^{p^{r-s}} \mod p^{r-s+1} .$$

Hence

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\xi - 1)^{j_0} \equiv \frac{p^{n_0}}{(j_0, p^{n_0})}(\xi^{p^s} - 1)^{p^{r-s} \cdot j} \mod p^{n_0-s+1} .$$

Since $p^{r-s} \cdot j \geq p \geq 3$ and $n_0 - s + 1 \geq n_0 - s$ we are done. □

Proposition 3.9. *Let A be a 2-primary algebra. Suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A "makes one jump" and let s be the subscript for which $n_s \leq n_{s-1} - 2$. The group* $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ *is cyclic; its order equals p to the power*

$$\begin{cases} \min\{s,\ n_0 - n_s - s\} & \text{if } n_s > 0; \\ \min\{s,\ n_0 - s - 1\} & \text{if } n_s = 0. \end{cases}$$

Proof. We describe here only changes which should be made in order to adopt the previous proof to the 2-primary case.

First suppose that $n_s > 0$.

The quotient $\mathrm{G}^2\Gamma K(X)$ has three generators:

$$u = 2^{n_0-1}(\xi - 1)^2 ;$$
$$v = 2^{n_0}(\xi - 1) \cdot 2^{n_s}(\xi^{2^s} - 1) ;$$
$$w = 2^{n_s-1}(\xi^{2^s} - 1)^2 .$$

The second one can be evidently excluded.

If $n_0 \geq n_s + 2s$ then we put

$$x = u - 2^{n_0-n_s-2s} w ;$$

otherwise we put

$$x = 2^{n_s+2s-n_0} u - w .$$

The element $x \in \mathrm{G}^2\Gamma K(X)$ generates the torsion subgroup. To verify the statement on its order we have to check that

$$2^{n_0+s-1}(\xi - 1)^2 - 2^{n_0-s-1}(\xi^{2^s} - 1)^2 \in \Gamma^3 K(X) \tag{1}$$

and

$$2^{n_0+s-2}(\xi - 1)^2 - 2^{n_0-s-2}(\xi^{2^s} - 1)^2 \notin \Gamma^3 K(X) \tag{2}$$

The inclusion (1) can be done in the same way as previously.

Let us do (2). We act in the polynomial ring $\mathbb{Z}[\xi]$ modulo 2^{n_0-s-1}. The element we are interested in is congruent to

$$2^{n_0-s-2}(\xi^{2^s}-1)^2\ .$$

Consider a generator of $\Gamma^3K(X)$:

$$(*)\qquad \frac{2^{n_0}}{(j_0,2^{n_0})}(\xi-1)^{j_0}\cdot\frac{2^{n_s}}{(j_s,2^{n_s})}(\xi^{2^s}-1)^{j_s}\ \text{ with }\ j_i\geq 0\ \text{ and }\ j_0+j_s\geq 3\ .$$

The proof is complete when we show that

$$(*)\equiv(\xi^{2^s}-1)^3\cdot f(\xi^{2^s})$$

where f is a polynomial.

If $j_0=0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0\neq 0$. Write down j_0 as $j_0=2^r\cdot j$ with odd j. If $n_0-r\geq n_0-s-1$ then

$$\frac{2^{n_0}}{(j_0,2^{n_0})}\equiv 0$$

and hence $(*)\equiv 0$. So, assume that $n_0-r<n_0-s-1$, i.e. that $r>s+1$. According to (3.4) we have:

$$(\xi-1)^{2^r}\equiv(\xi^{2^s}-1)^{2^{r-s}}\quad\operatorname{mod}\ 2^{r-s+1}\ .$$

Hence

$$\frac{2^{n_0}}{(j_0,2^{n_0})}(\xi-1)^{j_0}\equiv\frac{2^{n_0}}{(j_0,2^{n_0})}(\xi^{2^s}-1)^{2^{r-s}\cdot j}\quad\operatorname{mod}\ p^{n_0-s+1}\ .$$

Since $2^{r-s}\cdot j\geq 2^2\geq 3$ and $n_0-s+1\geq n_0-s-1$ we are done.

Now suppose that $n_s=0$.

The generators of $\mathrm{G}^2\Gamma K(X)$ are:

$$u=2^{n_0-1}(\xi-1)^2\ ;$$
$$v=2^{n_0}(\xi-1)\cdot(\xi^{2^s}-1)\ ;$$
$$w=(\xi^{2^s}-1)^2\ .$$

The second one can be evidently excluded.

If $n_0\geq 2s+1$ then we put

$$x=u-2^{n_0-2s-1}w\ ;$$

otherwise we put

$$x=2^{2s+1-n_0}u-w\ .$$

The element $x\in\mathrm{G}^2\Gamma K(X)$ generates the torsion subgroup. To verify the statement on its order we have to check that

$$(1)\qquad 2^{n_0+s-1}(\xi-1)^2-2^{n_0-s-1}(\xi^{2^s}-1)^2\in\Gamma^3K(X)$$

and

$$(2)\qquad 2^{n_0+s-2}(\xi-1)^2-2^{n_0-s-2}(\xi^{2^s}-1)^2\notin\Gamma^3K(X)$$

But it was done already (the assumption $n_s>0$ was not in use). ☐

Example 3.10. Let $\widetilde{A}$ be a "generic" division algebra of index p^n and exponent p^m (3.7). From (2.13), (3.8) and (3.9) it follows that $\operatorname{Tors}\mathrm{CH}^2(\widetilde{X})$ is a cyclic group of order p to the power

$$\begin{cases} \min\{m,\ n-m\} & \text{for an odd } p; \\ \min\{m,\ n-m-1\} & \text{for } p=2. \end{cases}$$

4. ALGEBRAS OF PRIME EXPONENT

Applying the above results to the case of a prime exponent we can state

Proposition 4.1. *Let A be an algebra of a prime exponent p. Then the group* $\operatorname{Tors}\mathrm{CH}^2(X)$ *is trivial or (cyclic) of order p. It is trivial if* $\operatorname{ind} A = p$ *or* $\operatorname{ind} A = 4$. *It is not if A is a "generic" division algebra of index p^n and exponent p (see (2.11) or (3.7)) where $n \geq 2$ in the case of odd p and $n \geq 3$ in the case when $p = 2$.*

□

It would be interesting to list all algebras A of prime exponent with trivial $\operatorname{Tors}\mathrm{CH}^2(X)$. We can only describe a class of such algebras. In [9] it was shown that any *decomposable* (into a tensor product of two smaller algebras) division algebra of index p^2 and exponent p has no torsion in $\mathrm{CH}^2(X)$ (in fact, there is no torsion in the whole graded group $\mathrm{G}^*\mathrm{T}K(X)$ [9, theorem 1]). The 2-analogy of this fact was obtained in [11, cor. 3.1]: any decomposable division algebra of index 2^3 and exponent 2 has no torsion in $\mathrm{CH}^2(X)$ (although non-trivial torsion may exist in $\mathrm{G}^*\mathrm{T}K(X)$). These facts can be generalized as follows:

Proposition 4.2. *Let A be a division algebra of prime exponent. If A decomposes then the group* $\mathrm{CH}^2(X)$ *is torsion-free.*

Proof. First consider the case when $p \neq 2$.

We have a surjection

$$\operatorname{Tors}\mathrm{G}^2\Gamma K(X) \twoheadrightarrow \operatorname{Tors}\mathrm{G}^2\mathrm{T}K(X) \simeq \operatorname{Tors}\mathrm{CH}^2(X)\ .$$

The group from the left-hand side is cyclic (3.8), its generator is represented by the element

$$x = p^n(\xi-1)^2 - p^{n-2}(\xi^p-1)^2 \in \Gamma^2 K(X) = \mathrm{T}^2 K(X)$$

where $p^n = \operatorname{ind} A$.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras. Assume that the base field F has no extensions of degree prime to p (otherwise we can replace F by a maximal extension of prime to p degree; such a change has no effect on $\mathrm{CH}^2(X)$). Take an extension E/F of degree $[E:F] = p^{n-2}$ such that $\operatorname{ind}(A_1)_E = \operatorname{ind}(A_2)_E = p$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1:F] = (\operatorname{ind} A_1)/p$ for which $\operatorname{ind}(A_1)_{E_1} = p$ and extending E_1 to E in such a way that $[E:E_1] = (\operatorname{ind} A_2)/p$ and $\operatorname{ind}(A_2)_E = p$). Consider an element

$$y = p^2(\xi-1)^2 - (\xi^p-1)^2 \in \mathrm{T}^2 K(X_E)\ .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index p^2 the group $\mathrm{CH}^2(X_E)$ is torsion-free [9, theorem 1]. Hence, $y \in \mathrm{T}^3K(X)$. Taking the transfer of y we get:

$$N_{E/F}(y) = p^n(\xi - 1)^2 - p^{n-2}(\xi^p - 1)^2 = x \in \mathrm{T}^3K(X)\ .$$

Consequently $\operatorname{Tors}\mathrm{CH}^2(X) = 0$.

Now consider the case $p = 2$.

If $\operatorname{ind} A = 4$ then $\operatorname{Tors}\mathrm{CH}^2(X) = 0$ (see e.g. (3.5) or use the Albert theorem and [9, theorem 1]). Suppose that $\operatorname{ind} A \geq 8$.

The group $\operatorname{Tors}\mathrm{G}^2\Gamma K(X)$ is cyclic (3.9), its generator is represented by the element

$$x = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 \in \Gamma^2K(X) = \mathrm{T}^2K(X)$$

where $2^n = \operatorname{ind} A$.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras and $\operatorname{ind} A_1 \geq \operatorname{ind} A_2$. Assume that the base field F has no extensions of odd degree. Take an extension E/F of degree $[E : F] = 2^{n-3}$ such that $\operatorname{ind}(A_1)_E = 4$ and $\operatorname{ind}(A_2)_E = 2$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1 : F] = (\operatorname{ind} A_1)/4$ for which $\operatorname{ind}(A_1)_{E_1} = 4$ and extending E_1 to E in such a way that $[E : E_1] = (\operatorname{ind} A_2)/2$ and $\operatorname{ind}(A_2)_E = 2$). Consider an element

$$y = 2^2(\xi - 1)^2 - (\xi^2 - 1)^2 \in \mathrm{T}^2K(X_E)\ .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index 2^3 the group $\mathrm{CH}^2(X_E)$ is torsion-free [11, cor. 3.1]. Hence, $y \in \mathrm{T}^3K(X)$. Taking the transfer of y we get:

$$N_{E/F}(y) = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 = x \in \mathrm{T}^3K(X)\ .$$

Consequently $\operatorname{Tors}\mathrm{CH}^2(X) = 0$. □

Corollary 4.3. *A "generic" algebra of prime exponent p and index p^n (3.7) is always indecomposable excluding the Albert case: $p = 2 = n$.*

Proof. Follows from (4.1) and (4.2). □

We would like to list some of works where the question of indecomposability for central simple algebras was considered previously: [1, 19, 21, 6, 10]. The method of [10] is close to but different from the one presented here; it does not cover the case $p = 2$.

References

[1] Amitsur, S. A., Rowen L. H., Tignol J.-P. *Division algebras of degree 4 and 8 with involution.* Israel J. Math. 33 (1979), 133–148.

[2] Artin, M. *Brauer-Severi varieties.* Lect. Notes Math. 917 (1982), 194–210.

[3] Blanchet, A. *Function fields of generalized Brauer-Severi varieties.* Comm. Algebra 19.1 (1991), 97–118.

[4] Fulton, W. *Intersection Theory.* Springer-Verlag, 1984.

[5] Hartshorne, R. *Algebraic Geometry.* Springer-Verlag, 1977.

[6] Jacob, B. *Indecomposible division algebras of prime exponent.* J. reine angew. Math. 413 (1991), 181–197.

[7] Karpenko, N. A. *Algebro-geometric invariants of quadratic forms.* Leningrad (St. Petersburg) Math. J. 2 (1991), no. 1, 119–138.

[8] Karpenko, N. A. *On topological filtration for Severi-Brauer varieties.* Proc. Symp. Pure Math. 58.2 (1995), 275–277.

[9] Karpenko, N. A. *On topological filtration for Severi-Brauer varieties II.* To appear in Adv. Soviet Math., 1995.

[10] Karpenko, N. A. *Torsion in* CH^2 *of Severi-Brauer varieties and indecomposability of generic algebras.* Manuscripta Math. 88 (1995), 109–117.

[11] Karpenko, N. A. *On topological filtration for triquaternion algebra.* To appear in Nova J. of Math., Game Theory and Algebra 4 (1995), no. 2.

[12] Karpenko, N. A. *Grothendieck's Chow motives of Severi-Brauer varieties.* To appear in St. Petersburg Math. J., 1996.

[13] Karpenko, N. A., Merkurjev, A. S. *Chow groups of projective quadrics.* Leningrad (St. Petersburg) Math. J. 2 (1991), no. 3, 655–671.

[14] Köck, B. *Chow motif and higher Chow theory of* G/P. Manuscripta Math. 70 (1991), 363–372.

[15] Manin, Yu. I. *Lectures on the K-functor in algebraic geometry.* Russian Math. Surveys 24 (1969), no. 5, 1–89.

[16] Merkurjev, A. S. *Certain K-cohomology groups of Severi-Brauer varieties.* Proc. Symp. Pure Math. 58.2 (1995), 319–331.

[17] Merkurjev, A. S., Panin I. A., Wadsworth, A. *A list of index reduction formulas.* Preprint 94-079, SFB preprint series, Bielefeld Univ., Germany, 1994.

[18] Quillen, D. *Higher algebraic K-theory: I.* Lect. Notes Math. 341 (1973), 85–147.

[19] Rowen, L. H. *Division algebras of exponent 2 and characteristic 2.* J. of Algebra 90 (1984), 71–83.

[20] Schofield, A., van den Bergh, M. *The index of a Brauer class on a Brauer-Severi variety.* Transactions of the Amer. Math. Soc. 333 (1992), no. 2, 729–739.

[21] Tignol, J.-P. *Algèbres indécomposibles d'exposant premier.* Adv. in Math. 63 (1987), 205–228.

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